HARCOURT MATHEMATICS 12

Geometry and Discrete Mathematics

Chapter 1

INTRODUCTION TO PROOF

The British detective Sherlock Holmes was famous for solving crimes by using deductive logic. Whether you work in law, medicine, politics, economics, or mathematics, you will use proof to establish the validity of your propositions.

In systems of logic and mathematics, a proof is a finite sequence of well-formed formulas. Our understanding of geometric proof has its roots in ancient history. In 300 B.C., the Greek mathematician Euclid put together all the geometric knowledge of the day in a set of books called *The Elements*. In these books, Euclid made such effective use of proof in geometry that the books became the standard in mathematics for over 2000 years. In the seventeenth century, a French mathematician, René Descartes, solved geometric problems by applying the newly developed skills in algebra. This new mathematics was called analytic geometry. It did not displace the geometry of Euclid, but became an additional tool for mathematicians to use. In this chapter, you will be introduced to the concept of proof and will learn how to construct a proof that presents a convincing mathematical argument.

CHAPTER EXPECTATIONS In this chapter, you will

- understand the principles of deductive proof, **Section 1.1, 1.2, 1.3, 1.4**
- prove some properties of plane figures using deduction, **Section 1.3**
- prove some properties of plane figures algebraically, **Section 1.4**

Rich Learning Link investigate

CHAPTER 1: FERMAT'S LAST THEOREM

Around 1637, Pierre de Fermat, a lawyer and amateur mathematician, conjectured that if n is a natural number greater than 2, the equation $x^n + y^n = z^n$ has no solutions where x, y, and z are non-zero integers. (Solutions where some of the integers are zero are possible but not interesting, and are known as trivial solutions.) Fermat wrote the statement in the margins of his Latin edition of a book called *Arithmetica*, written by the Greek mathematician Diophantus in the third century A.D. Fermat claimed, "I have discovered a truly marvellous proof of this, which, however, this margin is too small to contain." The result has come to be known as Fermat's Last Theorem because it was the last of his conjectures to remain unresolved after his papers were published. It became famous among mathematicians because for hundreds of years many great mathematicians attempted to prove it and achieved only partial results.

Investigate

Fermat's Last Theorem is closely related to the Pythagorean Theorem, and it is known that $x^2 + y^2 = z^2$ has integer solutions. For example, $3^2 + 4^2 = 5^2$. The numbers (3, 4, 5) are called a Pythagorean triple. There are an infinite number of Pythagorean triples. Just pick two numbers a and b with $a > b$ and set $x = 2ab$, $y = a^2 - b^2$, and $z = a^2 + b^2$. Then $x^2 + y^2 = z^2$. Try it. Then prove that it works for any a and b.

Similar to Fermat's Last Theorem is Leonhard Euler's conjecture that there are no non-trivial solutions to $x^4 + y^4 + z^4 = w^4$. This question remained unresolved for over 200 years until, in 1988, Naom Elkies found that

$$2\,682\,440^4 + 15\,365\,639^4 + 18\,796\,760^4 = 20\,615\,673^4$$

Since $x^2 + y^2 = z^2$ has infinitely many solutions and $x^4 + y^4 + z^4 = w^4$ has at least one solution, it is hard to believe that $x^n + y^n = z^n$ has no solution.

DISCUSSION QUESTIONS

1. Does $x^3 + y^3 = z^3$ have a non-trivial solution?
2. Why is there no need to consider negative integer solutions to $x^n + y^n = z^n$? That is, if we knew there were no solutions among the positive integers, how could we be sure there were no solutions among the negative integers?
3. When Fermat's Last Theorem was finally proven, its proof made headlines in newspapers around the world. Do you think the attention was justified?

Section 1.1 — What Is Proof?

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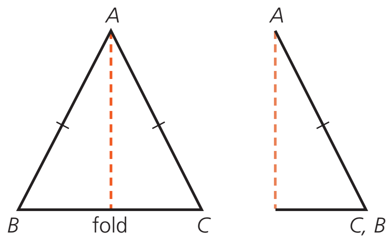
The concept of proof lies at the very heart of mathematics. When we construct a proof, we use careful and convincing reasoning to demonstrate the truth of a mathematical statement. In this chapter, we will learn how proofs are constructed and how convincing mathematical arguments can be presented.

Early mathematicians in Egypt "proved" their theories by considering a number of specific cases. For example, if we want to show that an isosceles triangle has two equal angles, we can construct a triangle such as the one shown and fold vertex B over onto vertex C. In this example, $\angle B = \angle C$; but that is only for this triangle. What if BC is lengthened or shortened? Even if we construct hundreds of triangles, can we conclude that $\angle B = \angle C$ for every isosceles triangle imaginable?



Consider the following example.

One day in class Sunil was multiplying some numbers and made the following observation:

$$1^2 = 1$$
$$11^2 = 121$$
$$111^2 = 12\,321$$
$$1111^2 = 1\,234\,321$$
$$11\,111^2 = 123\,454\,321$$

He concluded that he had found a very simple number pattern for squaring a number consisting only of 1s. The class immediately jumped in to verify these calculations and was astonished when Jennifer said, "This pattern breaks down." The class checked and found that she was right. How many 1s did Jennifer use?

From this example, we can see that some patterns that appear to be true for a few terms are not necessarily true when extended. The Greek mathematicians who first endeavoured to establish proofs applying to all situations took a giant step forward in the development of mathematics. We follow their lead in establishing the concept of proof.

In the example just considered, the pattern breaks down quickly. Other examples, however, are much less obvious. Consider the statement, *The expression $1 + 1141n^2$, where n is a positive integer, never generates a perfect square*. Is this statement true for all values of n? Does this expression ever generate a perfect square? We start by trying small values of n.

$1141(1)^2 + 1 = 1142$, which is not a square (use your calculator to verify this)
$1141(2)^2 + 1 = 4565$, which is not a square
$1141(3)^2 + 1 = 10\ 270$, which is not a square
$1141(4)^2 + 1 = 18\ 257$, which is not a square

Can we conclude that this expression never generates a perfect square? It turns out that the expression is not a perfect square for integers from 1 through to 30 693 385 322 765 657 197 397 207. It is a perfect square for the next integer, which illustrates that we must be careful about drawing conclusions based on calculations alone. It takes only one case where the conclusion is incorrect (a counter example) to prove that a statement is wrong.

We can use calculations or collected data to draw general conclusions. In 1854, John Snow, a medical doctor in London, England, was trying to establish the source of a cholera epidemic that killed large numbers of people. By examining the location of infection and analyzing the data collected, he concluded that the source of the epidemic was contaminated water. The water was obtained from the Thames River, downstream from sewage outlets. By shutting off the contaminated water, the epidemic was controlled. This type of reasoning, in which we draw general conclusions from collected evidence or data, is called **inductive reasoning**. Inductive reasoning rarely leads to statements of absolute certainty. (We will consider a very powerful form of proof called inductive proof later in this book.) After we collect and analyze data, the best we can normally say is that there is evidence either to support or deny the hypothesis posed. Our conclusion depends on the quality of the data we collect and the tests we use to test our hypothesis.

In mathematics, there is no dependence upon collected data, although collected evidence can lead us to statements we can prove. Mathematics depends on being able to draw conclusions based on rules of logic and a minimal number of assumptions that we agree are true at the outset. Frequently, we also rely on definitions and other ideas that have already been proven to be true. In other words, we develop a chain of unshakeable facts in which the proof of any statement can be used in proving subsequent statements. In writing a proof, it is important to explain our reasoning and to make sure that assumptions and definitions are clearly indicated to the person who is reading the proof. When a proof is completed and there is agreement that a particular statement can be useful, the statement is called a **theorem**.

A theorem is a proven statement that can be added to our problem-solving arsenal for use in proving subsequent statements. Theorems can be used to help prove other ideas and to draw conclusions about specific situations. Theorems are derived using **deductive reasoning**. Deductive reasoning allows us to prove a statement to be true. Inductive reasoning can give us a hypothesis, which might then be proved using deductive reasoning.

As an example of inductive reasoning, note that if we write triples of consecutive integers, say (11, 12, 13), exactly one of the three is divisible by 3. If a number of such triples are written (say two or three by everyone in the class), we can observe that every triple has exactly one number that is a multiple of 3. This provides strong evidence for us to conclude inductively that every such triple contains exactly one multiple of 3, *but it is not proof*. We will consider deductive proof in the other sections of this chapter.

Deductive reasoning is a method of reasoning that allows for a progression from the general to the particular.
Inductive reasoning is a method of reasoning in which specific examples lead to a general conclusion.

Exercise 1.1

Part A

In each of the following exercises, you are given a mathematical statement. Using inductive reasoning (that is, testing specific cases), determine whether or not the claim made is likely to be true. For those that appear to be true, try to develop a deductive proof to support the claim.

1. All integers ending in 5 create a number that when squared ends in 25. Test for the first ten positive integers ending in 5.

Knowledge/Understanding

2. The expression $f(n) = -n^3 + 5n^2 + 5n + 6$, where n is a positive integer, gives a composite number for all values of n. Test for $n = 1, 2, 3, 4, 5, 6$.

Thinking/Inquiry/Problem Solving

3. In every set of four positive integers such that the second, third, and fourth are each greater by 5 than the one preceding, there is always one divisible by 4. (One such set is 1, 6, 11, 16). Test using 5, 6, 7, 8, 11, 14, and 17 as first numbers in the set.

Part B

4. a. The expression $n^2 + n + 5$ generates a prime number for every positive integer value of n.

   b. The expression $n^2 + n + 11$ generates a prime number for every positive integer value of n.

   c. The expression $n^2 + n + 41$ generates a prime number for every positive integer value of n.

Application

5. Straight lines are drawn in a plane such that no two are parallel and no three meet in a common point. It is claimed that the nth line creates n new regions. For example, the first line divides the plane into two regions, creating one region in addition to the original one. Test this claim for $n = 1, 2, 3, 4, 5$.

6. If 9 is subtracted from the square of an even integer n greater than 2, the result is a number that is composite (has factors other than 1). Test this claim for $n = 4, 6, 8, 10, 12$.

7. If 9 is subtracted from the square of an odd integer n greater than 3, the result is a number that is divisible by 8. Test this claim for $n = 5, 7, 9, 13, 15$.

8. If 3 is subtracted from the square of a positive integer n greater than 4, the result is a composite number. Test this claim for $n = 5, 6, 7, 8, 9$.

Communication

9. John Snow, mentioned earlier, is often called the father of statistics. Discover more about his interesting work in the great cholera epidemic by searching on the London Cholera Epidemic on the Internet.

Part C

technology

10. It is claimed that the expression $n^2 - 79n + 1601$ generates a prime number for all integer values of n from 1 to 200. Write a computer program to determine whether or not a given number is prime and use it to test the claim.

Section 1.2 — An Introduction to Deductive Proof

In Section 1.1, we said that when we are proving something to be true we must clearly explain the steps in our reasoning and state any assumptions that we are making. The next example demonstrates these principles and provides a commentary to justify each step.

EXAMPLE 1 Prove that in any set of three consecutive positive integers exactly one of them is divisible by 3.

Proof

If a is any positive integer, then $a + 1$ and $a + 2$ are the next two integers, so the triple $(a, a + 1, a + 2)$ represents any set of three consecutive positive integers. If a is divided by 3, there must be a remainder of 0, 1, or 2.

If a leaves a remainder of 0, then it is divisible by 3. Further, $a + 1$ and $a + 2$ leave remainders of 1 and 2 on division by 3. Exactly one of the numbers is divisible by 3.

Next, suppose that a leaves a remainder of 1 when divided by 3. If a is the smallest of the three consecutive integers, then the next integer, $a + 1$, will leave a remainder of 2. The third integer, $a + 2$, will leave a remainder of 0, so $a + 2$ is divisible by 3. Again, exactly one of the numbers is divisible by 3.

Finally, suppose that the integer a when divided by 3 leaves a remainder of 2. Using the same argument as before, the next largest integer, $a + 1$, will leave a remainder of 0 when divided by 3, and the third integer, $a + 2$, will leave a remainder of 1, so $a + 1$ is divisible by 3. Exactly one of the numbers is divisible by 3.

We have considered all possibilities, and each one convinces us that when we take three consecutive positive integers there is always exactly one of them divisible by 3. We conclude that for any set of three consecutive positive integers exactly one of them will be divisible by 3.

There are a number of things to observe about this proof. First, the result is completely general. For any set of three consecutive positive integers, no matter how large or small, we are guaranteed that exactly one of them is divisible by 3. Second, we stated clearly the fact on which the proof was developed: when a positive integer is divided by 3, it leaves a remainder of 0, 1, or 2. Third, we considered all possibilities, leaving nothing to chance. Finally, and very importantly, the argument was presented completely in the proof. A good proof is complete, but we always strive to be brief in its presentation.

Mathematical proof enables us to construct a chain of proven facts. In the next example, we use the result of Example 1 to prove a new fact.

EXAMPLE 2 Prove that if n is an integer, $n \geq 2$, then $f(n) = n^3 - n$ is always divisible by at least 6.

Solution

Factoring the expression, we obtain

$$\begin{aligned} f(n) = n^3 - n &= n(n^2 - 1) \\ &= n(n - 1)(n + 1) \end{aligned}$$

If n is an integer and is 2 or greater, then $n - 1$, n, $n + 1$ is a set of three consecutive integers. Hence, one of these numbers is divisible by 3. Further, if we have three consecutive integers, at least one of them is even (divisible by 2) by an argument similar to that of Example 1. Then the product of the three numbers is divisible by 2×3 or 6.

Note that the fact proven in Example 1 makes our task in Example 2 simple once we factor $f(n)$.

EXAMPLE 3 If a, b, and c form an arithmetic sequence, prove that the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ has equal roots.

Proof

Two facts will help. First, if a, b, c form an arithmetic sequence, then there is a constant difference between consecutive terms. Second, for a quadratic equation having equal roots, the discriminant is 0.

Since a, b, c form an arithmetic sequence, they have a constant difference, say d. Then $b = a + d$ and $c = b + d$ or $c = a + 2d$.

$$\begin{aligned} D &= (c - a)^2 - 4(b - c)(a - b) \\ &= [(a + 2d) - a]^2 - 4[(a + d) - (a + 2d)][a - (a + d)] \\ &= (2d)^2 - 4(-d)(-d) \\ &= 4d^2 - 4d^2 \\ &= 0 \end{aligned}$$

Since $D = 0$, the roots of the equation are equal.

In these examples, we have constructed proofs that are entirely general in nature. We used no specific numerical examples in our proof, and no inductive reasoning. Note, however, that the numerical examples we considered in Section 1.1 did give us some assurance that the result is true.

Our next example uses the idea of **parity**. When two integers have the same parity, either they are both even or they are both odd. If two numbers have different parity, one of the numbers is even and the other is odd. We can say, for example, that the numbers 3 and 11 or 2 and 6 have the same parity, while 3 and 8 have different parity. Problems involving parity usually are posed for positive numbers only.

EXAMPLE 4 Prove that there is exactly one triple of numbers $(n, n + 2, n + 4)$ in which all three are prime numbers.

Proof

If n is an even number, then $n + 2$ and $n + 4$ are even also. But the only even prime is 2, so if n is an even prime, then $n = 2$. Now, $n + 2$ is 4, which is not a prime. It is not possible for n to be even.

If n is odd, then $n + 2$ and $n + 4$ are odd also. Now, one of n, $n + 2$, and $n + 4$ is always divisible by 3, because one of n, $n + 1$, and $n + 2$ is divisible by 3. If $n + 1$ is, then $n + 4 = (n + 1) + 3$ is also.

If one of the numbers n, $n + 2$, $n + 4$ is divisible by 3, that number is not a prime unless it is 3. Hence, $n = 3$ because that is the smallest odd prime, so there is one triple (3, 5, 7).

Any other triple $(n, n + 2, n + 4)$ with n odd must contain a number that is a multiple of 3, and hence is a composite number, so there is only one such triple.

When writing a proof, explain your steps and always state the facts on which you base the proof. Remember, the purpose is to convince a reader that you have constructed a clear, airtight argument.

Exercise 1.2

Part A

1. Prove that every positive integer, ending in 5 creates a number that when squared ends in 25.

2. Prove that if n is an odd positive integer, then one of the numbers $n + 5$ or $n + 7$ is divisible by 4.

Knowledge/Understanding

3. Prove that if n is an even positive integer, then $n^3 - 4n$ is always divisible by 48.

Application

4. Prove that the square of an odd integer is always of the form $8k + 1$, where k is an integer.

Part B

Communication

5. Observe that the last two digits of 7^2 are 49, the last two digits of 7^3 are 43, the last two digits of 7^4 are 01, and the last two digits of 7^5 are 07. Prove that the last two digits of 7^{201} are 07.

6. Prove that there are no integer solutions to the equation $2x + 4y = 5$.

Part C

Thinking/Inquiry/Problem Solving

7. Prove that $n^5 - 5n^3 + 4n$ is divisible by 120 for all positive integers $n \geq 3$.

8. Given that p and q are two consecutive odd prime integers, prove that their sum has three or more prime divisors (not necessarily distinct).

9. Let a_1, a_2, a_3, a_4, and a_5 be any distinct positive integers. Show that there exists at least one subset of three of these integers whose sum is divisible by 3. (Use the fact that every integer can be written in one of the forms $3k$, $3k + 1$, $3k + 2$, where k is an integer.)

10. Prove that if 4 is subtracted from the square of an integer greater than 3, the result is a composite number.

11. Prove that if 25 is subtracted from the square of an odd integer greater than 5, the resulting number is always divisible by 8.

Section 1.3 — Proof in Geometry

In Section 1.2, we illustrated some important ideas about the construction of a proof. We emphasized the following points:

1. A proof is based on assumptions and facts that we accept as true or that we have proven to be true.

2. Using these facts we can establish new conclusions, which we call theorems.

Now we'll continue our discussion of proof, in the context of geometry. Since we require a starting point, we begin by assuming three basic facts (or properties) to be true. As we proceed, you will realize that these properties are not difficult to prove. However, at this point they will be accepted as being true because they have been used in your past study of geometry and have become statements of fact that can be accepted as being true. They serve as a convenient starting point in our discussion of geometric proof. The proof of a property is called a theorem.

Angles in a Triangle **The sum of the interior angles in a triangle is 180°.**
Isosceles Triangle Property **In any isosceles triangle, the base angles are equal.**
Opposite Angles **When two straight lines intersect, opposite angles are equal.**

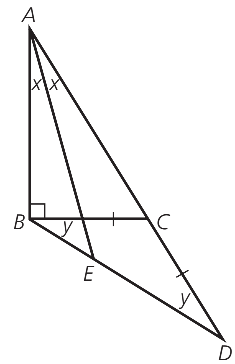
When we use theorems, we do not have to verify their correctness each time we wish to use them. The most important characteristic of theorems is that they are general in nature. For example, when we say, *The sum of the interior angles in a triangle is 180°*, we are describing every triangle. Given any triangle, we can be certain that its three angles will always add to this constant sum.

In geometric proofs, a diagram is imperative. On it we indicate all given information, and as we proceed we mark other facts as they are discovered.

EXAMPLE 1 Triangle ABC has a right angle at B. AC is extended to D so that $CD = CB$. The bisector of angle A meets BD at E. Prove that $\angle AEB = 45^\circ$.

Comment

We first draw a diagram, marking given information. Since $\angle A$ is bisected, we let $\angle BAE$ and $\angle EAD$ be x. Then $\angle BAC = 2x$. Because $CB = CD$, $\angle CDB = \angle CBD$, and they are each y. This is supplemental information that may be of use to us as we proceed. Note that we use the symbol $\angle CDB$ both as a name of the angle and to represent the measure of the angle.



Proof

In ΔBCD, $CB = CD$, so $\angle CDB = \angle CBD$.

Since the sum of the angles in a triangle is 180° in ΔABD, $(2x) + (y) + (90 + y) = 180$.

Then $\quad 2x + 2y = 90$

$\quad x + y = 45$

In ΔABE, $x + y + 90 + \angle AEB = 180$

Then $\quad 45 + 90 + \angle AEB = 180$

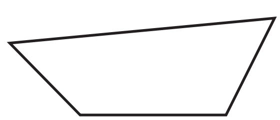
$\quad \angle AEB = 45$

Hence $\quad \angle \text{AEB} = 45^\circ$.

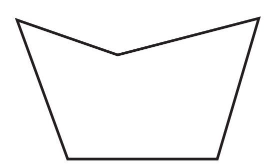
EXAMPLE 2 Prove that for any convex polygon of n sides, the sum of the interior angles is $180(n - 2)^\circ$.

Comment

First we clarify the term *convex*. A convex polygon is a polygon in which each of the interior angles is less than 180°.



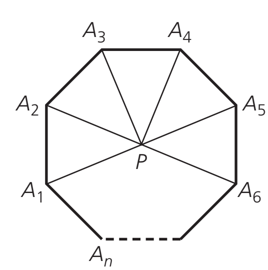
A convex polygon



A non-convex polygon

Proof

Let P be any point inside the polygon. Join P to each of the vertices. (This is why we specified a convex polygon. There are points inside a non-convex polygon that cannot be joined to all vertices by lines that lie completely inside the polygon.) There are n triangles, each having an angle sum of 180°, so the sum of the angles in the n triangles is $180n^\circ$. The sum of the n angles at P is 360°, a complete rotation. Then the sum of the angles at the vertices is $180n^\circ - 360^\circ$, or $180(n - 2)^\circ$.



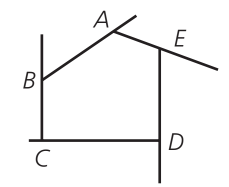
In the first example, we dealt with a specific situation, while in the second we proved a property of convex polygons in general. Since this property has general applicability, we can consider it to be a new theorem and remember it for future use.

Angles in a Convex Polygon Theorem In any convex polygon with n sides (or vertices), the sum of the interior angles is equal to $180(n - 2)^\circ$.

In a convex polygon, every interior angle has an exterior angle associated with it. If a side of the polygon is extended, the exterior angle is the angle between the extended and adjacent sides. There are two exterior angles associated with each vertex because there are two sides that can be extended. It is easy to see that these two angles are equal, because they are formed by intersecting lines. In considering the exterior angles of a polygon, we count only one angle at each vertex.

EXAMPLE 3

Prove that the sum of the exterior angles for any convex polygon is 360°.



Proof

At any vertex, the sum of the interior and exterior angles is 180°. If there are n vertices in the polygon, the sum of all interior and exterior angles is $180n^\circ$. The sum of the n interior angles is $180(n - 2)^\circ = (180n - 360)^\circ$. Then the sum of the exterior angles is $180n^\circ - (180n - 360)^\circ = 360^\circ$. Here we have another general result, one that is rather surprising. Regardless of the number of sides in a convex polygon, the sum of the exterior angles is 360°.

Exterior Angles in a Convex Polygon Theorem The sum of the exterior angles of any convex polygon is 360°.

EXAMPLE 4

For the star-shaped figure shown, prove that the sum of the angles at A, B, C, D, and E is 180°.

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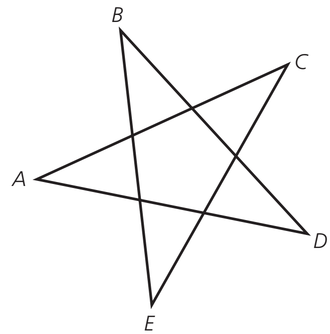


Proof

Since vertically opposite angles formed by intersecting lines are equal, we indicate equal pairs in the diagram. In each of the triangles having one of the required angles, the angle sum is 180°.

$(\angle A + v + x) + (\angle B + x + y) + (\angle C + y + z) + (\angle D + z + u) +$
$(\angle E + u + v) = 5(180)$

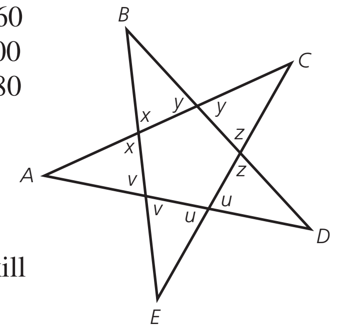
Then $\angle A + \angle B + \angle C + \angle D + \angle E + 2(x + y + z + u + v) = 5(180) = 900$

For the interior polygon, the sum of the exterior angles is 360°.

Then $\quad x + y + z + u + v = 360$

Therefore $\quad \angle A + \angle B + \angle C + \angle D + \angle E + 720 = 900$

$\angle A + \angle B + \angle C + \angle D + \angle E = 180$



The sum of the five angles is 180°.

In this section, we have tried to give you an idea of the nature of geometric proof and how to construct a proof. The exercises give you an opportunity to develop the skill of writing simple proofs.

Exercise 1.3

Part A

1. A convex polygon has 12 sides. Given that all interior angles are equal, prove that every angle is 150°.

2. In ΔKLM, P is the midpoint of the line segment LM.
Prove that if $PL = PK = PM$, $\angle LKM = 90°$.

Part B

Knowledge/Understanding

3. Prove that the sum of the exterior angles at opposite vertices of any quadrilateral is equal to the sum of the interior angles at the other two vertices.

4. In ΔABC, $\angle A$ is a right angle. The bisectors of $\angle B$ and $\angle C$ meet at D.
Prove that $\angle BDC = 135°$.

Application

5. In ΔPQR, $PQ = PR$. PQ is extended to S so that $QS = QR$.
Prove that $\angle PRS = 3(\angle QSR)$.

6. The number of degrees in one interior angle of a regular polygon is $x°$. Prove that a formula for the number of sides of the polygon is $\frac{360}{180 - x}$.
(Remember that a regular polygon has equal sides and equal interior angles.)

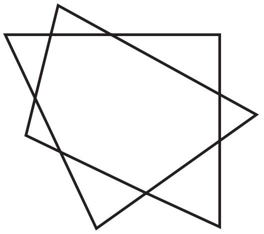
7. ΔABC is obtuse-angled at C. The bisectors of the exterior angles at A and B meet BC and AC extended at D and E, respectively. If $AB = AD = BE$, prove that $\angle ACB = 108^\circ$.

8. In ΔABC, the bisector of the interior angle at A and the bisector of the exterior angle at B intersect at P. Prove that $\angle APB = \frac{1}{2}\angle C$.

Thinking/Inquiry/
Problem Solving

9. a. For the following seven-pointed star, determine the sum of the angles at the tips of the star.

b. If a star has n points, where n is an odd number, find a formula for the sum of the angles at the tips of the star.



Section 1.4 — Proof With Analytic Geometry

In earlier grades, you used analytic geometry to solve problems. Since analytic geometry combines geometric properties with algebraic methods, it is useful in numeric problems and in problems requiring general proofs.

EXAMPLE 1 Determine the coordinates of the point that is equidistant from the three points $A(-2, 2)$, $B(6, 10)$, and $C(12, -2)$.

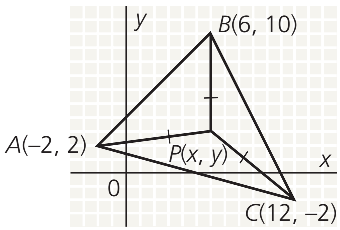
Solution

Let $P(x, y)$ be the required point.

Then $PA = PB = PC$

or

$$PA^2 = PB^2 = PC^2$$



$$PA^2 = (x + 2)^2 + (y - 2)^2 = x^2 + 4x + y^2 - 4y + 8 \qquad ①$$
$$PB^2 = (x - 6)^2 + (y - 10)^2 = x^2 - 12x + y^2 - 20y + 136 \qquad ②$$
$$PC^2 = (x - 12)^2 + (y + 2)^2 = x^2 - 24x + y^2 + 4y + 148 \qquad ③$$

Equating ① and ②, $x^2 + 4x + y^2 - 4y + 8 = x^2 - 12x + y^2 - 20y + 136$

$$16x + 16y = 128$$
$$x + y = 8 \qquad ④$$

Equating ① and ③, $x^2 + 4x + y^2 - 4y + 8 = x^2 - 24x + y^2 + 4y + 148$

$$28x - 8y = 140$$
$$7x - 2y = 35 \qquad ⑤$$

Multiply equation ④ by 2: $2x + 2y = 16$ ⑥

Adding ⑤ and ⑥ $9x = 51$

$$x = \frac{17}{3}$$

Then
$$y = 8 - x$$
$$= 8 - \frac{17}{3}$$
$$= \frac{7}{3}$$

The point $P\left(\frac{17}{3}, \frac{7}{3}\right)$ is equidistant from A, B, and C.

Using these same techniques, we can consider general proofs. Since we wish them to be general, we cannot use numeric values for points defining a figure. Instead, we name the coordinates of points in general terms. We begin by listing the basic facts with which you are familiar.

You know

1. how to determine the distance between two points
2. how to determine the equation of a line given its slope and a point on the line
3. how to determine the equation of a line given two points on the line
4. that two lines are parallel if their slopes are equal
5. that two lines are perpendicular if their slopes are negative reciprocals or if the product of the slopes is –1
6. how to determine the equation of a circle given its radius and centre

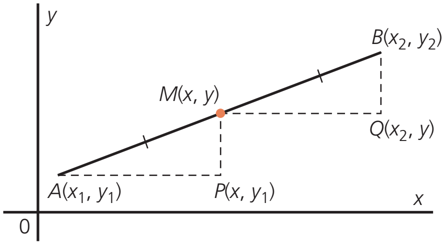
We can use these facts to consider the following examples.

EXAMPLE 2

Prove that the midpoint of the line segment connecting the points $A(x_1, y_1)$ and $B(x_2, y_2)$ has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Proof

Let the midpoint of the line segment be $M(x, y)$. Draw the run and the rise from A to M and from M to B. Then P has coordinates (x, y_1) and Q has coordinates (x_2, y).



In ΔAPM and ΔMQB, $AM = MB$

$\angle APM = \angle MQB$ (Right angle)

$\angle MAP = \angle BMQ$ (Lines parallel)

ΔAPM is congruent to ΔMQB.

Then $AP = MQ$ and $PM = QB$

Then $x - x_1 = x_2 - x$ and $y - y_1 = y_2 - y$

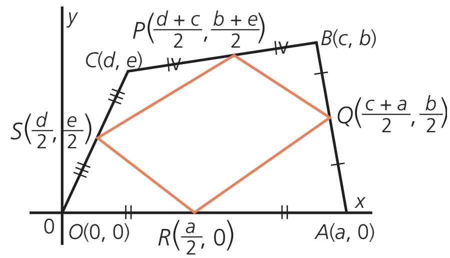
or $2x = x_1 + x_2$ and $2y = y_1 + y_2$

or $x = \frac{x_1 + x_2}{2}$ and $y = \frac{y_1 + y_2}{2}$

The coordinates of M are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

The midpoint of the line segment connecting points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

EXAMPLE 3 Prove that the lines joining consecutive midpoints of the sides of a convex quadrilateral form a parallelogram.



Proof

Let the coordinates for the quadrilateral be $O(0, 0)$, $A(a, 0)$, $B(c, b)$, and $C(d, e)$. (Note that we have placed one vertex at (0, 0) and one side along the x-axis. This will simplify calculations.)

Since we wish to show that the midpoints of the quadrilateral form a parallelogram, we first calculate the coordinates for the midpoints of the four sides.

The midpoint of OA is $R\left(\frac{a}{2}, 0\right)$.

Similarly, the midpoints of OC, BA, and CB are $S\left(\frac{d}{2}, \frac{e}{2}\right)$, $Q\left(\frac{c+a}{2}, \frac{b}{2}\right)$, and $P\left(\frac{d+c}{2}, \frac{b+e}{2}\right)$, respectively.

Now the slope of SP is $\dfrac{\frac{b+e}{2} - \frac{e}{2}}{\frac{d+c}{2} - \frac{d}{2}} = \dfrac{b}{c}$ and the slope of RQ is $\dfrac{\frac{b}{2} - 0}{\frac{c+a}{2} - \frac{a}{2}} = \dfrac{b}{c}$

The slopes are equal, so SP is parallel to RQ.

Also, the slope of RS is $\dfrac{\frac{e}{2} - 0}{\frac{d}{2} - \frac{a}{2}} = \dfrac{e}{d-a}$

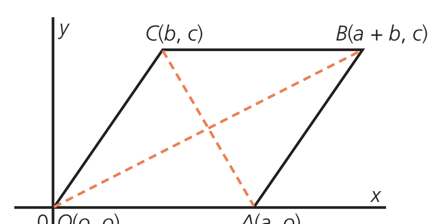
and the slope of QP is $\dfrac{\frac{b+e}{2} - \frac{b}{2}}{\frac{d+c}{2} - \frac{c+a}{2}} = \dfrac{e}{d-a}$

Then RS is parallel to QP.

Since the opposite sides are parallel, $PQRS$ is a parallelogram.

In Example 3, it was noted that convenient placing of a figure relative to the axes can simplify calculations. We must also be aware that geometric conditions will impose conditions on coordinates. This is important in the next example.

EXAMPLE 4 Prove that the diagonals of a rhombus bisect each other at right angles.



Proof

Let the rhombus have coordinates $O(0, 0)$, $A(a, o)$, $C(b, c)$, and $B(a + b, c)$. (This makes $CB = OA$.)
Since $OABC$ is a rhombus, $OA = OC$.
Then $a^2 = b^2 + c^2$, so $c^2 = a^2 - b^2$.

Now, slope $OB = \frac{c}{a + b}$ and slope $AC = \frac{c}{b - a}$, so the product of the slopes is $\frac{c}{a + b} \times \frac{c}{b - a} = \frac{c^2}{b^2 - a^2}$.

But $c^2 = a^2 - b^2$, so the product is $\frac{a^2 - b^2}{b^2 - a^2} = -1$,

and the diagonals are perpendicular.

The midpoint of OB is $P\left(\frac{a + b}{2}, \frac{c}{2}\right)$. The midpoint of AC is $Q\left(\frac{a + b}{2}, \frac{c}{2}\right)$.

Since these coordinates are the same, P and Q are the same point. The midpoint of one diagonal is also the midpoint of the other. Then the diagonals of a rhombus bisect each other at right angles.

Exercise 1.4

Part A

Always try to choose coordinates of points so as to simplify your work. It can sometimes be helpful to work a specific example before doing the general case.

1. Prove that the diagonals of a parallelogram bisect each other.

Knowledge/Understanding

2. Triangle ABC has vertices $A(-1, 3)$, $B(5, 5)$, and $C(7, -1)$.
   a. Prove that this triangle is isosceles.
   b. Prove that the line through $B(5, 5)$ perpendicular to AC passes through the midpoint of AC.

Part B

3. The ΔXYZ has its vertices at $X(5, 4)$, $Y(-2, 2)$, and $Z(9, -3)$.
   a. Determine the equation of the line drawn from vertex X to the midpoint of YZ. (The line in a triangle from a vertex to the midpoint of the opposite side is called a **median**.)
   b. Determine the equation of the median drawn from vertex Y.
   c. Prove that the two medians from parts **a** and **b** intersect at the point (4, 1).
   d. Verify that the point (4, 1) lies on the median drawn from vertex Z.
   e. What conclusion can be drawn about the medians of this triangle?

4. The ΔPQR has its vertices at $P(-6, 0)$, $Q(0, 8)$, and $R(4, 0)$.
   a. Determine the equation of a line drawn from R that is perpendicular to PQ. (This line is called the **altitude** from R to PQ.)

b. Determine the equation of the altitude drawn from P to QR.

c. Determine the coordinates of the point of intersection of the two altitudes found in parts **a** and **b**.

d. Show that the altitude from Q contains the point of intersection of the other two altitudes.

e. What conclusion can be drawn about the altitudes of this triangle?

Application

5. Prove that the diagonals in a square are equal.

6. Prove that the diagonals in a rectangle are equal.

Application

7. Prove that the line segments joining the midpoints of any rectangle form a rhombus.

8. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and has a length equal to one half of it.

9. Prove that the sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of the diagonals.

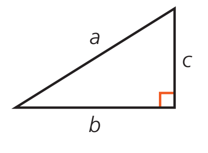
10. In ΔABC, where AD is the median, prove that $AB^2 + AC^2 = 2BD^2 + 2AD^2$. (Let the length of BC be $2a$ units and use B or D as the origin.)

11. a. In the interior of rectangle $ABCD$, a point P is chosen at random. Prove that $PA^2 + PC^2 = PB^2 + PD^2$.

b. Prove this result using the Pythagorean Theorem.

Pythagorean Theorem

If a triangle is right-angled, the square on the hypotenuse is equal to the sum of the squares on the other two sides; that is, $a^2 = b^2 + c^2$.



Part C

Thinking/Inquiry/ Problem Solving

12. Prove that the altitudes of a triangle are concurrent.

13. Prove that the medians of a triangle are concurrent.

Rich Learning Link wrap-up

investigate and apply

CHAPTER 1: FERMAT'S LAST THEOREM

Fermat's Last Theorem states that if n is a natural number greater than 2, the equation $x^n + y^n = z^n$ has no solutions where x, y, and z are non-zero integers. By 1992, mathematicians had proved the theorem for values of n up to four million. In 1994, over 350 years after Fermat first stated the theorem, Andrew Wiles, a Cambridge mathematician, finally proved it for all natural numbers n. It took Wiles more than seven years to produce the proof, including a final year to fix an error pointed out after he first announced that he had the proof. When the corrected proof was published it was well over 100 pages long. It is undoubtedly not the proof that Fermat himself claimed to have; most mathematicians working in number theory suspect that Fermat did not have a complete proof.

Investigate and Apply

1. Most proofs of Fermat's Last Theorem for the case $n = 4$ focus on the equation $x^4 + y^4 = z^2$. Show that if there are no non-trivial solutions to $x^4 + y^4 = z^2$, then there are no non-trivial solutions to $x^4 + y^4 = z^4$. (Or, equivalently, if there is a non-trivial solution to $x^4 + y^4 = z^4$, then there is a non-trivial solution to $x^4 + y^4 = z^2$. See the section on indirect proof in the next chapter.)
2. Prove that if n is a natural number greater than 2, the equation $x^n + y^n = z^n$ has no solutions where x, y, and z are non-zero rational numbers.

   Hint: Show that if there is a solution among the non-zero rational numbers then there is also a non-trivial integer solution (which we know is false).

INDEPENDENT STUDY

What kinds of questions are studied by mathematicians working in the branch of mathematics known as number theory?

What is the process by which a new mathematical result is accepted by the mathematical community?

Research other famous mathematicians who contributed to Fermat's Last Theorem, either by working out particular cases or by contributing ideas subsumed by Wiles' proof.

What are some other significant unresolved questions in mathematics?

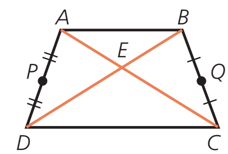
Chapter 1 Test

Achievement Category	Questions
Knowledge/Understanding	All
Thinking/Inquiry/Problem Solving	2, 7
Communication	1
Application	All

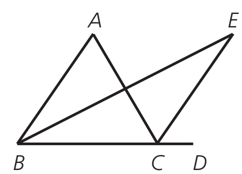
1. Given the two points $A(-2, 4)$ and $B(6, 4)$, what must be true about a point $C(x, y)$ that is equidistant from A and B?

2. Prove that the sum of the exterior angles for a convex hexagon is 360°. Be certain to explain the steps in your reasoning. (Do not use the formula for the sum of the exterior angles of a convex hexagon.)

3. $ABCD$ is a trapezoid with $AB \parallel CD$. The diagonals AC and BD have midpoints R and S. P is the midpoint of AD, and Q is the midpoint of BC. Prove, using analytic methods, that the points P, R, S, and Q lie in the same straight line.



4. In the diagram, EB bisects angle ABC, and EC bisects angle ACD. If $\angle A$ is 58°, determine $\angle E$.



5. Prove that when three consecutive even numbers are squared and the results are added, the sum always has at least three divisors.

6. In ΔABC, with vertices $A(0, a)$, $B(0, 0)$, and $C(b, c)$, prove that the right bisectors of the sides meet at a common point.

7. Julie writes the equation $x^2 + 2x + 3 = 0$. She observes that D, the discriminant, is $D = 2^2 - 4(1)(3) = -8$, which implies that the given quadratic equation has imaginary roots. She does another example, $5x^2 + 6x + 7 = 0$, and makes the same observation.
   a. Verify that the equation $8x^2 + 9x + 10 = 0$ has imaginary roots.
   b. Prove that if an equation has consecutive positive integers as its coefficients, such as $(n - 1)x^2 + nx + n + 1 = 0$, then this equation always has imaginary roots.

Chapter 2

PLANE FIGURES AND PROOF

Now that you have learned more about the basic concept of proof, you will expand your knowledge of proofs to include theorems and plane figures. In mathematics, you can put forward a theorem as something that can be proven from other propositions or formulas. As a rule or law, your theorem may be expressed as an equation formula. To develop our understanding of the properties of triangles, quadrilaterals, and other polygons, we continue to make use of the idea of proof in this chapter, extending it to include indirect proofs, as well.

CHAPTER EXPECTATIONS In this chapter, you will

- prove some properties of plane figures using deduction, **Section 2.1, 2.2, 2.5, 2.6**
- understand the principles of deductive proof, **Section 2.2, 2.3**
- prove some properties of plane figures using indirect methods, **Section 2.4**

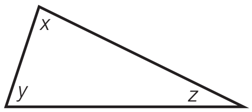
Review of Prerequisite Skills

In Chapter 1, we discussed some of the basic aspects of geometrical and non-geometrical proof. In this chapter, we continue with proof in geometry, showing how we can take basic definitions and theorems, accept them as true, and use them as building blocks for the development of further theorems. Using these theorems, we can prove new facts, constantly expanding our list of known facts.

THEOREMS ASSUMED IN CHAPTER 1

Angles in a Triangle

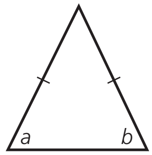
The sum of the angles in a triangle is 180º.



$x + y + z = 180º$

Isosceles Triangle Property

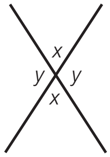
In any isosceles triangle, the base angles are equal.



$a = b$

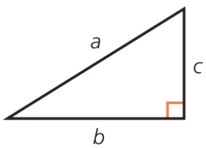
Opposite Angles

When two lines intersect, opposite angles are equal.



Pythagorean Theorem

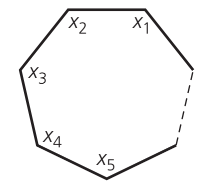
If a triangle is right-angled, the square on the hypotenuse is equal to the sum of the squares on the other two sides; in the diagram, $a^2 = b^2 + c^2$.



THEOREMS PROVED IN CHAPTER 1

Angles in a Convex Polygon

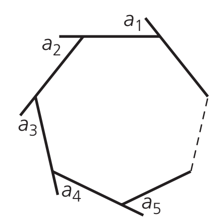
In a convex polygon of n sides,
the sum of the interior angles is $180(n - 2)°$.



$$x_1 + x_2 + x_3 + \ldots. + x_n = 180(n - 2)$$

Exterior Angles in a Convex Polygon

The sum of the exterior angles of a convex polygon is 360°.

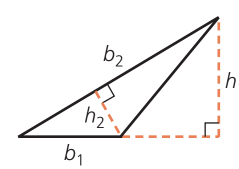


$$a_1 + a_2 + a_3 + \ldots + a_n = 360$$

THEOREMS ASSUMED IN CHAPTER 2

Area of a Triangle

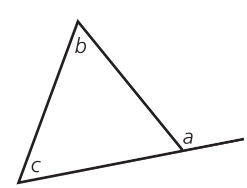
The area of a triangle having base b and height h is $A = \frac{1}{2}bh$.



$$A = \frac{1}{2}b_1h_1 = \frac{1}{2}b_2h_2$$

Exterior Angle Property

An exterior angle of a triangle is equal to the sum of the two interior and opposite angles.

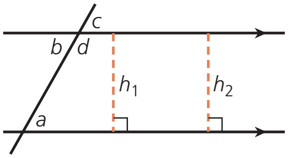


$$a = b + c$$

Parallel Line Properties

If two lines are parallel

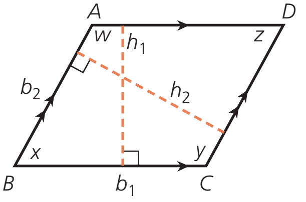
1. alternate angles a and b are equal
2. corresponding angles a and c are equal
3. the sum of co-interior angles a and d is 180º
4. distance between the lines is constant



Properties of a Parallelogram

In a parallelogram

1. opposite sides are equal
2. opposite angles are equal
3. the area is $A = bh$



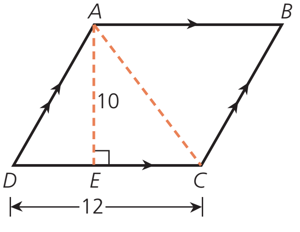
$AB = CD$ and $AD = BC$
$x = z$ and $w = y$
$A = b_1h_1 = b_2h_2$

These relationships are the basis for the theorems that we will prove in this chapter.

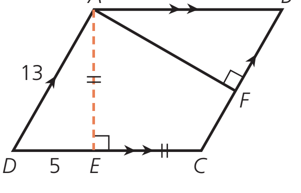
Exercise

Part A

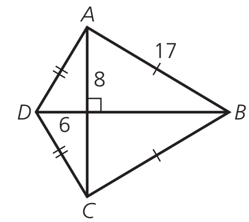
1. In $\|^{gm}$ $ABCD$, $DC = 12$ and $AE = 10$.
   a. Calculate the area of $\|^{gm}$ $ABCD$.
   b. What is the area of ΔABC?



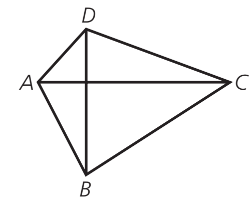
2. In $\|^{gm}$ $ABCD$, lengths are marked as shown.
   $AE = EC$.
   a. Determine the area of $\|^{gm}$ $ABCD$.
   b. Determine the length of AF.



3. Calculate the area of figure *ABCD*.

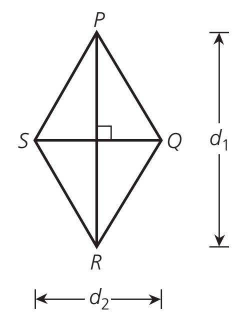


4. Quadrilateral *ABCD* has $BD = 12$ and $AC = 16$. $AC \perp BD$. Determine the area of quadrilateral *ABCD*.

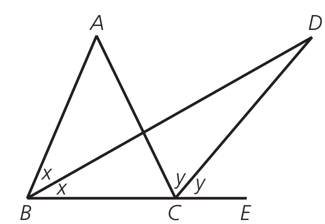


Part B

5. The rhombus *PQRS* has diagonals of length d_1 and d_2 as shown. Prove that the area of a rhombus is given by the formula $A = \frac{d_1 d_2}{2}$.

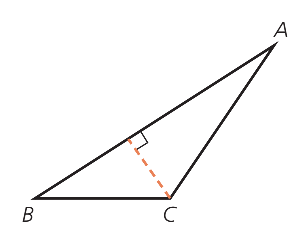


6. If $\angle ABC = 70°$ and $\angle A = 30°$, determine the measure of $\angle BDC$.



technology
APPENDIX P. 490

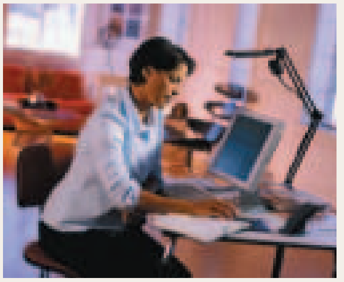
7. In ΔABC, $AB = 12$, $AC = 10$, and the altitude from *C* is 3. Determine the length of the altitude from *B* to *AC*.



Rich Learning Link

investigate

CHAPTER 2: VARIGNON PARALLELOGRAM

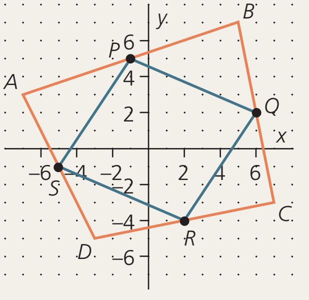


Parallelograms are quadrilaterals with opposite sides that are parallel. They are a common shape because rectangles are a type of parallelogram. Non-rectangular parallelograms are much less common, but they do arise as the solution to certain engineering and design problems. They are used in some jointed desk lamps, where they allow the light-bulb end to be moved without changing the direction in which the bulb points. For similar reasons, they are used in some binocular mounts for amateur astronomers and some motorcycle suspension systems. In movies, scenes that require a moving camera can be made jitter-free by using a Steadicam, a device which works because of the contribution of parallelograms.

Investigate and Inquire

In geometry, if we connect four points on a plane, the result is not likely to be a parallelogram. However, if we connect the midpoints of the sides of the quadrilateral, the resulting interior quadrilateral is a parallelogram.

Pick four points *A*(–7, 3), *B*(5, 7), *C*(7, –3), and *D*(–3, –5) on a Cartesian plane, as shown. The midpoints of *AB*, *BC*, *CD*, and *DA* are found to be *P*(–1, 5), *Q*(6, 2), *R*(2, –4), and *S*(–5, –1), respectively.



The slope of *PQ* is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 5}{6 - (-1)} = -\frac{3}{7}$.

A similar calculation shows that the slope of *SR* is also $-\frac{3}{7}$. Thus, *PQ* and *SR* are parallel.

Similarly, *QR* and *PS* can both be shown to have a slope of $\frac{3}{2}$, so they are also parallel. Thus, *PQRS* is a parallelogram.

Try this with four other points. Using Geometer's Sketchpad® will allow you to move your four points independently.

DISCUSSION QUESTIONS

1. If a quadrilateral has special properties, it usually has a special name. For example, a parallelogram is a special quadrilateral with opposite sides that are parallel. What are some other special quadrilaterals?
2. A rectangle is a special type of parallelogram. What are some other special parallelograms?
3. Consider a parallelogram frame that has one side held in position (e.g., mounted against the side of a wall). Describe the possible motions of the parallelogram.

Section 2.1 — Proofs Using Congruent Triangles

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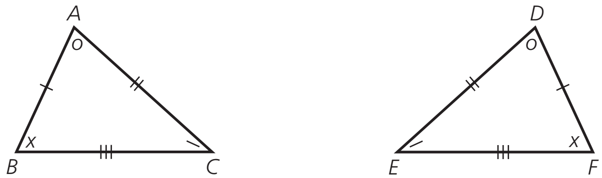
Two geometric figures are **congruent** if they are exactly the same except for position. That is, there is a way to match one figure with the other so that corresponding parts of the figures coincide. Thus measurable quantities, such as the lengths of corresponding line segments, the measures of corresponding angles, and the areas, are equal.



The two triangles shown above are congruent. We write $\Delta ABC \equiv \Delta DFE$. In this statement, the order of the letters gives the correspondence of the figures. Without referring to the diagram, we know from the statement that $\angle B = \angle F$ and $AB = DF$. Many other properties are equal because of the congruence. For example, the altitude from A and the altitude from D are equal. If we wish to indicate that the triangles are equal in area but are not necessarily congruent, we write $\Delta ABC = \Delta DEF$.

We use triangle congruence extensively in geometric proof, mainly because triangles are easily defined geometric objects. If we specify three properties (sides or angles) in the right combination, we generate only one triangle. All triangles with the same three properties must be congruent. The combinations that specify triangles are given by the following theorem. Since you have seen this result earlier, we omit the proof.

Triangle Congruence Theorem

Two triangles are congruent if

1. **(side-angle-side) two sides and the contained angle of one triangle are respectively equal to two sides and the contained angle of the second triangle; or if**
2. **(side-side-side) three sides of one triangle are respectively equal to three sides of the second triangle; or if**
3. **(angle-side-angle) two angles and a side of one triangle are respectively equal to two angles and the corresponding side of the second triangle; or if**
4. **(hypotenuse-side) the hypotenuse and one other side of one right triangle are respectively equal to the hypotenuse and one other side of a second right triangle.**

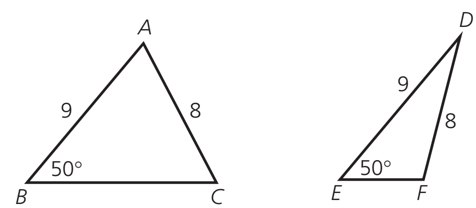
Here are three additional observations:

1. Angle-angle-angle is *not* a triangle-congruence combination. Since the total of the angles is fixed, we know that when two pairs of angles are equal, the third pair must be equal; saying so does not give a third property.

2. The angle-side-angle combination can have the equal sides in any position relative to the pairs of equal angles, as long as they are the same position in both triangles.

3. If there are two pairs of equal sides and one pair of equal angles in two triangles, the angles must be contained (where the pairs of sides meet) or right angles (the third property in the hypotenuse-side combination). Recall from your study of trigonometry that there can be two triangles in some cases; this was referred to as the ambiguous case.

   For example, $\angle ABC$ and $\angle DEF$ have two pairs of equal sides and a pair of equal angles, but they are not congruent. The equal angles are neither right angles nor contained by the equal sides.



In writing a proof using congruent triangles, we usually

1. identify the triangles we are considering (it is a matter of preference whether we are careful about the correspondence of vertices at this stage)

2. list the three pairs of equal components, giving reasons, especially where the equality is not obvious

3. state the congruence, implying the correspondence by the order of the vertices (it is customary to refer to the triangle congruence combination being used)

Once we have established that the triangles are congruent, we can draw conclusions about components of the triangles that were not previously known to be equal. Note that if it is necessary, in order to establish one of the three properties, to appeal to facts not relevant to the triangle congruence, we will generally do this before identifying the triangles.

EXAMPLE 1 Prove that the median to the base of an isosceles triangle bisects the vertical angle.

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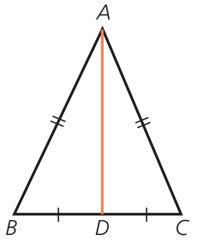
Solution

On the diagram we indicate what is known at the beginning.
Thus we have an isosceles triangle ABC, with $AB = AC$. We have drawn the median AD, so $BD = DC$. We need to prove that $\angle BAD = \angle CAD$.



Proof

In ΔABD and ΔACD,

	$AB = AC$	(given)
	$BD = CD$	(median)
	$AD = AD$	(common)
Then	$\Delta ABD \equiv \Delta ACD$	(side-side-side)
Therefore	$\angle BAD = \angle CAD$	(triangles congruent)

Then the median bisects the vertical angle.

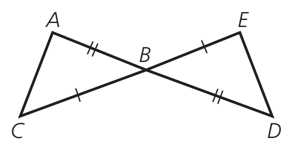
Notice that the same pair of congruent triangles show that $\angle B = \angle C$, the Isosceles Triangle Property. Had we used this instead of the common side AD, the triangle congruence combination would have been side-angle-side.

EXAMPLE 2 In the diagram, $AB = DB$ and $CB = EB$. Prove $\Delta ABC \equiv \Delta DBE$.

Proof

In ΔABC and ΔDBE,

$BA = BD$	(given)
$\angle ABC = \angle DBE$	(vertically opposite angles)
$BC = BE$	(given)
Therefore, $\Delta ABC \equiv \Delta DBE$	(side-angle-side)

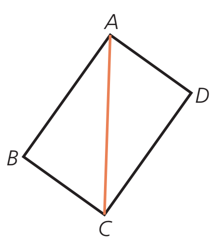


Exercise 2.1

Part A

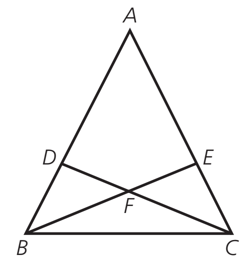
Communication

1. For the diagram given, state all properties that must be present for ΔABC to be congruent with ΔADC. Use
   a. the side-side-side property
   b. the side-angle-side property
   c. the angle-angle-side property
   d. the hypotenuse-side property



Communication

2. In the given diagram, $AB = AC$.



a. State one additional fact that would prove that $\Delta ABE \equiv \Delta ACD$.

b. If $\Delta ABE \equiv \Delta ACD$, could you state enough reasons for ΔDBC to be congruent with ΔECB?

c. Could you now prove $\Delta FDB \equiv \Delta FEC$?

3. Given the statement $\Delta ABC \equiv \Delta DEF$, draw a diagram illustrating the triangles.

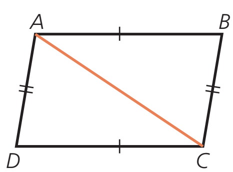
a. Which angle in ΔDEF is equal to $\angle ABC$ in ΔABC?

b. Which side in ΔABC is equal to DF in ΔDEF?

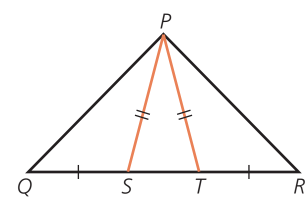
Knowledge/
Understanding

4. In the following diagrams, we have marked the equal angles and sides. Find a pair of congruent triangles in each diagram, and write a proof explaining why the triangles are congruent.

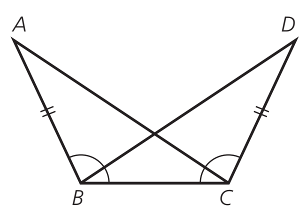
a.



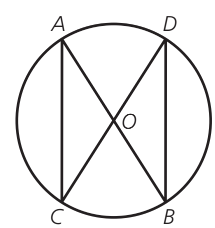
b.



c.

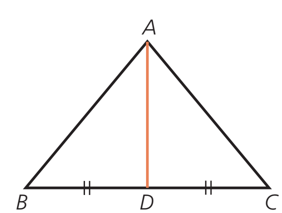


d.



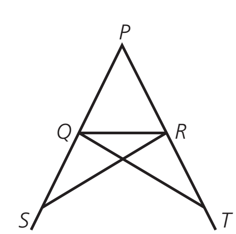
AB and *CD* are diameters

5. Is the statement that follows true or false? Justify your answer.
The median of a triangle divides a triangle into two congruent triangles.



Part B

6. In the given diagram, $PQ = PR$ and $PS = PT$. Prove that $\angle QRS = \angle RQT$.



Application

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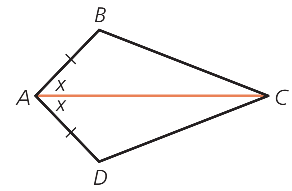
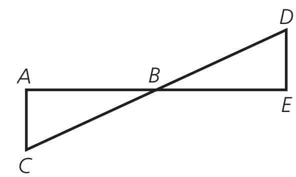
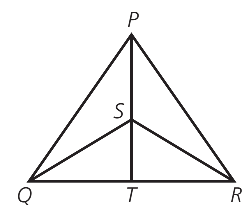
7. In quadrilateral $ABCD$, the diagonal AC bisects $\angle DAB$ and $AB = AD$. Prove that AC bisects $\angle BCD$ and $BC = DC$.



8. In the given diagram, ABE and CBD are straight lines. If $AB = BE$, AC and AB are perpendicular, and DE and BE are also perpendicular, prove that $AC = DE$.



9. In ΔPQR, $PQ = PR$ and $\angle QPT = \angle RPT$. Prove that $\angle SQT = \angle SRT$.



10. $PQRS$ is a quadrilateral in which $PQ = SR$. If the diagonals PR and QS are equal, then prove that $\angle PQR = \angle SRQ$.

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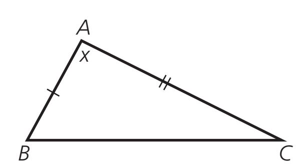
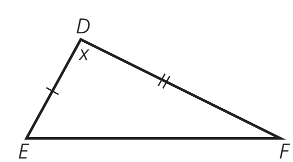
11. Prove that if the opposite sides of a quadrilateral are equal then the diagonals bisect each other.

12. S is the midpoint of side QR of ΔPQR. QT and RW are drawn perpendicular to PS or PS extended. Prove that $QT = RW$.

13. We know that two triangles are congruent if two sides and the contained angle of one are respectively equal to two sides and the contained angle of a second triangle. Using ΔABC and ΔDEF, write an informal proof showing that this statement is true.





Thinking/Inquiry/ Problem Solving

14. Discuss the conditions needed for two quadrilaterals to be congruent. Is there a possible statement about quadrilaterals that parallels

a. the triangle congruent situation side-angle-side?

b. the triangle congruent situation angle-side-angle?

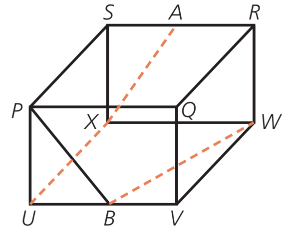
c. the triangle congruent situation side-side-side?

Part C

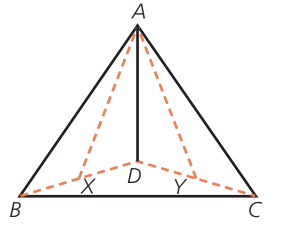
15. The diagram shows a rectangular solid in which A is the midpoint of SR, and B is the midpoint of UV.

a. Prove that $AP = WB$.

b. Prove that $AX = PB$.



16. A tetrahedron has four identical equilateral faces. X and Y are points on BD and CD, respectively, so that $BX = CY$. Prove that $AX = AY$.



Section 2.2 — Conditional Statements

We often hear people make statements in which one action is a consequence of another. We might hear someone say, *If you tip that glass the water in it will run out*, or *If you study diligently, then your knowledge will increase*. Statements in which the first part implies the second part as a natural conclusion are called **conditional statements.** One example of a mathematical conditional statement is, *If a triangle is drawn having exactly two equal sides then the triangle will have exactly two equal angles.* In this section, we examine the structure of conditional statements and their role in mathematical proof.

Before considering further mathematical examples we examine the structure of conditional statements using examples from everyday life. Consider the following statements:

p: I throw a stone into a pond.
q: Ripples are produced in the pond.

If we write $p \rightarrow q$, we are saying, *If I toss a stone into a pond then ripples are produced in the pond*. When we write $p \rightarrow q$, we read it as *If* p*, then* q. Is this statement true? Our experience tells us that ripples are produced when a stone is tossed into a pond. The conclusion follows as a natural consequence of the premise that a stone is tossed into the pond. In mathematical terms, we are saying that the conclusion q follows as a result of p being true. The combination of the two simple statements, together with *If… then…* creates a conditional statement.

When considering the truth of q in the statement $p \rightarrow q$, we are asking, *If we accept* p *as being true, does the conclusion* q *follow absolutely?* The truth of such statements depends on the premises that we accept. For example, suppose we let p and q represent the following:

p: My family pet has four legs.
q: It is a dog.

In this example, we read the statement $p \rightarrow q$, as *If my family pet has four legs then it is a dog.* This statement is not necessarily true. The fact that my family pet has four legs does not guarantee that it is a dog.

We use this type of logical inference frequently in everyday speech. When we apply it in mathematics we are attempting a similar kind of reasoning, but with a higher degree of precision. Consider the statement, *If a prime number is squared then the resulting square has exactly three divisors.* This statement implies that if we choose a prime and square it the result is always a number having three divisors. As we know, this statement is true. If we square a prime number p, the resulting number p^2 has exactly three divisors: 1, p, and p^2. Verify that this statement is true using the primes 3, 5, and 7.

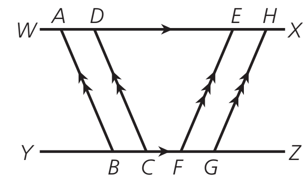
In mathematical applications, if we can prove that a statement q is always true if a given statement p is known, then the conditional statement defines a theorem.

THEOREM

Consider the following statements:

p: Two parallelograms have equal bases and lie between the same parallel lines.
q: The parallelograms have the same area.

If we can show that the conditional statement $p \rightarrow q$ is always true, we will always be able to make an immediate conclusion when we encounter such parallelograms. We can do so using the theorems stated at the beginning of the chapter.



Proof

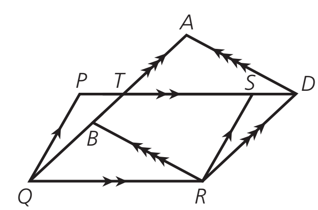
Let the parallelograms be $ABCD$ and $EFGH$ with $BC = FG$. Both parallelograms have opposite sides in the two parallel lines WX and YZ. For each parallelogram, the height is the distance between the parallel lines. Using the theorem of Parallel Line Properties, we know that such distances are equal.

Since the parallelograms have equal bases and equal heights, their areas are equal.

Parallelogram Area Property **Two parallelograms having equal bases (or the same base) and lying between two parallel lines have the same area.**

EXAMPLE 1

In the given diagram, $PQ \| RS$, $PD \| QR$, $QA \| RD$, and $BR \| AD$. Prove that the area of $\|^{gm}\ PQRS = \|^{gm}\ BRDA$.



Proof

$\|^{gm}\ PQRS$ and $\|^{gm}\ TQRD$ have QR as a common base and lie between parallel lines QR and PD.

Then area $\|^{gm}\ PQRS$ = area $\|^{gm}\ TQRD$. (Parallelogram Area Property Theorem)

$\|^{gm}\ TQRD$ and $\|^{gm}\ BRDA$ have RD as a common base and lie between parallel lines RD and QA.

Then area $\|^{gm}\ TQRD$ = area $\|^{gm}\ BRDA$. (Parallelogram Area Property Theorem)

Then area $\|^{gm}\ PQRS$ = area $\|^{gm}\ BRDA$.

Notice that here we have used extended reasoning. Since $p = q$ and $q = r$, then $p = r$. This type of reasoning is used frequently in the construction of proofs.

EXAMPLE 2

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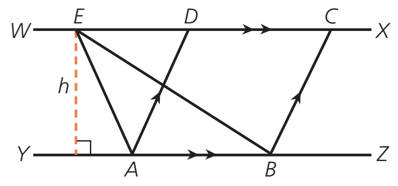
Prove that if a triangle and a parallelogram are on the same base and between the same parallels, then the area of the triangle equals $\frac{1}{2}$ that of the parallelogram.

Proof



Let the triangle EAB and the parallelogram $DABC$ have the common base AB and lie between the parallel lines WX and YZ.

Since the distance between parallel lines is constant, represent this distance by h. (Parallel Line Properties Theorem)

Since h is the height of ΔEAB,

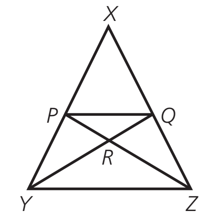
$$\text{area } \Delta EAB = \frac{1}{2}(AB)h$$

$$\text{area } \|^{gm}\, ABCD = (AB)h$$

Thus area $\Delta EAB = \frac{1}{2}(\text{area } \|^{gm}\, ABCD)$.

EXAMPLE 3

In the diagram given, $XY = XZ$ and P and Q are on the sides of ΔXYZ. Prove that if the area of ΔPRY equals the area of ΔQRZ, then the distance from P to XZ is equal to the distance from Q to XY.



Proof

Let the distance from Q to XY be h_1 and the distance from P to XZ be h_2. Note that h_1 is the altitude in ΔXQY and that h_2 is the altitude in ΔXPZ.

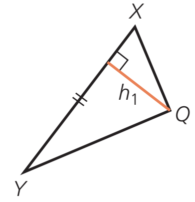
Also $\Delta XQY = \Delta PRY + \text{quad } XPRQ$

$\Delta XPZ = \Delta QRZ + \text{quad } XPRQ$

But $\Delta PRY = \Delta QRZ$

Then $\Delta XQY = \Delta XPZ$

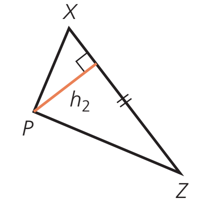
$\frac{1}{2}(XY)h_1 = \frac{1}{2}(XZ)h_2$



But $XY = XZ$

Then $h_1 = h_2$

The distances are equal.



In this section we have introduced a number of proofs that involve simple conditional reasoning. When we write the statement $p \rightarrow q$, we have a conditional statement or an *If … then…* statement. We start with a premise p that we assume to be correct and attempt to prove that the condition q follows.

Exercise 2.2

Part A

Communication

1. For each of the following statements,

a. identify the premise and conclusion;

b. determine whether or not the conclusion follows from the premise;

c. if the conclusion is incorrect, state an amendment that makes it correct.

(i) If an integer ends in a 0 or a 5, then the integer is divisible by 0 or 5.

(ii) If an odd integer is squared, then the resulting number will always have an even number of divisors.

(iii) If a positive integer is divided by a positive integer n, then the remainder is an integer in the set 0, 1, 2, …, $n - 1$.

(iv) If a parallelogram has an area of t square units, then a parallelogram of $2t$ square units has a base that is twice as large as the first parallelogram.

(v) If a five-sided figure has 3 angles each equal to 90º, then the remaining two angles must both be obtuse.

(vi) If an integer is squared, then the resulting number always gives a remainder of 1 when divided by 4.

Knowledge/Understanding

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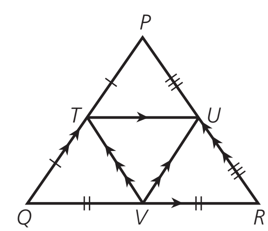
2. In ΔPQR, T, U, and V are the midpoints of PQ, PR, and QR, respectively. If $TU \parallel QR$, $TV \parallel PR$, and $VU \parallel QP$, name three parallelograms having equal area.



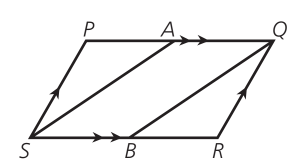
Knowledge/Understanding

3. Given a parallelogram $PQRS$, draw a rectangle equal in area to the parallelogram.

4. Given a parallelogram $ABCD$, draw a second parallelogram having an area double that of parallelogram $ABCD$.

Part B

5. $PQRS$ is a parallelogram. If A is the midpoint of PQ and B is the midpoint SR, prove that the area of $\|^{gm}\ ASBQ = \frac{1}{2}$ area of $\|^{gm}\ PQRS$.



Application

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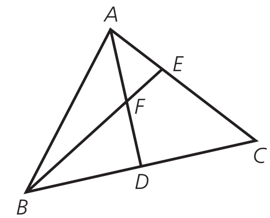
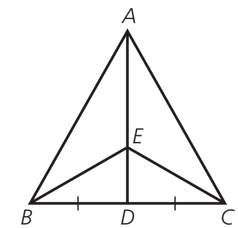
6. Prove that if a line in a triangle is a median, then the line bisects the area of the triangle.

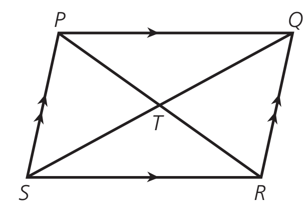
7. Prove that if AD and BE are medians in the given triangle, then $\Delta AEF = \Delta FBD$.



8. AD is the median of ΔABC. Prove that $\Delta ABE = \Delta ACE$.

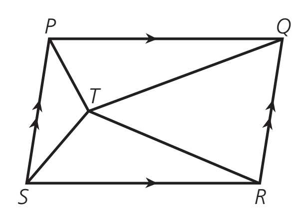


9. $PQRS$ is a parallelogram. Prove that ΔPQT, ΔPTS, ΔTSR, and ΔTRQ all have equal area.



10. Prove that if the diagonals of a quadrilateral divide it into four triangles of equal area, then the quadrilateral is a parallelogram.

11. $PQRS$ is a parallelogram and T is any point inside the parallelogram. Prove that $\Delta TSR + \Delta TQP = \frac{1}{2} \|^{gm} PQRS$.



12. $ABCD$ is a quadrilateral whose area is bisected by the diagonal AC. Prove that BD is bisected by AC.

13. BE and CF are two medians of ΔABC, intersecting at O. Prove that $\Delta OFA = \Delta OEA$.

14. A parallelogram $ABCD$ has diagonal AC extended to X. Prove that $\Delta ABX = \Delta ADX$.

15. A point O is on the side BC of $\|^{gm} ABCD$. Prove that $\Delta AOB + \Delta DOC = \frac{1}{2}\|^{gm} ABCD$.

Part C

Thinking/Inquiry/
Problem Solving

16. For a quadrilateral *ABCD*, describe a procedure for the construction of a triangle that is equal in area to the quadrilateral.

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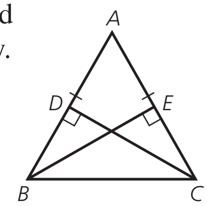
17. *ABC* is an isosceles triangle in which *AB* = *AC*. *BE* and *CD* are drawn perpendicular to *AC* and *AB* respectively. Prove that $DE \parallel BC$.



18. In ΔABC, *AB* = *AC* and *D* is any point on *AB*. *AC* is extended to *E* so that *CE* = *BD*. *DE* cuts *BC* at *K*. Prove that *DK* = *KE*.

Section 2.3 — The Converse of a Conditional Statement

technology
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The theorems we have accepted and the results we have proved from them are examples of conditional statements. If one statement (the premise) is true, then another statement (the conclusion) is true. Symbolically, if p is the premise and q is the conclusion, we write $p \rightarrow q$, which is read, "p implies q," or "if p then q."

An idea closely related to a conditional statement $p \rightarrow q$ is that of its converse, which we write as $q \rightarrow p$. We cannot conclude that the converse is true because the original statement is true. Consider the statement, *If the traffic light is red then we stop the car.* This statement is true for both legal and safety reasons. The converse is *If we stop the car, then the light is red*. This statement is not necessarily true; we can stop the car for a variety of reasons.

If a statement is true ($p \rightarrow q$) and its converse is also true ($q \rightarrow p$), we write $p \leftrightarrow q$, which is read, "p if and only if q (or q if and only if p)." We sometimes write this as "p iff q."

When we write $p \leftrightarrow q$ we recognize that the truth of either of the statements depends upon the truth of the other. This type of statement is said to be **biconditional.**

EXAMPLE 1 For each of the following statements

a. state the converse
b. determine whether the converse is a true statement
c. if the converse is true, restate the sentence as an *. . .if and only if. . . .* statement

1. If one side of a balance falls, there is more weight on that side than on the other.
2. If one of two integers is even and the other is odd, then the sum of the integers is odd.
3. If it is spring, then the grass is green.

Solution

1. The converse of this statement is, *If there is more weight on one side of a balance than on the other, then one side of the balance falls.* This is certainly true. The biconditional statement is, *One side of a balance falls if and only if there is more weight on that side than on the other.*

2. The converse of this statement is, *If the sum of two integers is odd, then one of the integers is even and one is odd.* This is certainly true. The biconditional statement is, *The sum of two integers is odd if and only if one of them is even and the other is odd.*

3. The converse of this statement is, *If the grass is green, then it is spring.* This statement is certainly not true.

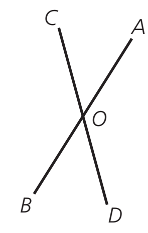
EXAMPLE 2 Is it true that $p \leftrightarrow q$, for these two statements?

p: Two angles are vertically opposite. q: The two angles are equal.

Solution

Part 1 $p \rightarrow q$

The first statement we will prove is, *If two angles are vertically opposite then they are equal*.



Since $\angle COA + \angle COB = 180º$, $\angle COB = 180º - \angle COA$

Since $\angle COA + \angle AOD = 180º$, $\angle AOD = 180º - \angle COA$

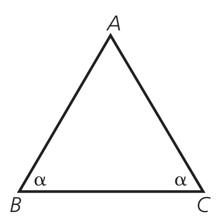
Therefore $\angle COB = \angle AOD$

The statement is true.

Part 1 $q \rightarrow p$

The second statement is, *If two angles are equal then they are vertically opposite angles.*

We can demonstrate that this statement is not true by constructing an example showing it to be false.



In isosceles triangle *ABC*, $\angle ABC = \angle ACB$ but these two angles are not vertically opposite each other. Then $p \leftrightarrow q$ is not true because $p \rightarrow q$ and $q \rightarrow p$ are not both true.

THEOREM Prove the biconditional statement, *A point is on the right bisector of a given line segment if and only if it is equidistant from the ends of the segment.*

Solution

First we prove that any point on the right bisector of a line segment is equidistant from the end points of the line segment.

Proof

Let AB be any line segment and XY be its right bisector, cutting AB at D. Let P be any point on XY. Join P to A and P to B.

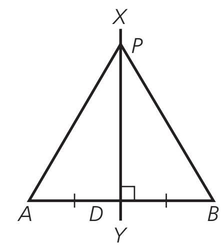
In triangles PAD and PBD, $AD = BD$ (right bisector)

$\angle PDA = \angle PDB$ (right angles)

$PD = PD$ (same line)

Then $\Delta PAD \equiv \Delta PBD$ (side-angle-side)

Then $PA = PB$



Now we prove that if a point is equidistant from the endpoints of a line segment, then it is on the right bisector of the line segment.

Proof

We are given $PA = PB$. We join P to M, the midpoint of AB.
In triangles PAM and PBM,

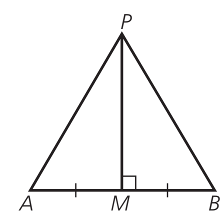
$PA = PB$ (given)

$AM = BM$ (constructed)

$PM = PM$ (same line segment)

Then $\Delta PAM \equiv \Delta PBM$ (side-side-side)

Then $\angle PMA = \angle PMB$ and, since their sum is 180º,
$\angle PMA = \angle PMB = 90º$



Then PM is the right bisector of AB.
We combine the results of these two properties in the Right Bisector Theorem.

Right Bisector Theorem A point is on the right bisector of a given line segment if and only if it is equidistant from the end points of the line segment.

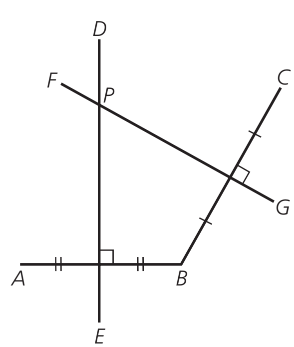
EXAMPLE 3 Determine the position of a point P equidistant from three given points A, B, and C that are not in a straight line.

Solution

If P is equidistant from A and B it must lie on the right bisector of AB, so P is a point on DE, the right bisector of AB.

Similarly, P is a point on FG, the right bisector of BC.

The point P is equidistant from A, B, and C, and so $PA = PB = PC$.

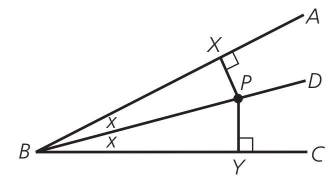


THEOREM A point is on the bisector of an angle if and only if it is equidistant from the arms of the angle.

Proof

Part 1

If a point is on the bisector of an angle, it is equidistant from the arms of the angle.



Let BD be the bisector of $\angle ABC$ and let P be any point on BD.

From P draw perpendicular lines to meet the sides BA and BC at X and Y respectively. We will prove that $PX = PY$.

In ΔPXB and ΔPYB,

$\angle PBX = \angle PBY$ (given)

$PB = PB$ (same line)

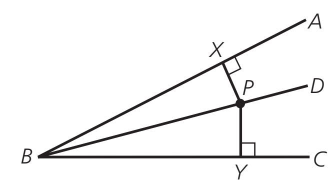
$\angle PXB = \angle PYB = 90^\circ$ (construction)

Therefore $\Delta PBX \equiv \Delta PBY$ (angle-angle-side)

Then $PX = PY$ and P is equidistant from the arms of the angle.

Part 2

If a point is equidistant from the angle arms, then it is on the bisector of the angle.



Let P be any point on line BD such that perpendiculars PX and PY are equal.

In ΔPBX and ΔPBY

$PX = PY$ (given)

PB is common

$\angle PXB = \angle PYB = 90^\circ$ (given)

Then $\Delta PXB \equiv \Delta PBY$ (hypotenuse-side)

Then $\angle PBX = \angle PBY$ and BP is the bisector of the angle.

Angle Bisector Theorem A point is on the bisector of an angle if and only if it is equidistant from the sides of the angle.

Exercise 2.3

Part A

Knowledge/Understanding

1. State the converse of each of the following statements.

a. If a triangle has three unequal sides, then it has three unequal angles.

b. If it rains, then we will get wet.

c. If a four-sided figure has four equal angles, then it is a square.

d. If the fruit is yellow, then it is a banana.

e. If today is Saturday, then it is the weekend.

f. If all answers on a test are incorrect, then several errors have been made.

g. If an integer is a prime, then it is not divisible by 2, 3, 5, or 7.

Communication

2. For each of the statements in Question 1, determine whether

a. the statement is true

b. the converse is true

c. the statement and its converse form a biconditional statement

3. Determine which of the following statements are true.

a. A positive integer is prime if and only if it is odd.

b. An integer is divisible by 2 if and only if it is even.

c. Figures are congruent if and only if they are similar.

d. A four-sided figure is a parallelogram if and only if it is a rectangle.

e. An integer is divisible by 5 if and only if it ends in a 5.

f. An animal is a cat if and only if it has four legs.

Application

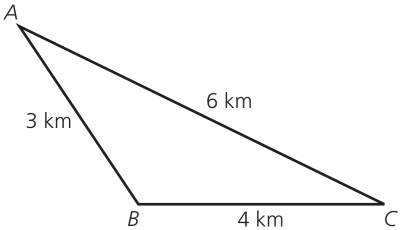
4. a. State the converse of the Isosceles Triangle Property Theorem.

b. Prove this converse.

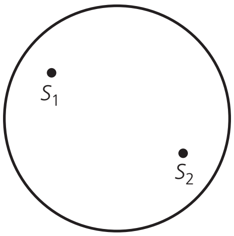
We now consider the property to be an *if and only if* statement.

Thinking/Inquiry/Problem Solving

5. A new school is being built so that it will be equidistant from three small towns A, B, and C. If the distances between the towns are as shown in the diagram, determine an approximate location for the new school.



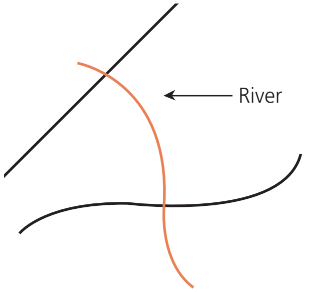
6. Two sheds, S_1 and S_2, are located in a circular compound as shown. Two gates are to be built so that each gate is equidistant from the two sheds. Where should the gates be located? Provide justification for your answer.



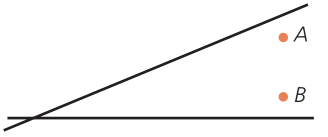
Part B

Thinking/Inquiry/ Problem Solving

7. A river crosses two roads as shown. Determine the approximate location of a pumping station if the pumping station is equidistant from the two roads.



8. Find the location of a point that is equidistant from the two intersecting lines and is also equidistant from the two given points. Provide a justification for your answer.



Thinking/Inquiry/ Problem Solving

9. Suppose that you are given four points, A, B, C, and D. Explain why it is not likely that a circle would pass through the four points.

10. Y and Z are two given points on the circumference of a circle. Find a point X also on the circumference such that ΔXYZ is isosceles.

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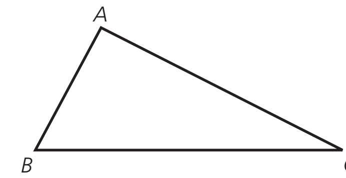
11. Prove that the right bisectors of the sides of a triangle pass through a common point.

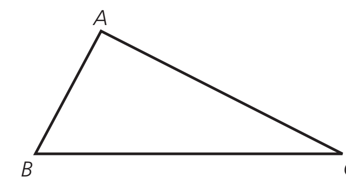
12. Draw a series of five circles that pass through two given points.
    a. Where do the centres of these circles lie?
    b. What can you say about the centres of all circles that pass through the two given points? Why?

Part C

13. Suppose we are given ΔABC as shown. Show how to draw a line parallel to BC that meets AB at D and AC at E so that $DE = DB + EC$.



14. Suppose we are given ΔABC as shown. Find a point D in side AB that is equidistant from A and the midpoint of BC.



15. Prove or disprove the following statement:
The angles in a triangle are in arithmetic sequence if and only if one of the angles equals 60º.

16. Prove that one of the roots of $x^3 + ax^2 + bx + c = 0$ is the negative of another if and only if $c = ab$.

Section 2.4 — Indirect Proof and Parallel Lines

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Sherlock Holmes, the great fictional detective, was famous for remarking that after we have eliminated the impossible, whatever remains, however improbable, must be the truth.

This method of arriving at a conclusion is used frequently in real life by a variety of people. Auto mechanics, by eliminating things that cannot be wrong, deduce the source of a problem. Doctors, by eliminating possibilities, decide on treatments to prescribe. Certain neurological illnesses are diagnosed only by eliminating all other possibilities.

In mathematics, some things that we sense to be true but cannot prove directly are approached in the same way. We list all possible outcomes and examine those that seem incorrect. If we can eliminate them by showing them to be impossible, then, using Sherlock's approach, we arrive at the conclusion that the only possible truth is the remaining possibility.

This line of reasoning is called **Indirect Proof** or **Proof by Contradiction.** It depends on a complete listing of all possible outcomes and the elimination of all but one. Consider an example.

EXAMPLE 1 Ten different teams are playing in a championship tournament in which each team plays every other team exactly once. Thus far in the tournament, eleven games have been played. Prove that one team has played at least three games.

Proof

There are only two possibilities: either there is a team that has played at least three games or every team has played two games or less. These two possibilities cannot both be true. If no team has played more than two games, the maximum number of games is $\frac{10 \times 2}{2} = 10$. Since eleven games have been played, this is not possible. The only other possibility is that at least one team has played three games or more. Notice that we cannot say that two teams have played three games, or that one team has played more than three.

In this example, we identified the two possibilities and showed that one of them led to a contradiction. This method of indirect proof was first developed by Euclid in 300 B.C. Our approach is to consider all possibilities and show that all but the one we wish to prove leads to a false conclusion or a contradiction.

EXAMPLE 2 If x and y are different numbers and $a \neq 0$, prove that $\frac{x}{y} \neq \frac{x+a}{y+a}$.

Solution

There are only two possibilities: either $\frac{x}{y} \neq \frac{x+a}{y+a}$ or $\frac{x}{y} = \frac{x+a}{y+a}$.

Since we wish to prove $\frac{x}{y} \neq \frac{x+a}{y+a}$, we start by assuming that $\frac{x}{y} = \frac{x+a}{y+a}$.

This assumption leads to a contradiction.

If $\frac{x}{y} = \frac{x+a}{y+a}$, $x(y+a) = y(x+a)$

$$xy + xa = xy + ya$$
$$xa = ya$$

Then $x = y$ or $a = 0$.

But we are given that $x \neq y$ and $a \neq 0$.

Thus we conclude that $\frac{x}{y} = \frac{x+a}{y+a}$ is not true. We conclude that $\frac{x}{y} \neq \frac{x+a}{y+a}$.

We previously assumed certain properties of parallel lines. We can now prove these properties using indirect proof.

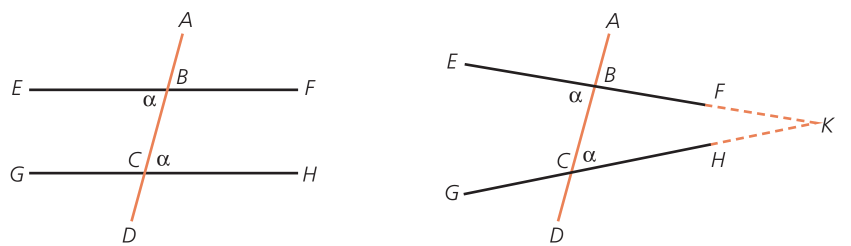
THEOREM

Part 1

Prove that if a straight line crosses two straight lines in a plane so that the alternate angles are equal, then the two straight lines are parallel.

Proof

Let EF and GH be two lines cut by the line AD at B and C so that $\angle EBC = \angle BCH$. We wish to prove that EF is parallel to GH.



There are only two possibilities. Either $EF \parallel GH$ or $EF \nparallel GH$. We make the assumption that $EF \nparallel GH$. If the lines are not parallel, then they must meet at some point K so that ΔBCK is formed, with $\angle CKB > 0°$.

In ΔKBC, $\angle EBC$ is an exterior angle.

Then $\angle EBC = \angle BCK + \angle CKB$ (Exterior Angle Property Theorem)

But $\angle EBC = \angle BCK$

Then $\angle EBC \neq \angle BCK + \angle CKB$

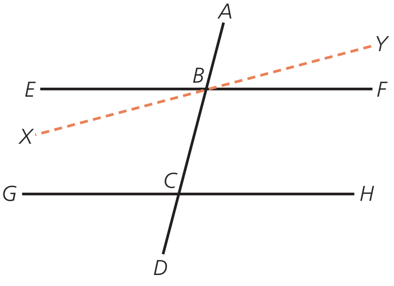
The assumption that $EF \nparallel GH$ is not true, since $< CKB > 0°$.

Then $EF \parallel GH$.

Part 2
If a straight line cuts two parallel lines, then the alternate angles formed are equal.

Proof

Let EF and GH be parallel lines cut by AD at B and C. We wish to prove that $\angle EBC = \angle BCH$.



There are only two possibilities:
$\angle EBC = \angle BCH$ or $\angle EBC \neq \angle BCH$

Assume that $\angle EBC \neq \angle BCH$.

Draw $\angle XBC = \angle BCH$ and extend XB to Y.

Then $XY \parallel GH$. (alternative angles equal)

Then XY and EF are intersecting lines and are both parallel to GH, which is impossible.

The assumption that $\angle EBC \neq \angle BCH$ is false.

Then $\angle EBC = \angle BCH$.

Using the alternate angle properties it is easy to prove similar results for corresponding angles, and we can state the complete parallel line theorem.

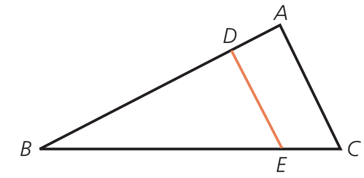
The Parallel Line Theorem
Two straight lines are parallel if and only if
1. alternate angles are equal
or
2. corresponding angles are equal

Summary of Indirect Proof
1. List all the possibilities including the one that must be proved.
2. Remove the alternative that is to be proved and then consider each of the other possibilities. Show that each of them leads to an incorrect conclusion or a contradiction.
3. Conclude that the one remaining possibility must be true.

Exercise 2.4

Part B

1. In ΔABC, $AB > AC$ and $DE \parallel AC$. Use indirect reasoning to show that $\angle DBE \neq \angle DEB$.



Communication

2. Describe a real-life situation in which you have used indirect reasoning.

3. Prove that the line whose equation is $y = 2x - 1$ does not intersect the curve with equation $y = x^4 + 3x^2 + 2x$.

4. A quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are not 0, has real roots. Prove that a, b, and c cannot be consecutive terms of a geometric sequence.

Communication

5. The lines CD and EF are each parallel to the line AB. Prove that $CD \parallel EF$.

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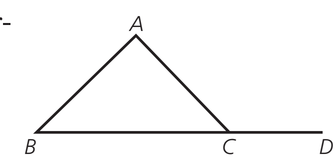
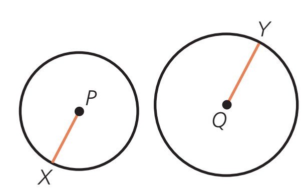
6. Prove that if the bisector of exterior $\angle ACD$ is parallel to AB, then ΔABC has two equal angles.



Application

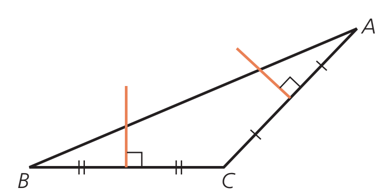
7. The medians BD and CE in ΔABC are produced to X and Y respectively so that $BD = DX$ and $CE = EY$. Prove that X, A, and Y lie in a straight line.

8. PX and QY are radii in the given circles such that $PX \parallel QY$. When X and Y are joined, the line cuts the circles at M and N, with M being on the circle with centre P. Prove that $PM \parallel QN$.

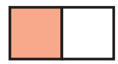


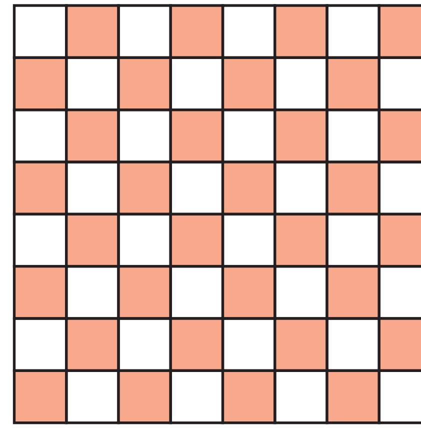
Thinking/Inquiry/ Problem Solving

9. It is often assumed that the right bisectors of two sides of a triangle meet. Prove that they do by using indirect proof.



Part C

10. You are given an 8×8 checkerboard as shown in the diagram. If we cut out two opposite white corners, is it possible to tile the new board with dominoes that look like  ?

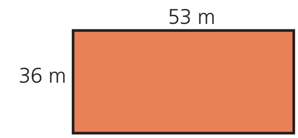


Knowledge/Understanding

11. If x is a real number such that $x^9 + 7x < 10$, prove that $x < 1.1$.

Section 2.5 — Ratio and Proportion

Farmer Collins wishes to divide a rectangular field into two parts so that the ratio of the smaller part to the larger is 1:3. How can he do this? How confident can he be that he is correct in the division?



If he divides the 36 m side into parts of 9 m and 27 m and divides the field with a line parallel to the 53 m side, he can accomplish his goal. Can we justify this statement?

Does the same thinking apply for a triangle or other geometric shapes? In this section we examine some of the mathematical properties that use ratios.

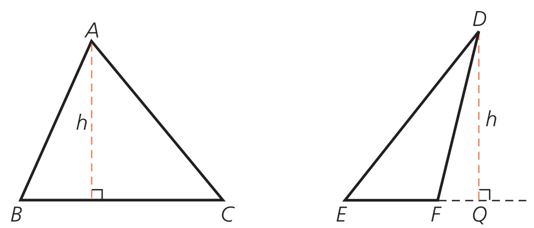
THEOREM Triangles having equal heights have areas proportional to their bases.

Proof

Two triangles, ABC and DEF, have height h. We wish to calculate the ratio of their areas.

The area of $\Delta ABC = \frac{1}{2}h(BC)$

The area of $\Delta AEF = \frac{1}{2}h(EF)$



Comparing areas, $\frac{\Delta ABC}{\Delta AEF} = \frac{\frac{1}{2}h(BC)}{\frac{1}{2}h(EF)} = \frac{BC}{EF}$

Then $\Delta ABC : \Delta DEF = BC : EF$

When referring to the area of a triangle, we usually use the triangle name. The context makes it clear that we are referring to the area.

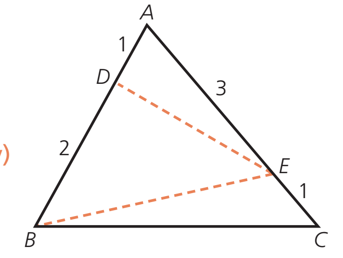
By a similar approach, we can show that the areas of triangles having equal bases are proportional to their heights.

Triangle Area Property
If triangles have equal heights, their areas are proportional to their bases and
If triangles have equal bases, their areas are proportional to their heights.

EXAMPLE 1

In ΔABC, D is a point on AB such that $AD = 1$ and $DB = 2$. E is a point on AC such that $AE = 3$ and $EC = 1$. What is the ratio $\Delta ADE : \Delta ABC$?

Solution



Join B to E and E to D.
Since ΔADE and ΔABE have the same height, and $AB = 3AD$,

$$\frac{\Delta ADE}{\Delta ABE} = \frac{1}{3} \qquad \text{(Triangle Area Property)}$$

or $\quad \Delta ADE = \frac{1}{3}\Delta ABE$

Since $\quad \Delta ABE$ and ΔABC have the same height for bases AE and AC,

$$\Delta ABE = \frac{3}{4}\Delta ABC \qquad \text{(Triangle Area Property)}$$

Substituting $\Delta ADE = \frac{1}{3}\left[\frac{3}{4}\,\Delta ABC\right] = \frac{1}{4}\,\Delta ABC$

Then $\quad \dfrac{\Delta ADE}{\Delta ABC} = \dfrac{1}{4}$

EXAMPLE 2

Prove that if $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{b} = \frac{c+d}{d}$ and $\frac{a+3b}{b} = \frac{c+3d}{d}$.

Proof

Since $\quad \frac{a}{b} = \frac{c}{d},\ \frac{a}{b} + 1 = \frac{c}{d} + 1$

Then $\quad \dfrac{a+b}{b} = \dfrac{c+d}{d}$

Also $\quad \frac{a}{b} + 3 = \frac{c}{d} + 3$

Then $\quad \dfrac{a+3b}{b} = \dfrac{c+3d}{d}$

THEOREM

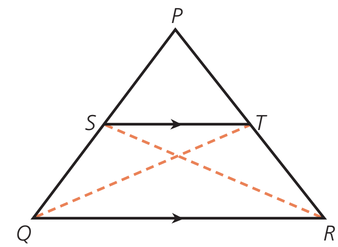
A line in a triangle is parallel to a side of the triangle if and only if it divides the other sides in the same proportions.

Proof

Part 1

Let ST be a line in ΔPQR that is parallel to QR. We will prove that $\frac{PS}{SQ} = \frac{PT}{TR}$.
Join SR and QT.

Since ΔPST and ΔSTQ have the same altitude with bases PS and SQ,



$$\frac{\Delta PST}{\Delta STQ} = \frac{PS}{SQ}$$ (Triangle Area Property)

Since ΔPTS and ΔSTR have the same altitude with bases PT and TR,

$$\frac{\Delta PST}{\Delta STR} = \frac{PT}{TR}$$ (Triangle Area Property)

Because $ST \parallel QR$, ΔSTQ and ΔSTR have the same base ST and equal altitudes.

Therefore $\Delta STQ = \Delta STR$

Then $$\frac{\Delta PST}{\Delta STQ} = \frac{\Delta PST}{\Delta STR}$$

Therefore $$\frac{PS}{SQ} = \frac{PT}{TR}$$

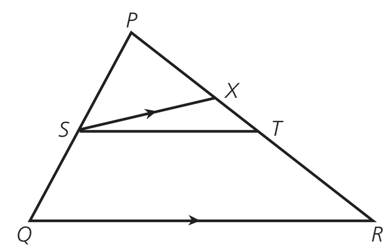
Note as an extension that we easily obtain

$$\frac{PS}{SQ} + 1 = \frac{PT}{TR} + 1$$

$$\frac{PS + SQ}{SQ} = \frac{PT + TR}{TR}$$

$$\frac{PQ}{SQ} = \frac{PR}{TR}$$

Part 2



Let PQR be a triangle and let S and T be points in the sides so that $\frac{PS}{SQ} = \frac{PT}{TR}$. We can prove that $ST \parallel QR$, and we do so by indirect proof.
Either $ST \parallel QR$ or $ST \nparallel QR$.

If $ST \nparallel QR$, then draw $SX \parallel QR$ meeting PR at X.

Because $SX \parallel QR$, $\frac{PS}{SQ} = \frac{PX}{XR}$ (Part 1 of theorem)

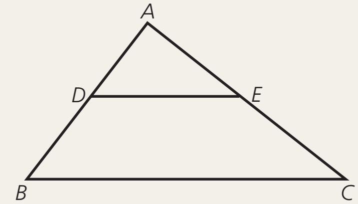
Then $\frac{PT}{TR} = \frac{PX}{XR}$

This means that the points T and X divide PR in the same ratio, which is impossible.
Then the statement $ST \nparallel QR$ is not true. Therefore $ST \parallel QR$.

Triangle Proportion Property Theorem

A line in a triangle is parallel to a side of the triangle if and only if it divides the other sides in the same proportion.

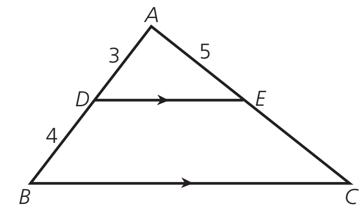
In ΔABC, $DE \parallel BC$ if and only if $\frac{AD}{DB} = \frac{AE}{EC}$.



$DE \parallel BC$

EXAMPLE 3 If $DE \parallel BC$, calculate the length of EC.

Solution



Since $DE \parallel BC$, $\frac{AD}{DB} = \frac{AE}{EC}$ (Triangle Proportion Property)

Then $\frac{3}{4} = \frac{5}{EC}$

$3EC = 20$

$EC = \frac{20}{3}$

EXAMPLE 4 If A, B, C, and D are the midpoints of the sides of quadrilateral $PQRS$, as shown, prove that $ABCD$ is a parallelogram. (Recall that we proved this using coordinates in Example 3 of Section 1.4).

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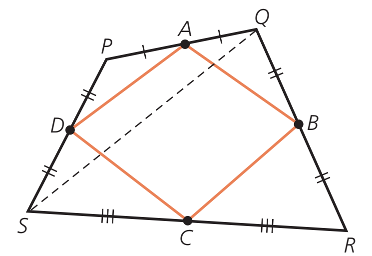
Proof



Join SQ.

In ΔPSQ, $PD = DS$ and $PA = AQ$

Then $\frac{PD}{DS} = \frac{PA}{AQ}$

Therefore, $DA \parallel SQ$ (Triangle Proportion Property)

In ΔRSQ, $RC = CS$ and $RB = BQ$

Then $\frac{RC}{CS} = \frac{RB}{BQ}$

Therefore $CB \parallel SQ$ (Triangle Proportion Property)

Then $CB \parallel DA$

By joining PR and using the same approach, $AB \parallel CD$.

Since opposite sides are parallel, $ABCD$ is a parallelogram.

Exercise 2.5

Part A

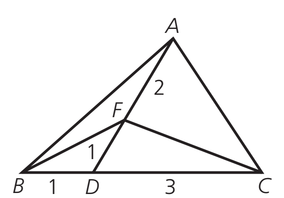
Knowledge/Understanding

1. In the given triangle, lengths are as shown. Determine the following:

a. $\Delta ABD{:}\Delta ADC$ b. $\Delta ABD{:}\Delta ABC$

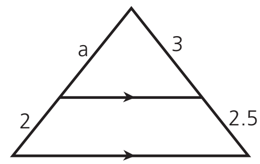
c. $\Delta BFD{:}\Delta BFA$ d. $\Delta BFD{:}\Delta BDA$

e. $\Delta CDF{:}\Delta CFA$ f. $\Delta CFD{:}\Delta ABC$

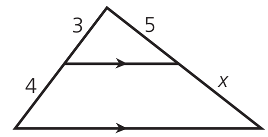


2. Calculate the unknown in each of the following.

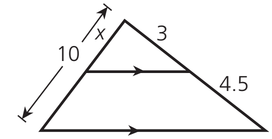
a.



b.

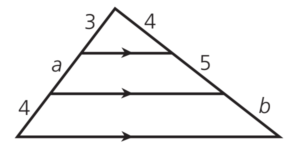


c.



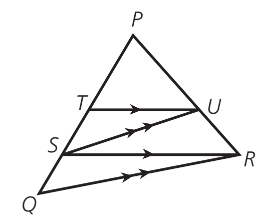
Communication

3. Find the lengths of a and b in the given triangle.



Application

4. If $\frac{PT}{TS} = \frac{2}{1}$, determine $\frac{PT}{SQ}$.



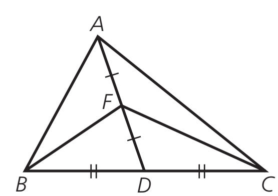
Part B

5. If $BD = DC$ and $AF = FD$, prove that

a. $\frac{\Delta ABF}{\Delta ABC} = \frac{1}{4}$

b. $\frac{\Delta AFC}{\Delta ABC} = \frac{1}{4}$

c. Prove that $\Delta ABF = \Delta AFC$.



6. If $\frac{a}{b} = \frac{c}{d}$, prove

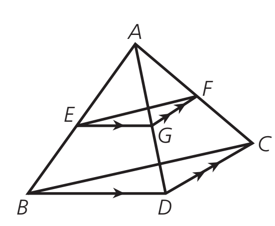
a. $\frac{a - b}{b} = \frac{c - d}{d}$

b. $\frac{ma + nb}{b} = \frac{mc + nd}{b}$

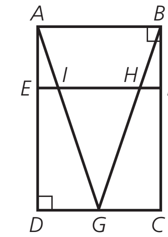
Thinking/Inquiry/ Problem Solving

7. ΔABC is an isosceles right-angle triangle in which $AB = BC = 3$. D is a point on AB such that $AD = 2$ and $DB = 1$. Through D a line is drawn parallel to BC to meet AC at E. Determine trapezoid $DECB : \Delta ADE$.

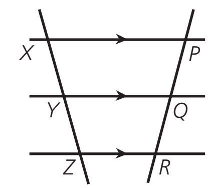
8. In the diagram, $EG \parallel BD$ and $GF \parallel DC$. Prove that $EF \parallel BC$.



9. *ABCD* is a rectangle in which $\frac{AE}{ED} = \frac{1}{2}$ and *G* is the midpoint of *DC*. If $EF \parallel AB$, determine ΔBHF : rectangle *ABCD*.



10. In the diagram, three parallel lines are cut by lines at *X*, *Y*, *Z*, and *P*, *Q*, *R*, respectively. Prove that $\frac{XY}{YZ} = \frac{PQ}{QR}$.



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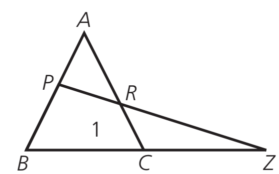
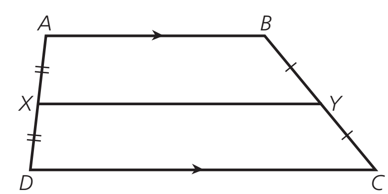
11. In $\parallel^{gm}$ *ABCD*, *X* and *Y* are the midpoints of *AD* and *BC*, respectively. Prove that *BX* and *DY* trisect *AC*.

Part C

12. If $\frac{AP}{PB} = \frac{3}{4}$ and $\frac{AR}{RC} = \frac{3}{2}$, prove that *C* is the midpoint of *BZ*.



13. *ABCD* is a trapezium in which $AB \parallel DC$, $AX = XD$, and $BY = YC$. If *X* is joined to *Y*, prove that $XY \parallel DC \parallel AB$.



14. *ABC* and *BDC* are two triangles on the same side of *BC*. From *K*, any point in *BC*, *KE* is drawn parallel to *BA* to meet *AC* at *E*, and *KF* is drawn parallel to *BD* to meet *CD* at *F*. Prove that $EF \parallel AD$.

Section 2.6 — Similar Figures

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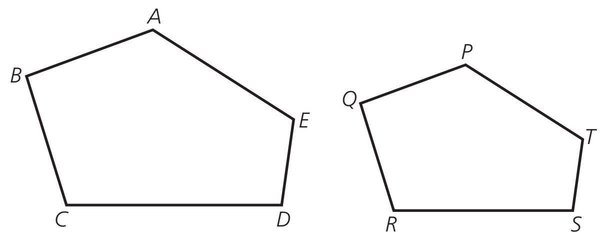
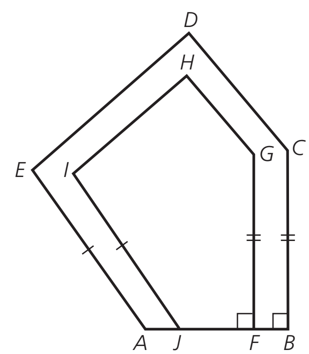
Similarity is one of the most important properties of plane figures. If we take a photograph of any shape and enlarge the photo, we create a new shape similar to the original. Two figures are similar if the angles of one, taken in order, are respectively equal to the angles of the other, in the same order, and the corresponding sides are proportional.

Thus, if polygons *ABCDE* and *PQRST* are similar, then $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$, $\angle D = \angle S$ and $\angle E = \angle T$; and $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{AE}{PT}$.



The converse of this statement is also true.

The two pentagons *ABCDE* and *JFGHI* are equiangular but their side lengths are not in proportion, so the pentagons are not similar.



The rectangle *ABCD* and the parallelogram *EFGH* have sides that are proportional (in this case equal) but the figures are not equiangular. Again, the figures are not similar.

We saw earlier that there are a number of conditions on two triangles that make the triangles congruent. There are three sets of conditions that make two triangles similar. We examine these in the **Similar Triangle Theorem.**

Similar Triangle Theorem

Two triangles are similar if

1. they are equiangular, or if

2. their sides are proportional, or if

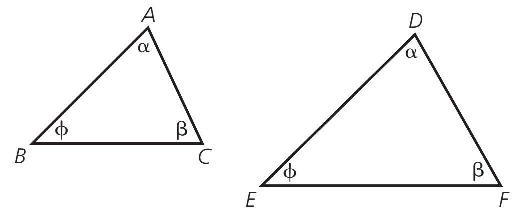
3. two pairs of sides are proportional, and the angles contained by these sides are equal.

Proof

Part 1

Let ΔABC and ΔDEF have $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

We prove that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.



Translate the smaller triangle ΔABC onto ΔDEF so that $\angle A$ fits on $\angle D$.

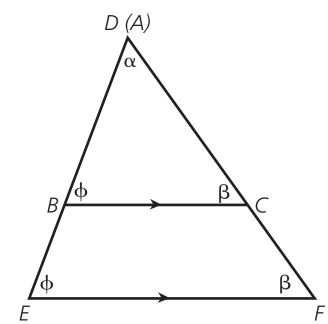
Then AB falls along DE and B is on DE.

Also AC falls along DF and C is on DF.

Since $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$,

$BC \parallel EF$ (Parallel Line Property)

Then $\frac{AB}{DE} = \frac{AC}{DF}$ (Triangle Proportion Property)



By repeating this process with $\angle C$ on $\angle F$,

$\frac{AC}{DF} = \frac{BC}{EF}$

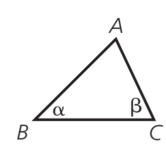
Then $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

The triangles are similar.
To indicate that figures are similar, we write $\Delta ABC \sim \Delta DEF$.

Part 2

Let ΔABC and ΔDEF be such that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.

We prove that $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

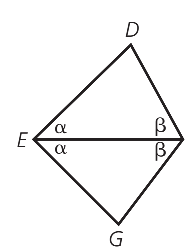


Construct ΔEFG equiangular to ΔBCA.

Since $\angle ABC = \angle FEG$ and $\angle ACB = \angle EFG$,

$\Delta ABC \sim \Delta GEF$

Then $\frac{AB}{EG} = \frac{BC}{EF} = \frac{AC}{FG}$



We are given that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

Since $\frac{BC}{EF}$ is common to both proportions,

$$\frac{AB}{EG} = \frac{AB}{DE} \text{ and } \frac{AC}{FG} = \frac{AC}{DF}$$

Then $EG = DE$ and $FG = DF$
In ΔEFG and EFD,

$$EG = DE$$
$$FG = FD$$
$$EF = FE$$

Therefore $\Delta EFG \equiv \Delta EFD$ (side-side-side)

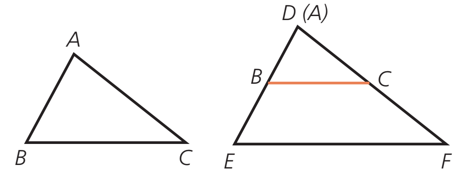
Then $\angle EFG = \angle EFD$, $\angle FEG = \angle FED$, and $\angle EGF = \angle EDF$

But $\angle EFG = \angle BCA$, $\angle FEG = \angle ABC$, and $\angle EGF = \angle BAC$

Therefore $\angle BCA = \angle EFD$, $\angle ABC = \angle FED$, and $\angle BAC = \angle EDF$

Then $\Delta ABC \sim \Delta DEF$.

Part 3
Let ΔABC and ΔDEF be triangles in which $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$.
We will prove that the triangles are similar.



Translate ΔABC so that $\angle A$ coincides with $\angle D$, AB falls along DE and AC falls along DF, as shown.

Since $\frac{AB}{DE} = \frac{AC}{DF}$,

$BC \parallel EF$ (Triangle Proportion Property)

Then $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$ (Parallel Lines Property)

Then $\Delta ABC \sim \Delta DEF$ (equal angles)

We now have all conditions for similar triangles.

EXAMPLE 1 Prove that if two triangles are similar, corresponding altitudes have the same ratio as a pair of corresponding sides.

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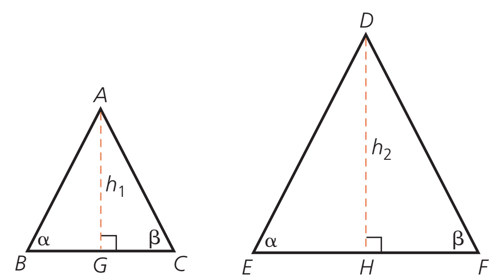
Proof

Let ΔABC and ΔDEF be similar. Let h_1 be the altitude from A, meeting BC at G, and h_2 be the altitude from D, meeting EF at H.



In ΔABG and ΔDEH,

$$\angle ABG = \angle DEH \quad \text{(similar triangles)}$$
$$\angle AGB = \angle DHE \quad \text{(right angles)}$$
$$\angle BAG = \angle EDH \quad \text{(angle sum)}$$

Then $\Delta ABG \cong \Delta DEH$ (Similar Triangle Theorem)

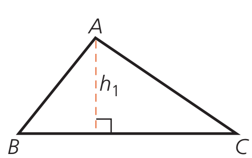
Therefore $\dfrac{AB}{DE} = \dfrac{AG}{DH} = \dfrac{h_1}{h_2}$

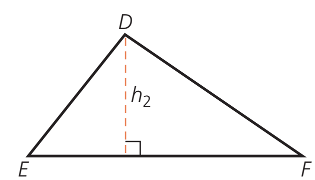
But $\Delta ABC \sim \Delta DEF$, so $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$

Then $\dfrac{h_1}{h_2} = \dfrac{AB}{DE} = \dfrac{AG}{DF} = \dfrac{BC}{EF}$

If two figures are similar, can we make any statement about the ratio of their areas? Suppose, for example, that $\Delta ABC \sim \Delta DEF$ and $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF} = \dfrac{1}{2}$. Can we determine the ratio of their areas? The answer is that we can. We cannot determine the area of either triangle, but from Example 1 we can say that $\dfrac{h_1}{h_2}$ is also $\dfrac{1}{2}$, and using this we can determine the ratio of the triangle areas.

$$\text{Then } \frac{\Delta ABC}{\Delta DEF} = \frac{\frac{1}{2}(BC)h_1}{\frac{1}{2}(EF)h_2}$$
$$= \frac{BC}{EF} \times \frac{h_1}{h_2}$$
$$= \frac{1}{2} \times \frac{1}{2}$$
$$= \frac{1}{4}$$





Now we can prove the result in general.

EXAMPLE 2

Prove that the areas of similar triangles are proportional to the squares on corresponding sides.

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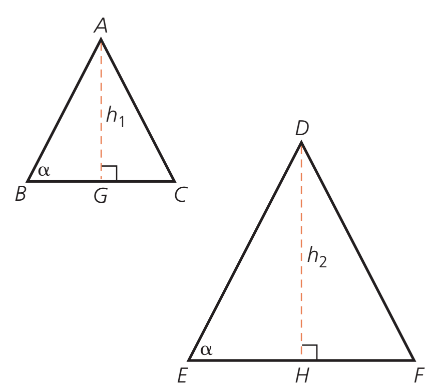
Proof

Let ΔABC and ΔDEF be two similar triangles and let h_1 and h_2 be their altitudes.

Since $\Delta ABC \sim \Delta DEF$,

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} = \frac{h_1}{h_2}$$

$$\frac{\Delta ABC}{\Delta DEF} = \frac{\frac{1}{2}(BC)h_1}{\frac{1}{2}(EF)h_2}$$



$$= \frac{BC}{EF} \times \frac{h_1}{h_2}$$

$$= \frac{BC}{EF} \times \frac{BC}{EF}$$

Then $\frac{\Delta ABC}{\Delta DEF} = \frac{BC^2}{EF^2}$

EXAMPLE 3 If $\Delta ABC \sim \Delta DEF$, $\Delta ABC = 60$, $AB = 12$, and $DE = 9$, determine ΔDEF.

Solution

$$\frac{\Delta ABC}{\Delta DEF} = \left(\frac{AB}{DE}\right)^2$$

$$\frac{60}{\Delta DEF} = \left(\frac{12}{9}\right)^2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$$

$$16\ \Delta DEF = 9 \times 60$$

Then $\Delta DEF = \frac{9}{16} \times 60 = 33.75$

Exercise 2.6

Part A

Communication

1. In the following list, which pairs are similar figures?

a. a quadrilateral and a rectangle

b. two hexagons that have equal angles

c. two squares

d. two isosceles triangles

e. two circles

f. two equilateral triangles

g. two right-angled triangles, each having an acute angle of 30º

h. two congruent pentagons

Knowledge/Understanding

2. ΔABC and ΔDEF are similar. If $DE = 2$, $DF = 4$, $EF = 5$, and $AB = 12$, determine the lengths of AC and BC.

Application

3. The sides of a triangular field are 40 m, 50 m, and 65 m. In a map of the region containing the field, the scale is 1:100. What are the dimensions of the field in the map?

Application

4. A man 2 m tall casts a shadow 5.2 m long. How tall is a pole that casts a shadow 9.1 m long?

Part B

5. A point X is chosen in side AB of ΔABC such that $\frac{AX}{XB} = \frac{2}{3}$. From X, a line is drawn parallel to BC and meets AC at Y. If $BC = 15$, calculate the length of XY.

6. In ΔABC, D and E are in AB and AC respectively such that $DE \parallel BC$, $AD = 3$, $DB = 4$, and $\Delta ADE = 81$. Determine each of the following:

   a. ΔABC

   b. quadrilateral $DBCE$

Knowledge/ Understanding

7. If two similar triangles have areas in the ratio 4:49, how do the side lengths of the two triangles compare?

8. Prove that if two triangles are equiangular, the bisectors of two corresponding angles have the same ratio as any pair of corresponding sides.

Thinking/Inquiry/ Problem Solving

9. Prove that if two triangles are equiangular, their perimeters have the same ratio as a pair of corresponding sides. (*Hint:* Let the ratio between a pair of corresponding sides be k: 1.)

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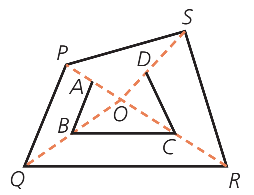
10. In quadrilateral $PQRS$, O is any point and is joined to each vertex. In OP, a point A is chosen. From A, the line AB is drawn parallel to PQ, then BC is drawn parallel to QR, and CD is drawn parallel to RS. Prove that $AD \parallel PS$.



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11. a. In the diagram, name all pairs of similar triangles.

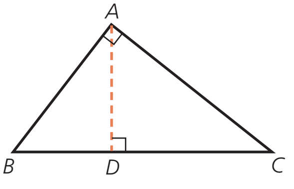
   b. Prove each of the following:

      (i) $AD^2 = (BD)(DC)$

      (ii) $AC^2 = (BC)(DC)$

      (iii) $AB^2 = (BC)(BD)$

   c. Using parts **(ii)** and **(iii)**, prove that $AC^2 + AB^2 = BC^2$.



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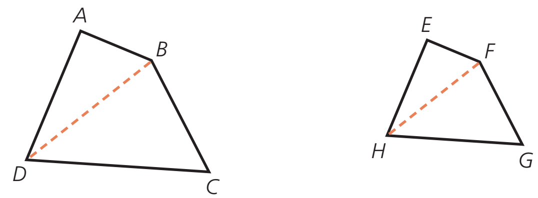
12. The medians BD and CE of ΔABC intersect at F. Prove that $BF = 2FD$ and $CF = 2FE$.

13. In ΔPQR, S is any point on QR and S is joined to P. A line is drawn parallel to QR, meeting PQ in A, PS in B, and PR in C. Prove that $\frac{AB}{BC} = \frac{QS}{SR}$.

Part C

14. In the figures below, quadrilateral $ABCD \sim$ quadrilateral $EFGH$.



Prove each of the following:

a. $\Delta ABD \sim \Delta EFH$

b. $\Delta BDC \sim \Delta FHG$

c. $\frac{\Delta ABD}{\Delta EFH} = \frac{\Delta DBC}{\Delta HFG} = \left(\frac{DC}{HG}\right)^2$

d. $\frac{\text{quad } ABCD}{\text{quad } EFGH} = \frac{DC^2}{HG^2}$

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15. Prove that two similar pentagons are proportional in area to the squares on corresponding sides.

16. In ΔABC, D is on AB and E on AC so that $DE \parallel BC$. If trapezoid $DBCE = \frac{24}{25}\Delta ABC$, determine $\frac{AD}{DB}$.

Key Concepts Review

In Chapter 2, we introduced you to different methods of proof through a discussion of plane figures. One of the main features of this chapter was the introduction of conditional, biconditional, and indirect reasoning. Conditional reasoning, or $p \rightarrow q$, means showing q to be correct as a condition of p being true. We noted that the two statements $p \rightarrow q$ and $p \leftrightarrow q$ are not the same. If $q \rightarrow p$ is correct, for instance, there is no guarantee that p necessarily implies q. In addition, you were introduced to indirect reasoning. In this type of reasoning, you list all possible outcomes for a certain premise and then eliminate all possibilities except one, thereby implying that the remaining one must be correct.

In addition, we also introduced properties of area and similarity of plane figures.

IMPORTANT CONCEPTS

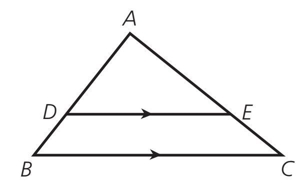
1. Parallelograms having the same bases and between the same parallels have the same area.
2. If two triangles have the same bases but different heights, then their areas are proportional to their corresponding heights.
3. Triangles are similar under the following conditions:
   (i) corresponding angles are equal
   (ii) corresponding sides are proportional
   (iii) an angle of one triangle equals an angle in the second triangle, and sides about the angle are proportional
4. If $\Delta ABC \sim \Delta DEF$, then $\frac{\Delta ABC}{\Delta DEF} = \left(\frac{AB}{DE}\right)^2 = \left(\frac{AC}{DF}\right)^2 = \left(\frac{BC}{EF}\right)^2$.

Review Exercise

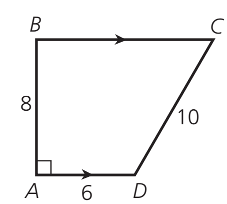
1. Which of the following pairs are similar figures?
   a. a rectangle and a parallelogram
   b. two rectangles
   c. two congruent triangles
   d. two rectangles that measure 8×3 and 32×12
   e. two right-angled triangles
   f. two quadrilaterals each containing two angles of 100º and 120º

2. The area of a rectangle is A and it has a length of L. Write an expression for the perimeter of this rectangle in terms of A and L.

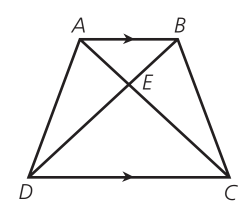
3. If $DE \parallel BC$ and trapezoid $DECB = \frac{7}{16}\Delta ABC$, determine $DE{:}BC$.



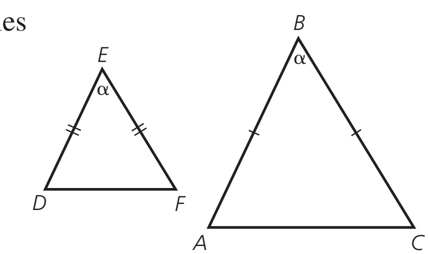
4. Determine the area of figure $ABCD$.



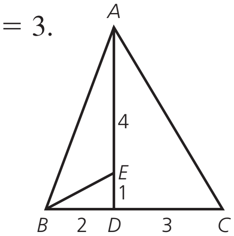
5. In the trapezoid $ABCD$, $AB{:}DC = 2{:}5$. The two diagonals DB and CA intersect at E. Determine the following:
   a. $AE{:}EC$
   b. $\Delta ABE{:}\Delta CD$E



6. ΔABC and ΔDEF are two isosceles triangles in which the vertical angles $\angle ABC$ and $\angle DEF$ are equal. Prove that the two triangles are similar.



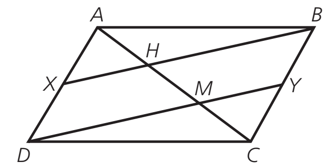
7. In ΔABC, D is a point on BC such that $BD = 2$ and $DC = 3$. The point E is on DA such that $DE = 1$ and $EA = 4$.



a. Determine ΔBED:ΔABC.

b. Determine ΔBED:ΔADC.

8. $ABCD$ is a parallelogram where X and Y are the midpoints of AD and BC, respectively. Prove



a. $BX \parallel DY$

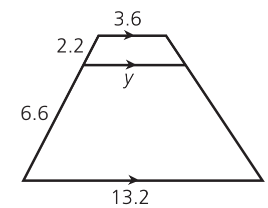
b. $\frac{AX}{XD} = \frac{AH}{HM}$

c. $AH = HM$

d. $CM = HM$

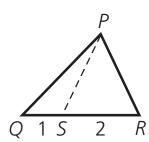
e. BX and DY trisect AC

9. Determine the value of y in the diagram shown.

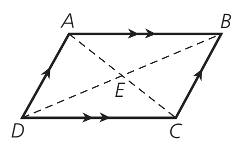


10. Into what fractional parts do the dotted lines divide each of the following?

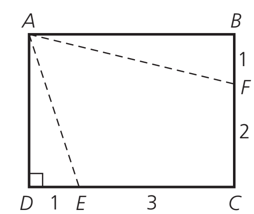
a.



b.

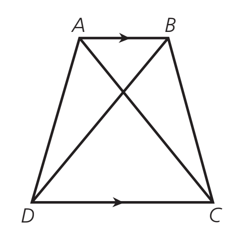


c.

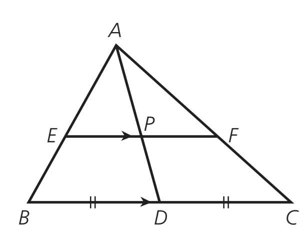


11. The legs of a right-angled triangle are 5 and 10, while the hypotenuse of a similar triangle is 15. What is the area of the larger triangle?

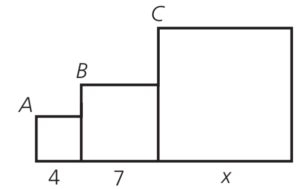
12. In the trapezoid $ABCD$, $AB = 1$ and $DC = 2$. Prove that the diagonals of the trapezoid trisect each other.



13. In ΔABC, a median is drawn from A to the point D on BC. Through any point P on AD, a line is drawn parallel to BC that meets AB and AC at E and F, respectively. Prove that $EP = PF$.



14. In the adjacent squares shown, the vertices A, B, and C lie in a straight line. Determine the value of x.

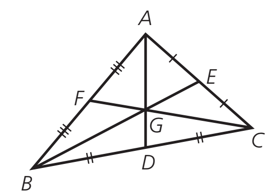


technology
APPENDIX P. 527

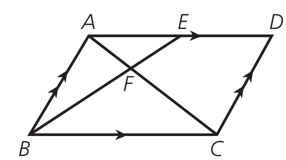
15. Prove that the three altitudes of a triangle are equal if and only if the triangle is equilateral.

16. If a and b are the lengths of the two parallel sides of a trapezoid and h is the distance between them, prove that the area of the trapezoid is $\frac{1}{2}(a + b)h$.

17. In ΔABC, the medians AD, BE, and CF intersect at the point G. Prove that the six interior triangles in ΔABC have equal areas.

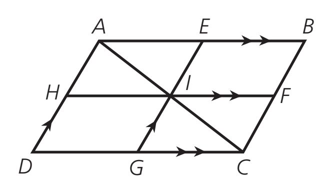


18. In the parallelogram $ABCD$, $AE{:}ED = 1{:}2$. Determine the ratio of the area of ΔABF to quadrilateral $EFCD$.



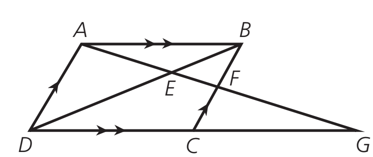
19. The base of a triangle is four times as long as a side of a square. If the triangle and the square have equal areas, find the ratio of the altitude of the triangle to a side of the square.

20. $ABCD$ is a parallelogram, and point I is any point on the diagonal AC. A line is drawn through I such that EG is parallel to AD and BC. Similarly, a line is drawn through I that is parallel to DC and AB. Prove that parallelogram $EBFI$ = parallelogram $HIGD$.



21. In ΔABC, D divides AB in the ratio 1:2, and E divides BC in the ratio 3:4. If $\Delta BDE = 6$, find the area of ΔABC.

22. $ABCD$ is a parallelogram. A straight line through A cuts BD at E, and BC at F, and meets DC extended at G. Prove $\frac{AE}{EF} = \frac{AG}{AF}$.



CHAPTER 2: VARIGNON PARALLELOGRAM

Varignon's Parallelogram Theorem states that the figure formed by joining the midpoints of any convex quadrilateral is a parallelogram. The theorem is named after Pierre Varignon, a French mathematician who lived from 1654 to 1722. He was the first to prove the theorem. The parallelogram formed by joining the midpoints of a quadrilateral is called a **Varignon parallelogram**. Like so many others, Varignon accidentally came across a copy of Euclid's *Elements* and was inspired to a career in mathematics.

Investigate and Apply

Geometer's Sketchpad® may be useful as you carry out some of these investigations.

1. a) Prove Varignon's Parallelogram Theorem using the methods of this chapter.

   b) Consider four points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, and $D(x_4, y_4)$. Prove Varignon's Parallelogram Theorem using the midpoint and slope formulas from analytic geometry.

   c) Compare and contrast the proofs from parts **a** and **b.** Which is more convincing? Which is easier to understand?

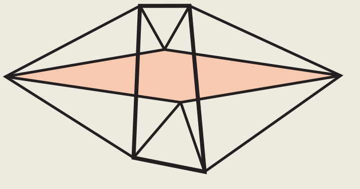
   A third method of proof will be encountered in Chapter 6.

2. If the initial quadrilateral is a square, does the interior Varignon Parallelogram have any special properties? What if the initial quadrilateral is a rectangle, a rhombus, a trapezoid, or a parallelogram?
3. Prove that the area of a Varignon Parallelogram is one half the area of the initial quadrilateral. *Hint:* Consider the initial quadrilateral as two triangles.

INDEPENDENT STUDY

What special characteristics must the initial quadrilateral have in order for its Varignon Parallelogram to be a rhombus, a rectangle, or a square?

If the initial quadrilateral is already a parallelogram, what special properties must it have in order to be similar to its Varignon Parallelogram?

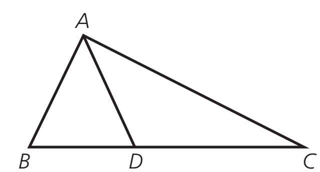


Prove that if equilateral triangles are drawn on the sides of a quadrilateral, alternately inwards and outwards, their vertices will form a parallelogram.

Chapter 2 Test

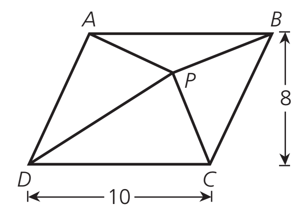
Achievement Category	Questions
Knowledge/Understanding	all
Thinking/Inquiry/Problem Solving	7
Communication	1
Application	2, 3

1. In ΔABC, D is a point on BC.
   Consider the following statement:
   If $\Delta ADC = \Delta ADB$, then AD is a median.



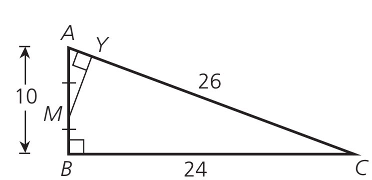
   a. State the converse of this theorem.
   b. Prove that the converse of this theorem is true.
   c. Restate the original statement and its converse in "if and only if" form.

2. Answer each of the following. (You do not need to provide proofs.)
   a. A rhombus has diagonals of length 12 and 20. What is the area of the rhombus?
   b. In ΔABC, $\angle C = 90^\circ$. The midpoint E of the median AD is joined to B. If $AC = 8$ and $BC = 6$, determine the area of ΔDEB.
   c. $ABCD$ is a parallelogram in which $DC = 10$ with a height of 8 units as shown. If P is any point inside the parallelogram, calculate $\Delta APB + \Delta CPD$.



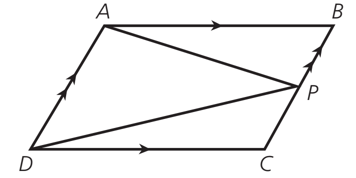
   d. The lengths of the parallel sides of a trapezoid are 8 and 12 units, respectively. If the area of the trapezoid is 120, determine the distance between the parallel lines.

3. The sides of a right-angled triangle ABC are 10, 24, and 26, with the right angle at B as shown. A line segment MY is drawn from M, the midpoint of AB, perpendicular to AC. Determine the length of MY.

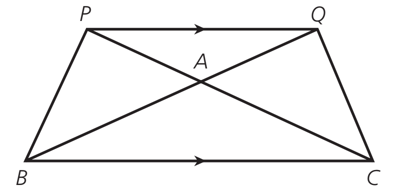


4. AD is a median of ΔABC. E is on AD so that $AE = \frac{1}{4}AD$. If $\Delta AEC = 36$, determine ΔABC.

5. In parallelogram $ABCD$, P is any point on BC. Prove that $\Delta APD = \Delta ABP + \Delta DCP$.

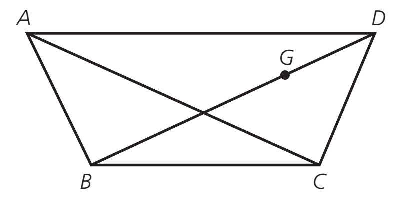


6. $PBCQ$ is a trapezoid in which $PQ \parallel BC$ and $PQ{:}BC = 2{:}3$. PC and QB intersect at A. If $\Delta ABC = 36$, calculate the area of trapezoid $PBCQ$.



7. Prove the following statement using indirect reasoning. ΔABC and ΔDBC have the same base BC and equal areas. Prove that the line joining A to D is parallel to BC.

(*Hint:* Select a point G on BD, and assume that $AG \parallel BC$.)

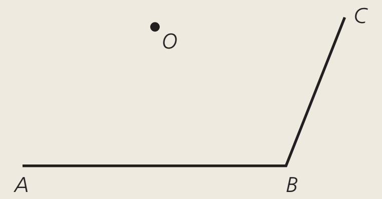


Extending and Investigating

COMPUTER INVESTIGATIONS

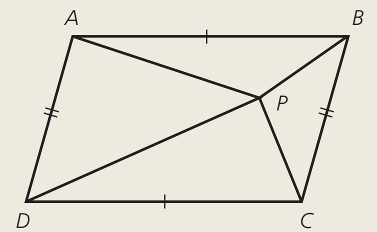
These investigations are provided for enrichment activity and as an introduction to investigation in mathematics. They are designed to be used with Geometer's Sketchpad®. You are encouraged to develop other questions of interest for research. Keep in mind that after you've made a conjecture, it must always be proved.

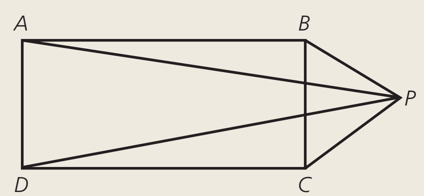
1. On a blank screen, select any three non-collinear points A, B, and C. Determine by experimentation the location of a point O that is equidistant from A, B, and C. Verify that point O is at the point of intersection of the right bisectors of AB and BC using the **Perpendicular Line** function. Draw the circle which passes through A, B, and C.



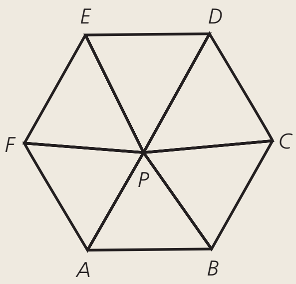
2. On a blank screen, select any three non-collinear points A, B, and C such that these points form an obtuse-angled triangle. Locate the position of point O, which is equidistant from A, B, and C. If three points are chosen that form an obtuse-angled triangle, what can we say about the location of a point equidistant from the three points? Is the same conclusion true if the three points form the vertices of an acute-angled triangle? a right-angled triangle?

3. In Exercise 1.4, you were asked to prove that for any rectangle $ABCD$, $PA^2 + PC^2 = PB^2 + PD^2$, where P is an interior point in the rectangle.
   a. Verify that this result is correct using Geometer's Sketchpad®.
   b. In Chapter 9, we will ask you to prove this same result where the point P is either in the interior or the exterior of the rectangle. Move the point P to the outside of the rectangle and verify that this result is still true.
   c. $ABCD$ is a parallelogram and P is any point in either the interior or exterior of the parallelogram. Prove or disprove the conjecture that $PA^2 + PC^2 = PB^2 + PD^2$.

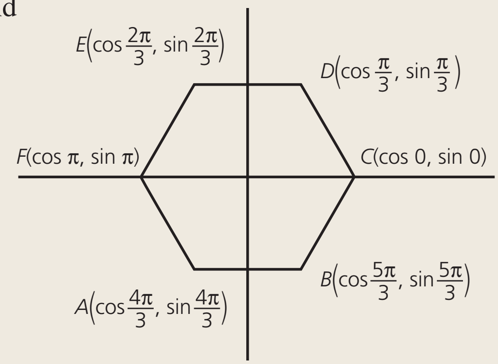




4. Using Geometer's Sketchpad®, construct a regular hexagon *ABCDEF*. The point *P* is any point on the plane that contains the given hexagon.

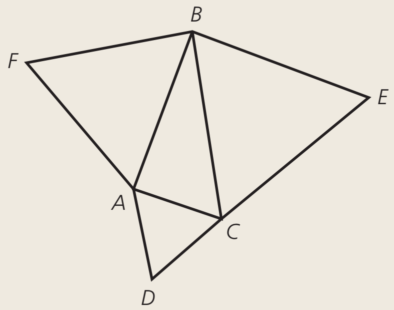


   a. Determine the location of all points on the plane such that $PA + PC + PE = PB + PD + PF$.
   b. Using Geometer's Sketchpad®, verify that $PA^2 + PC^2 + PE^2 = PB^2 + PD^2 + PF^2$.
   c. Prove that this relationship is true by setting up and coordinatizing in the following way.



   d. If the point *P* is not on the plane containing the hexagon, explain why this relationship is still true.

5. a. Construct a **scalene triangle** *ABC*. (A scalene triangle is one in which there are three unequal sides.)
   b. On each side of ΔABC, construct equilateral triangles, as shown.



   c. Locate the circumcentres of ΔDAB, ΔEAC, and ΔFCB and label these three points as *P*, *Q*, and *R*, respectively.
   d. Verify that ΔPQR is equilateral.

Chapter 3

PROPERTIES OF CIRCLES

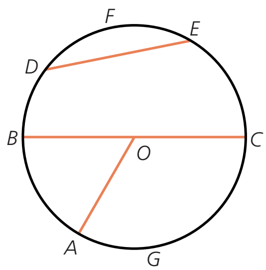
In Chapter 3, we will consider the closed plane curve of a circle. The circle is the most symmetrical shape, and very important in art and design. The properties of circles were fundamental to the ancient Greeks, and especially important were the formulas for circumference and area. These formulas, $C = 2\pi r$ and $A = \pi r^2$, have been part of the study of mathematics throughout history. Why do we continue to study them? Aside from the practical value that they have for architects, designers, and engineers, these formulas introduce the number π (pi).

CHAPTER EXPECTATIONS In this chapter, you will

- prove some properties of plane figures using deduction, **Section 3.1, 3.2, 3.3, 3.4, 3.5**
- prove some properties of plane figures using indirect methods, **Section 3.2, 3.3**
- prove some properties of plane figures algebraically, **Section 3.4. 3.5**

Review of Prerequisite Skills

DEFINITIONS RELATING TO THE CIRCLE



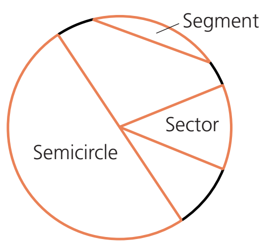
A *circle* is the locus of a point that moves so that it is always a constant distance from a fixed point. The fixed point is the *centre* and the constant distance is the *radius*. In the diagram, *O* is the *centre*, and *OA* is the *radius.*

DE is a *chord*; a chord is a line segment connecting two points on the circumference.

BC is a *diameter*; a diameter is a chord passing through the centre.

The curved line enclosing the circle is the *circumference.*

DFE and *DGE* are *arcs*; an arc is a portion of the circumference. *DFE* is a *minor arc*, and *DGE* is a *major arc.*



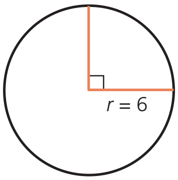
BGC is a *semicircle.* A semicircle is that part of a circle bounded by a diameter and an arc.

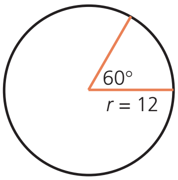
A circle can also be considered as the surface area enclosed by the circumference.

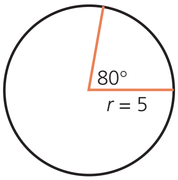
A *sector* is that part of a circle bounded by two radii and an arc. A *segment* is that part of a circle bounded by a chord and an arc.

Exercise

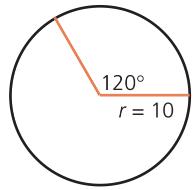
1. Calculate the area of the circle and the area of the sector for each of the following.

a. 

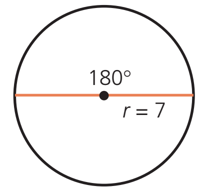
b. 

c. 

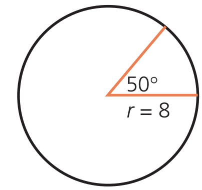
d.



e.



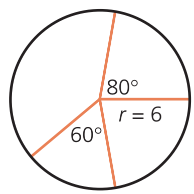
f.



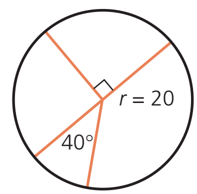
2. State a relationship between a sector area and the circle area in terms of the sector angle.

3. Calculate the ratio of the areas of the two sectors in each of the following.

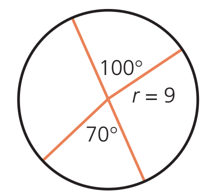
a.



b.

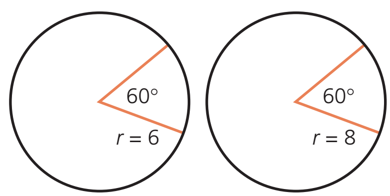


c.

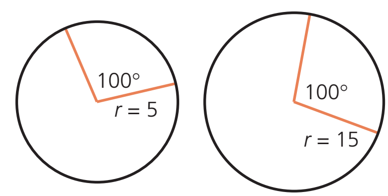


4. Calculate the ratio of the areas of the two sectors for each of the following pairs of circles.

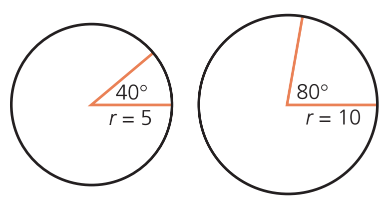
a.



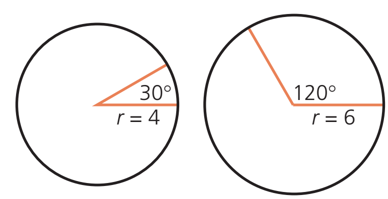
b.



c.



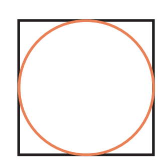
d.



5. The area of a given circle is 16π cm^2.

   a. What is the radius of this circle?

   b. What is the circumference of this circle?



6. If the area of the given square is 81 cm^2, determine the area of the inscribed circle.

7. The circumference of a circle, in cm, is equal to its area, in cm^2. What is the radius of the circle?

8. A set of *n* circles, each with a diameter of 1 cm, has a total area equal to that of a circle with radius 3 cm. What is the value of *n*?

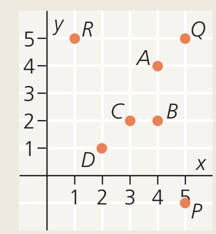
Rich Learning Link investigate

CHAPTER 3: GEOGRAPHIC PROFILING

People who commit crimes sometimes seem difficult to catch. When there is no clear relationship between the criminal and the victims, there can be no short list of suspects. Instead, investigators must look for relationships between the victims. This means analyzing vast quantities of data, most of which is likely to be insignificant. To help police around the world, mathematicians have developed a field called **geographic profiling.** Geographic profiling analyzes crime locations to find where the perpetrator is likely to live.

Investigate

Three crimes have occurred at locations indicated on the map by the points *P*, *Q*, and *R*. Which of the points *A*, *B*, *C*, or *D* most likely marks the residence of the criminal?



City map with superimposed Cartesian grid

We assume that the criminal's home is in a central position relative to the crime locations.

The perpendicular bisector of the segment *PQ* is the line of all points equidistant from *P* and *Q*. Given points *P* and *Q* as shown, the perpendicular bisector is the line $y = 2$. Similarly, the line $x = 3$ marks all points equidistant from *Q* and R.

The intersection of these two lines is the point (3,2). This is the point *C*. It is the same distance from *P* and *Q*, and the same distance from *Q* and *R*. Therefore the point *C* is equidistant from all three crime sites.

We conclude that *C* marks the most likely location of the criminal's residence.

Verify, using the distance formula, that *C* is equidistant from all three crime sites *P*, *Q*, and *R*; that is, *C* is the centre of a circle through the points *P*, *Q*, and *R*.

DISCUSSION QUESTIONS

1. How would you find the perpendicular bisector of *PR*?
2. Is there always a point equidistant from any three given points? What about four points, or more than four points?
3. Is it reasonable to assume that the criminal's home is in a central position relative to the crime locations?

Section 3.1 — Properties of Chords

technology
APPENDIX P. 530

In the Review of Prerequisite Skills, we reviewed basic properties of circles. Here we examine the properties of chords in a circle.

THEOREM

A line drawn from the centre of a circle to the midpoint of a chord is perpendicular to that chord if and only if it bisects the chord.

Part 1

Let O be the centre of the circle and AB be a chord. From O draw OM where M is the midpoint of AB. We will prove that $OM \perp AB$.

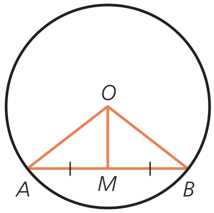
Proof

In ΔOAM and ΔOBM,

$OA = OB$ (Radii)

OM is common

$AM = BM$ (Given)



Therefore	$\Delta OAM \equiv \Delta OBM$	(Side-side-side)
Then	$\angle OMA = \angle OMB$	
But	$\angle OMA + \angle OMB = 180^\circ$	
Then	$\angle OMA = \angle OMB = 90^\circ$	
	$OM \perp AB$	

Part 2

Let AB be a chord in the circle having centre O. From O draw OM perpendicular to AB. We prove that $AM = MB$.

Proof

In ΔOAM and ΔOBM,

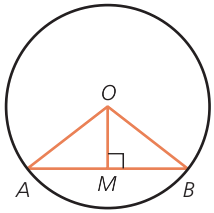
$OA = OB$ (Radii)

OM is common

$\angle OMA = \angle OMB = 90^\circ$

Therefore $\Delta OMA \equiv \Delta OBM$ (Hypotenuse-side)

Then $AM = MB$



The Chord Right Bisector Property

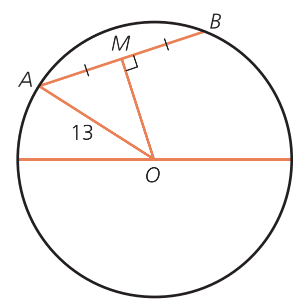
1. **The right bisector of a chord passes through the centre.**
2. **The perpendicular from the centre to a chord bisects the chord.**
3. **The line joining the centre to the midpoint of a chord is perpendicular to the chord.**
4. **The centre of a circle is the intersection of the right bisectors of two non-parallel chords.**

EXAMPLE 1

A circle has a diameter of length 26. If a chord in the same circle has a length of 10, how far is the chord from the centre?

Solution

Let the circle centre be O and let the chord be AB.



Since the diameter is 26, $OA = 13$.
If OM is perpendicular to AB, $\angle AMO = 90°$
and $AM = MB = 5$. (Chord Right Bisector Property)

Then $AM^2 + MO^2 = OA^2$ (Pythagoras)

$$5^2 + MO^2 = 13^2$$
$$MO^2 = 169 - 25$$
$$= 144$$
$$MO = 12\ (MO > 0)$$

THEOREM

Two chords are of equal length if and only if they are the same distance from the centre of the circle.

Part 1

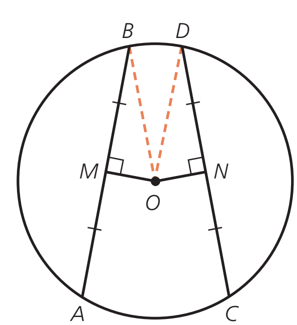
Let AB and CD be equal chords in a circle whose centre is O. We will prove that the chords are equidistant from the centre.

Proof

From O draw OM and ON to the midpoints of the chords AB and CD. Join O to B and O to D.
In ΔOMB and ΔOND,

$OB = OD$ (Radii)
$\angle OMB = \angle OND$ (Chord Right Bisector)
$MB = ND$ (Chord Right Bisector)
$\Delta OMB \equiv \Delta OND$ (Hypoteneuse-side)

Then $OM = ON$.



Part 2

Let O be the centre of a circle and let AB and CD be chords that are equidistant from O. We will prove that $AB = CD$.

Proof

Let $OM = ON$ be the distance to the chords.

In ΔOMA and ΔONC,

$OA = OC$ (Radii)

$OM = ON$ (Given)

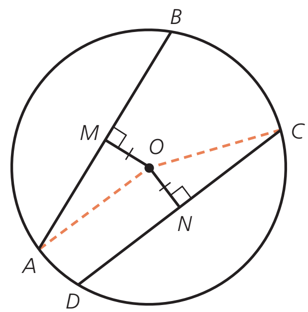
$\angle OMA = \angle ONC$ (Right angles)

Therefore $\Delta OMA \equiv \Delta ONC$ (Hypotenuse-side)

Then $MA = NC$

But $MA = \frac{1}{2}AB$ and $NC = \frac{1}{2}CD$ (Chord right bisector)

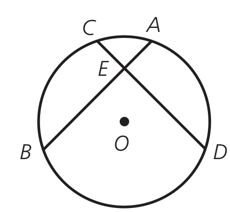
Then $AB = CD$.



> **Equal Chords Property**
> 1. Chords that are equal are equidistant from the centre.
> 2. Chords that are equidistant from the centre are equal.

EXAMPLE 2

In a circle with centre O, AB and CD are equal chords intersecting at E. Prove that $BE = DE$.



Solution

Let OX be perpendicular to AB and OY be perpendicular to CD. Join OE.

In ΔOXE and ΔOYE,

$OX = OY$ (Equal Chords Property)

$OE = OE$ (Same line)

$\angle OXE = \angle OYE$ (90°)

$\Delta OXE \equiv \Delta OYE$ (Hypotenuse-side)

Then $XE = EY$

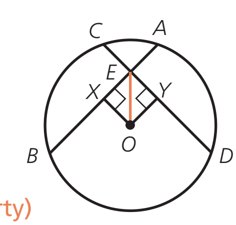
Since $OX \perp AB$, $BX = \frac{1}{2}AB$ (Chord Right Bisector Property)

Also $OY \perp CD$, $DY = \frac{1}{2}CD$ (Same Property)

But $AB = CD$, so $BX = DY$

Then $BX + XE = DY + YE$

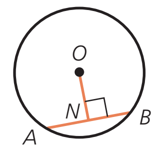
or $BE = DE$



Exercise 3.1

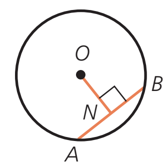
Part A

1. Determine the length of the chord AB if $OA = 5$ and $ON = 3$.



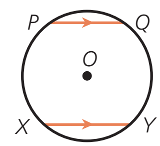
Application

2. If $AB = 10$ and $OA = 13$, determine the length of ON.

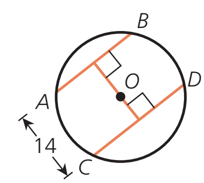


Knowledge/ Understanding

3. Calculate the distance between the parallel chords PQ and XY if $PQ = 6$, $XY = 8$, and the radius of the circle is 5.

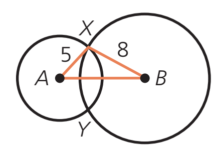


4. The two parallel chords AB and CD are a distance of 14 units apart. If AB has length 12 and the radius of the circle is 10, calculate the length of CD.

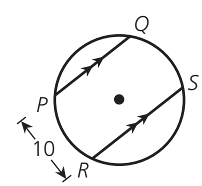


Part B

5. Two circles with centres A and B have radii 5 and 8, respectively. The circles intersect at the points X and Y. If $XY = 8$, determine the length of AB, the distance between the centres.



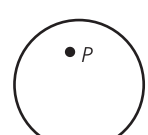
6. The distance between the parallel chords PQ and RS is 10. If $PQ = 8$ and $RS = 12$,
determine the length of the radius of the circle.



7. A chord of a circle is 4 units away from the centre of the circle, which has a radius of 10. What is the length of the chord?

Communication

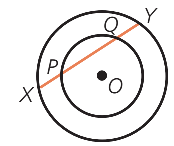
8. The point P is any point placed inside a circle. Is it always possible to draw a line through P that is bisected at the point P? Explain.



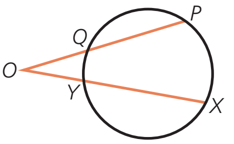
9. Prove that a straight line drawn from the centre of a circle perpendicular to a chord will, if extended, bisect the arc cut off by this chord.

Thinking/Inquiry/ Problem Solving

10. A line is drawn through two concentric circles as shown. Prove that $PX = QY$.



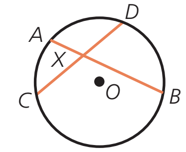
11. PQ and XY are two equal non-parallel chords in a circle. When extended, they meet at a point O outside the circle. Prove that $OQ = OY$.



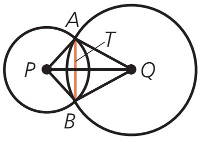
12. Two circles with centres X and Y intersect at P and Q. Prove that XY is the right bisector of PQ.

Part C

13. AB and CD are two chords of equal length that intersect at the point X in the circle with centre O. Prove that when O is joined to X, $\angle OXC = \angle OXB$.



14. In the diagram, $PA = 13$ cm and $QA = 20$ cm, where P and Q are the centres of the circles. Determine the length of AB if $PQ = 21$ cm.

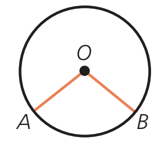


Section 3.2 — Angles in a Circle

If an arc subtends an angle at the centre of a circle, we can determine the size of the angle.

EXAMPLE 1

In the diagram, the radius of the circle is 6 and the length of the arc is π. Determine the measure of $\angle AOB$.

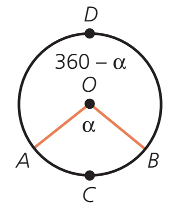


Solution

The circumference of the circle is $C = 2\pi r = 12\pi$.
If $\angle AOB = x^{\circ}$, then

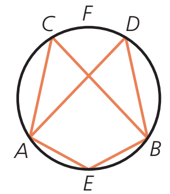
$$\frac{x}{360} = \frac{\pi}{12\pi}$$
$$x = \frac{360\pi}{12\pi} = 30$$
$$\angle AOB = 30^{\circ}$$

In this example, we have created a central angle. A **central angle** is an angle that has its vertex at the centre of a circle and stands on an arc (or chord) of the circle. In the diagram, two central angles are shown: $\angle AOB$, which is less than 180°, and reflex $\angle AOB$. The angle α stands on the minor arc ACB (or chord AB). In the same way, the angle $360 - \alpha$ stands on the major arc ADB. Each of the two angles is described as a central angle because each has its vertex at the centre of the circle and each stands on an arc of the circle.



Central Angles

An *angle at the circumference* of a circle is an angle that has its vertex on the circumference of a circle and that stands on an arc (or chord) of a circle. $\angle ADB$ and $\angle ACB$ are examples of two such angles which stand on the minor arc AEB (or chord AB). $\angle AEB$ is an example of an angle which stands on the major arc AFB (or chord AB).

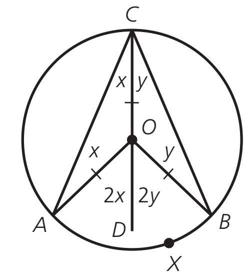


Angles at the circumference

THEOREM The angle at the centre of a circle is twice an angle at the circumference of the circle that stands on the same arc.

Proof

Let A and B be two points on a circle having centre O, let X be a point on minor arc AB, and let $\angle ACB$ be an angle at the circumference, standing on arc AXB. Join CO and extend it to D.



In ΔOCA, $OC = OA$ (Radii)

Then $\angle OCA = \angle OAC$ (Isosceles triangle)

Then $\angle DOA = 2\angle OCA$ (Exterior angle)

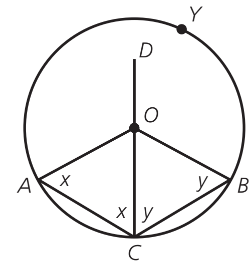
Similarly, in ΔOCB,

$\angle DOB = 2\angle OCB$

Then $\angle DOA + \angle DOB = 2(\angle OCA + \angle OCB)$

$\angle AOB = 2\angle ACB$

By choosing a point Y on the major arc AB, it can be shown that reflex $\angle AOB = 2\angle ACB$.



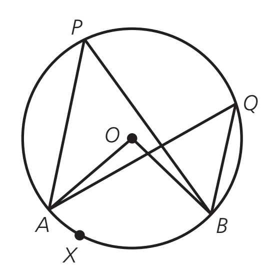
Angle at the Circumference Property
An angle at the centre of a circle is twice the angle at the circumference standing on the same arc.

This theorem leads to two immediate conclusions.

THEOREM Angles in the same segment of a circle are equal.

Proof

Let A and B be points on a circle having centre O, and let $\angle APB$ and $\angle AQB$ be angles in the same segment, standing on arc AXB.
Join OA and OB.



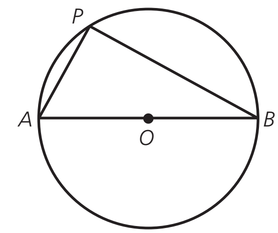
Then $\angle APB = \frac{1}{2}\angle AOB$ (Angle at the circumference)

and $\angle AQB = \frac{1}{2}\angle AOB$, (Same)

so $\angle APB = \angle AQB$.

Equal Angles in a Segment Property
Angles in the same segment of a circle are equal.

THEOREM The angle in a semicircle is a right angle.



Proof

Let AOB be a diameter in a circle and let $\angle APB$ be an angle on AB.

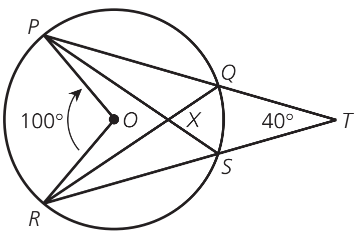
Then $\angle AOB = 2\angle APB$ (Angle at the circumference)

But $\angle AOB = 180^\circ$

Then $\angle APB = 90^\circ$

> **Angle in a Semicircle Property**
> **The angle in a semicircle is a right angle.**

EXAMPLE 2 If O is the centre of a circle such that $\angle POR = 100^\circ$, and $\angle PTR = 40^\circ$, determine the size of $\angle QXS$.



Solution

Since $\angle POR = 100^\circ$ and is at the centre of the circle, $\angle PQR = \angle PSR = 50^\circ$.

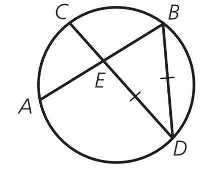
Since $\angle PQR$ and $\angle RQT$ are supplementary, $\angle RQT = 130^\circ$.

In a similar fashion, $\angle PST = 130^\circ$.

The sum of the interior angles of the quadrilateral $QXST$ is 360°.

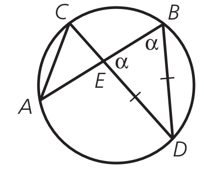
Then $\angle QXS = 360^\circ - (130 + 130 + 40)^\circ = 60^\circ$.

EXAMPLE 3 CD and BA are two chords of a circle that intersect at E. If $DB = DE$, prove that ΔACE is isosceles.



Proof

Join A to C, and for convenience, let $\angle DBE = \alpha$



Since $BD = ED$, ΔDEB is isosceles.

Then $\angle DBE = \angle BED = \alpha$

Since CD and AB are two intersecting straight lines,
$\angle CEA = \angle BED = \alpha$

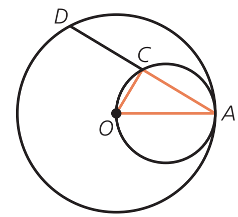
Also $\angle ACD = \angle DBA = \alpha$ (Equal angles in a segment)

Then $\angle CEA = \angle ACE = \alpha$

Therefore ΔACE is isosceles, with $AC = AE$.

EXAMPLE 4

In the diagram, the two circles are tangent at A. If AO is the diameter of the smaller circle and the radius of the larger circle, prove that $AC = CD$.



Proof

Join OC.

$\angle OCA = 90^\circ$ (Angle in a semicircle)

Then $OC \perp AD$

Then $AC = CD$ (Chord right bisector)

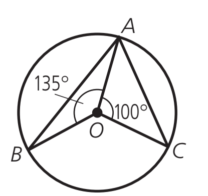
Exercise 3.2

Part A

Application

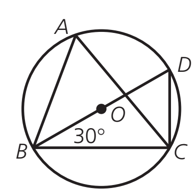
1. Determine the measure of the indicated angle, for each of the following.

a.



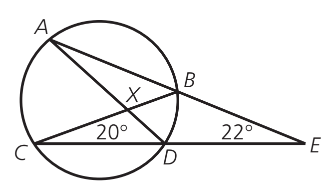
Determine $\angle BAC$.

b.



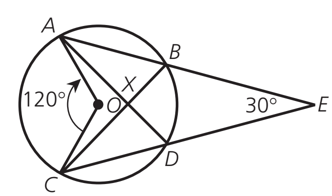
Determine $\angle BAC$.

c.



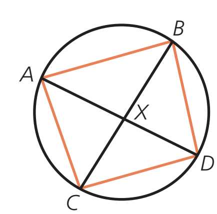
Determine $\angle ADC$ and $\angle AXB$.

d.



Determine $\angle BXD$.

2. The quadrilateral $ABCD$ has its vertices on a circle. The diagonals of the quadrilateral meet at the point X. Prove that ΔDCX and ΔABX are similar.



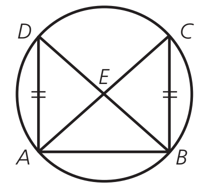
Communication

3. AB is a chord of a circle. Points D and C are chosen on the circumference of the circle so that $AD = BC$.
We want to prove $\Delta ABC \equiv \Delta ABD$.

a. State two equalities in the triangles.

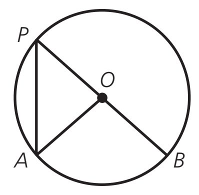
b. Discuss other necessary conditions for the congruence of the triangles.

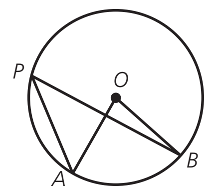
c. Prove $\Delta ABC \equiv \Delta ABD$.



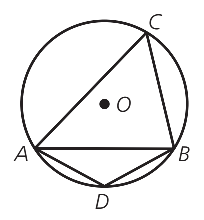
Part B

4. Prove that the Angle at the Circumference Property is true for circles with centre O where P is a point on the circumference as shown.

a. 

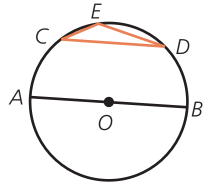
b. 

5. The length of chord AB is equal to the radius of the circle with centre O. D is a point on the circumference between A and B. Prove that $\angle ADB = 5\angle ACB$.



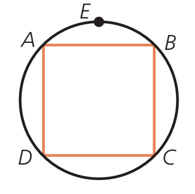
6. Prove that if the parallelogram $ABCD$ has each of its vertices on a circle then $ABCD$ must be a rectangle.

7. AB is the diameter of a circle with centre at O. Prove that if $CD < AB$, then $\angle CED > 90°$.

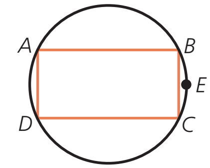


Knowledge/
Understanding

8. A square $ABCD$, with side length 2, is inscribed in a circle. The point E is on the arc AB as shown. Determine the numerical value of $AE^2 + BE^2 + CE^2 + DE^2$.



9. The rectangle $ABCD$ has its vertices on a circle as shown. Point E is taken to be any point on the circumference of the circle. Prove that $AE^2 + BE^2 + CE^2 + DE^2 = 2d^2$ where d is the length of the diagonal of the rectangle.

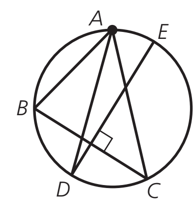


Part C

Thinking/Inquiry/ Problem Solving

10. Two circles intersect at A and B. KAL is drawn perpendicular to AB and meets the circles at the points K and L. The lengths KB and LB are extended to meet the circles at P and Q respectively. Prove that AB bisects $\angle PAQ$.

11. ABC has its vertices on a circle as shown. The bisector of the angle at A meets the circumference at D. From D, a line is drawn perpendicular to the chord BC so that it meets the circumference at E. Prove that DE is a diameter of the circle.



12. AXB and CXD are two perpendicular chords of a circle with centre O. Prove that $\angle AOD + \angle BOC = 180°$.

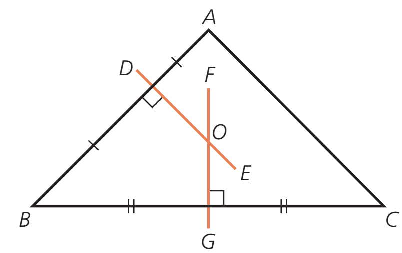
13. For ΔABC, two circles are drawn. The first circle has AB as its diameter and the second has BC as its diameter. The circles meet at the midpoint of AC. Prove that ΔABC is isosceles.

Section 3.3 — Cyclic Quadrilaterals

We can draw a circle that passes through the vertices of any triangle. This circle is the circumscribed circle of the triangle.

EXAMPLE 1

Describe a method of obtaining a circle that passes through the vertices of a triangle.



Solution

Let ABC be any triangle.

Draw DE, the right bisector of AB.

Draw FG, the right bisector of BC.

Let DE and FG intersect at O.

Then $OA = OB$ (Right bisector)
$OB = OC$ (Same)

Therefore $OA = OC$.

Then a circle with centre O and radius OA passes through A, B, and C.

Every triangle has a circumscribing circle. Are there quadrilaterals for which a circle can be drawn that passes through the four vertices? Yes, there are, but this is not true for every quadrilateral. The quadrilateral must meet special criteria. In this section we discuss the conditions necessary.

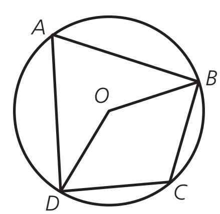
Concyclic points are points that lie on a circle. Because any three non-collinear points lie on a circle, we usually use the term to refer to a set of four or more points. A set of such points is also referred to as a cyclic set of points.

A cyclic quadrilateral is a quadrilateral whose vertices lie on a circle. In other words, the four points of a cyclic quadrilateral are concyclic.

THEOREM

In a cyclic quadrilateral, opposite angles are supplementary.

Proof



$ABCD$ is a quadrilateral with vertices on a circle with centre O. Join BO and DO.

In the diagram, $\angle BAD$ is an inscribed angle standing on the minor arc BD, for which $\angle BOD$ is the central angle.

Hence $\angle BAD = \frac{1}{2}\angle BOD$ (Angles in a circle)

Likewise, $\angle BCD = \frac{1}{2}(\text{reflex } \angle BOD)$ (Angles in a circle)

$$\therefore \angle BAD + \angle BCD = \frac{1}{2}(\angle BOD + \text{reflex}\angle BOD)$$
$$= \frac{1}{2}(360^\circ)$$
$$= 180^\circ$$

Note that it does not matter which arc BD is minor and which is major, or even if both are semicircles; the proof is the same.

The converse of the theorem is also true.

THEOREM If opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.

Proof

Let $ABCD$ be a quadrilateral in which $\angle ABC + \angle ADC = 180^\circ$.

There is a circle passing through A, B, and C.

The point D is inside the circle, or outside the circle, or on the circle. Assume D is inside the circle (Diagram 1).

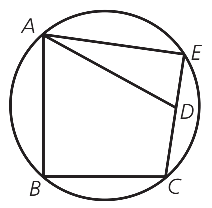


Diagram 1

Extend CD to meet the circle at E.

Now $ABCE$ is a cyclic quadrilateral, so $\angle ABC + \angle AEC = 180^\circ$.

But $\angle ABC + \angle ADC = 180^\circ$.

So $\angle ADC = \angle AEC$.

But $\angle ADC = \angle AEC + \angle EAD$ (Exterior angle)

$\angle ADC > \angle AEC$.

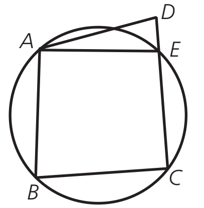
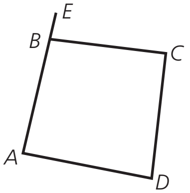


Diagram 2

This is a contradiction. Then D cannot be inside the circle. Assuming D is outside the circle (Diagram 2) leads to a similar contradiction. Hence D is on the circle, so $ABCD$ is cyclic.

A corollary worth noting is that if we extend one side of a cyclic quadrilateral, the exterior angle so formed is equal to the interior angle at the opposite vertex. You will be asked to prove this in Exercise 3.3.



Angles in a Cyclic Quadrilateral Property

A quadrilateral is cyclic if and only if its opposite angles are equal. The exterior angle of a cyclic quadrilateral is equal to the interior angle at the opposite vertex.

EXAMPLE 2 In a cyclic quadrilateral $ABCD$, $AB = AD$, $\angle BCD = 110^\circ$, and $\angle BAC = 30^\circ$. What is $\angle ABC$?

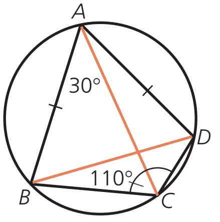
Solution

$\angle BCD + \angle BAD = 180^\circ$ (Angles in a cyclic quadrilateral)
$\angle BAD = 70^\circ$
$\angle DAC = 40^\circ$ (Subtraction)
$\angle DBC = 40^\circ$ (Angles in a circle)

$\angle ABD + \angle ADB = 110^\circ$ (Angles in a triangle)

Since $\angle ABD = 55^\circ$, (Isosceles triangle)

then $\angle ABC = 40^\circ + 55^\circ$
$= 95^\circ$



EXAMPLE 3 In ΔABC, $AB = AC$. D and E are on AB and AC respectively so that $DE \parallel BC$. Prove that $DECB$ is a cyclic quadrilateral.

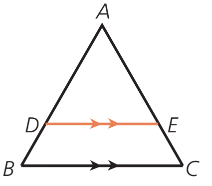
Proof

$\angle ABC = \angle ACB$ (Isosceles triangle)
$\angle AED = \angle ACB$ (Parallel lines)

Then $\angle ABC = \angle AED$
$\angle AED + \angle DEC = 180^\circ$ (Straight line)

Therefore $\angle ABC + \angle DEC = 180^\circ$

Since $\angle DBC$ and $\angle ABC$ are the same angle, $DECB$ is cyclic.

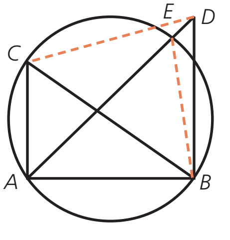


Other properties of a circle also lead to theorems that prove that points lie on a circle. The proofs are similar to the indirect proof given above. We give two of the more useful results below.

THEOREM If one side of a quadrilateral subtends equal angles at the two remaining vertices, the quadrilateral is cyclic.

Proof

Let $ABDC$ be a quadrilateral such that $\angle ACB = \angle ADB$. We will prove that $ABDC$ is a cyclic quadrilateral. Draw a circle that passes through A, B, and C. This circle either passes through D or it does not. Suppose that it does not, and let the circle cut AD at E. Join EB.



Then $\angle ACB = \angle AEB$ (Angles in a cyclic quadrilateral)

But $\angle ACB = \angle ADB$.

Therefore $\angle AEB = \angle ADB$.

Then $EB \parallel DB$ (Parallel lines property)

But this is impossible, since EB and DB meet at B.

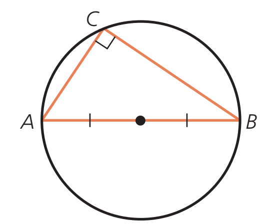
Then the supposition is false.

By a similar proof, it is not possible that the circle meets AD extended. Then the circle must pass through D, and $ABDC$ is a cyclic quadrilateral.

Cyclic Quadrilateral Property
A quadrilateral is cyclic if and only if one side subtends equal angles at the remaining vertices or opposite angles are supplementary.

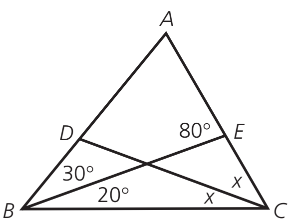
THEOREM

The circle having as its diameter the hypotenuse of a right-angled triangle passes through the third vertex of the triangle. This theorem is the converse of the result that the angle in a semicircle is a right angle. Its proof is similar to the one above.



EXAMPLE 4

In the diagram at the right, the angles are given. Show that $BCED$ is cyclic.



Solution

$\angle DAE = 70^\circ$ (Angles in ΔABE)

Then $\angle ACB = 60^\circ$ (Angles in ΔABC)

$\angle ACD = 30^\circ$

$\angle BDC = 100^\circ$ (Exterior angles, ΔADC)

$\angle BEC = 100^\circ$ (Straight line)

$\therefore \angle BCED$ is cyclic (Cyclic quadrilateral property)

EXAMPLE 5

In ΔABC, D is on BC and E on AC so that $AD \perp BC$ and $BE \perp AC$. AD and BE meet at O. Prove that $\angle OCD = \angle BAD$.

Proof

In quadrilateral $OECD$, $\angle OEC + \angle ODC = 180^\circ$ (Right angles)
Then $OECD$ is cyclic

Therefore $\angle OCD = \angle OED$ (Angles in a circle)

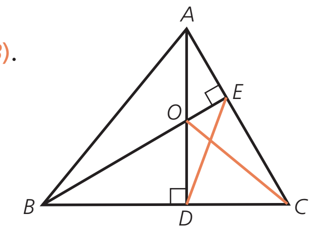
Also $\angle AEB = \angle ADB(90^\circ)$,

so $EDBA$ is cyclic. (Equal angles on chord AB).

Then $\angle BAD = \angle BED$ (Angles in a circle)

But $\angle OCD = \angle BED$ (From above)

Therefore $\angle BAD = \angle OCD$



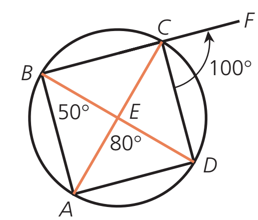
Exercise 3.3

Part A

Communication

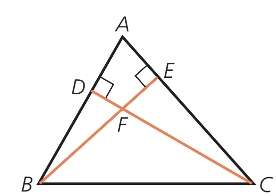
1. Is it true that any square is cyclic? Any rectangle? Any parallelogram? Explain your answer.

2. In the cyclic quadrilateral $ABCD$, BC is extended to F and $\angle DCF = 100^\circ$. The diagonal AC intersects BD at E. If $\angle ABD = 50^\circ$ and $\angle AED = 80^\circ$, determine each of the remaining angles in the diagram.

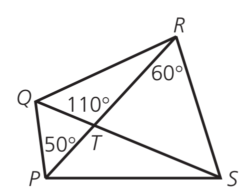


Knowledge/ Understanding

3. In the given diagram, what four points are concyclic?

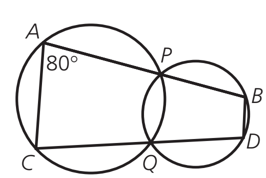


4. Prove that the points P, Q, R, and S are concyclic.



Application

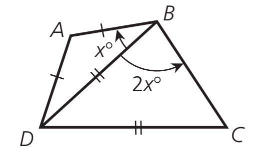
5. Determine the size of $\angle PBD$ in the given diagram.



Part B

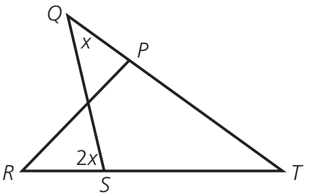
6. Prove that if one side of a cyclic quadrilateral is extended, any exterior angle equals the opposite interior angle of the quadrilateral.

7. In the quadrilateral $ABCD$, $AB = AD$ and $\angle ABD = x°$. Also, $DB = DC$ and $\angle DBC = 2x°$. Prove that $ABCD$ is a cyclic quadrilateral.



8. Prove that if a trapezoid has its vertices on a circle, then its base angles are equal, as are its diagonals.

9. In the given diagram, $PT = PR$. Prove that the points P, Q, R, and S are concyclic.



10. In isosceles ΔPQR, $PR = PQ$. The point T is chosen on PR, and TS is parallel to RQ where S is a point on PQ. Prove that $\angle QTS = \angle SRQ$.

Thinking/Inquiry/Problem Solving

11. PQ and PR are two chords of a circle with centre O. OT is perpendicular to PQ and OS is perpendicular to PR. If $OT = OS$, prove that the points T, S, R, and Q are concyclic.

Part C

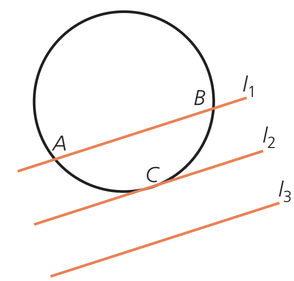
12. A cyclic regular octagon $ABCDEFGH$ has its vertices on a circle. Points X_1, X_2, . . . , X_7, and X_8 are chosen on successive arcs. Determine the numerical value of $\angle AX_1B + \angle BX_2C + ... + \angle HX_8A$.

13. A chord ST of constant length slides around a semicircle with diameter AB. M is the mid-point of ST, and P is the foot of the perpendicular from S to AB. Prove that the angle SPM is constant for all positions of ST.

Section 3.4 — Tangents to a Circle

A straight line can intersect a circle in two places, can touch the circle, or make no contact at all. There are no other possibilities.

A **secant** to a circle is a line that intersects the circle in two points. Thus a secant contains a chord of the circle. In the diagram, the line l_1 is a secant, containing the chord AB.



A **tangent** to a circle is a line that intersects the circle at only one point; it is said to touch the circle. In the diagram, the line l_2 is a tangent, touching the circle at the point C. C is the point of contact.

The line l_3 in the diagram, which does not intersect the circle, is said to be skew to the circle. Since they have no particular relation to the circle, skew lines are of no current interest.

If you have taken any calculus, you probably know that the above definition of a tangent is insufficient to describe tangents to more general curves. It is, however adequate for circles. We will not do a formal study of slopes and limits in this book, but we can use some of these concepts to investigate properties of tangents. We can use geometry software in two ways to compare properties of secants as they approach tangency:

technology

1. Create a circle and a secant PAB, with P outside the circle and A and B on the circle (start with A and B some distance apart on the circle). Now let the point B move so that it gets closer to A. This requires P to move as well. What happens to the radii to A and B? As B gets closer to A, what do you observe about the secant?
2. Create a circle and a secant $PABQ$ as in the example above. This time, allow the secant to move towards the edge of the circle, remaining parallel to the original secant. As the chord AB gets smaller, what do you observe about the secant?

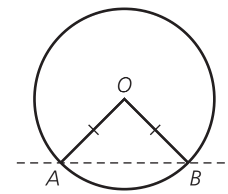
Most of the properties of tangents can be demonstrated using one of these techniques.

DEMONSTRATION

Using a computer (or using ruler and pencil), make a diagram with radii *OA* and *OB* and the secant through *A* and *B*. ΔOAB is isosceles, so

$$\angle OAB = \angle OBA = \frac{1}{2}(180° - \angle AOB).$$

What happens to $\angle OAB$ as the secant moves towards tangency?



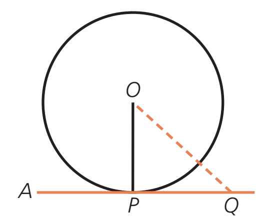
THEOREM

Part 1

A tangent to a circle is perpendicular to the radius at the point of contact.

Proof

Let *AB* be a tangent to a circle with centre *O*, touching the circle at *P*. Using indirect proof we will prove that $\angle OPA = 90°$.



Either $\angle OPA = 90°$ or $\angle OPA \neq 90°$.

Assume that $\angle OPA \neq 90°$.

Then one of $\angle OPA$ and $\angle OPB$ is greater than 90°, the other less than 90°. If $\angle OPB < 90°$, then there is a point *Q* on *AB* such that $\angle OPQ = \angle OQP$.

But then $OP = OQ$. (Isosceles triangle)
This is impossible because *Q* is outside the circle, so $OQ > OP$.

It is not true that $\angle OPA \neq 90°$.

Then $\angle OPA = 90°$.

Part 2

A straight line drawn at right angles to a radius of a circle at the circumference is a tangent to the circle.

Part 3

A line drawn at right angles to a tangent at the point of contact passes through the centre of the circle.

Tangent Radius Property

For a given circle,

1. a tangent is perpendicular to the radius at the point of contact;

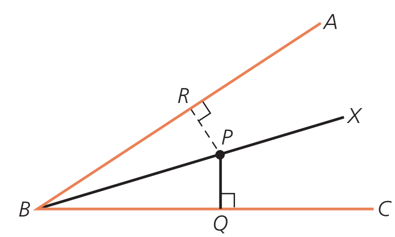
2. a line at right angles to a radius at the circumference is a tangent;

3. a perpendicular to a tangent at the point of contact passes through the centre.

EXAMPLE 1

The line BX is a bisector of $\angle ABC$, P is a point on BX, and $PQ \perp BC$. Prove that a circle with centre P and radius PQ will have BA as a tangent.

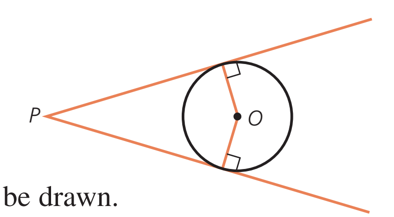
Proof



Let PR be perpendicular to BA.

Then $PQ = PR$ (Angle bisector)

Therefore a circle with centre P and radius PQ passes through R. Since $PR \perp AB$, and PR is a radius, BA is tangent to the circle. (Tangent radius property)

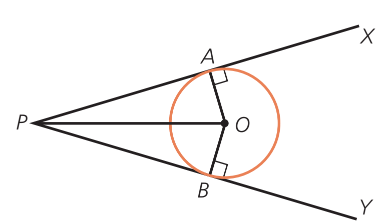


From a point outside a circle, two tangents can be drawn.

THEOREM

If tangents are drawn to a circle from an external point, the segments to the points of contact are equal.

Proof



Let PX and PY be tangents to a circle with centre O, and let A and B be the contact points of the tangents. We will prove that $PA = PB$. Join PO.

In ΔPOA and ΔPOB,

$PO = PO$ (Same line)

$OA = OB$ (Radii)

$\angle OAP = \angle OBP$ (Tangent radius property)

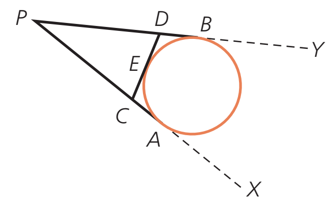
$\Delta POA \equiv \Delta POB$ (Hypotenuse-side)

Then $PA = PB$

Tangent from a Point Property
Tangent segments from an external point to a circle are equal.

EXAMPLE 2

In the given diagram, PX is a tangent contacting the circle at A, PY is a tangent contacting the circle at B, and CD is a tangent contacting the circle at E. If $PA = 15$, determine the perimeter of ΔPCD.



Solution

$PA = PB = 15$ (Tangents from a point)
$CA = CE$ (Tangents from a point)
$DB = DE$ (Same)

Now perimeter $\Delta PCD = PC + CE + ED + DP$
$= PC + CA + BD + DP$
$= PA + PB$
$= 30$

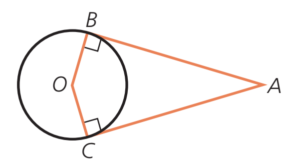
The perimeter is 30.

Exercise 3.4

Part A

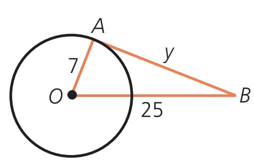
Knowledge/Understanding

1. In the diagram, the radius is 5 and tangent AB has length 12. Determine the length of tangent AC and the length of OA.



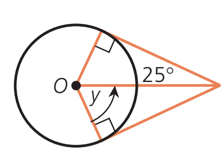
2. Determine the value of the variable(s) in each of the following. In each circle, the centre is marked as 0.

a.

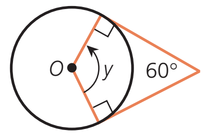


AB is a tangent to the circle.

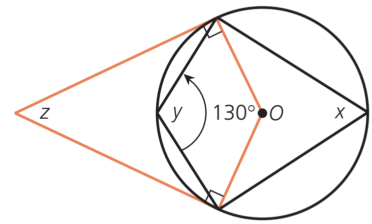
b.



c.

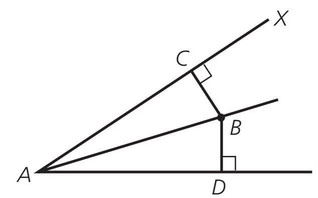


d.

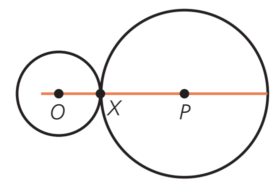


Communication

3. In the diagram, what condition is necessary in order that a circle with centre B and radius BC will have AX and AY as tangents?



4. Circles with centres O and P touch at X, where X is on the line OP.



   a. Will a tangent to the circle with centre O, drawn at X, also be tangent to the second circle?

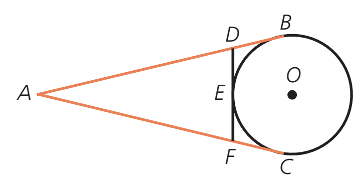
   b. If the smaller circle is placed inside the larger, will a tangent to one circle be a tangent to the other?

Part B

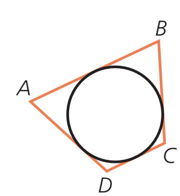
5. Prove that tangents to a circle drawn at the ends of a diameter are parallel.

6. Concentric circles have the same centre but different radii. Prove that in concentric circles, two chords of the larger circle that are tangent to the smaller circle are equal.

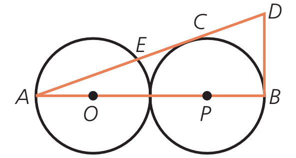
7. AB, AC, and DF are tangents to a circle at points B, E, and C, as shown. If $AB = 10$, determine the perimeter of ΔADF.



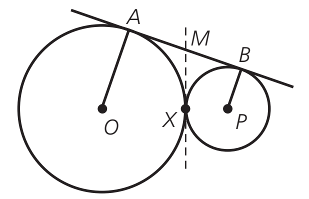
8. The quadrilateral $ABDC$ is circumscribed about a circle; that is, each side of the quadrilateral is tangent to the circle. Prove that $AB + CD = AD + BC$.



9. Two circles with centres O and P, each with a radius of 2, are tangent to each other. The line OP cuts the circles at A and B. Lines AC and BD are tangent to the circle with centre P and intersect at D, as shown. Determine the length of BD.



10. Two circles with centres O and P touch externally at X. The tangent at X meets the direct common tangent AB at M, as shown. Prove that M is the midpoint of AB.



11. Tangents at two points P and Q on a circle with centre O meet at an external point X. Prove that $\angle XPQ = \angle XOP$.

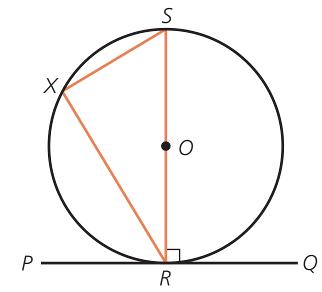
Part C

Thinking/Inquiry/ Problem Solving

12. ΔPQR is right-angled at Q. $PQ = 16$ cm and $QR = 30$ cm. A circle is drawn inside the triangle, touching all sides. (This is called an inscribed circle or incircle). Find the radius of the circle.

Section 3.5 — More About Tangents

We have seen that a line perpendicular to a tangent passes through the centre of a circle. If this line is extended, it becomes a diameter of the circle. In the diagram, PQ is a tangent at R to the circle with centre O, and by extending RO to S we have RS as a diameter. For any point X on the circle, we create $\angle RXS = 90^\circ$. Now $\angle RXS = \angle SRQ$. What happens if RS is any chord, rather than a specific one? Is it still true that $\angle RXS = \angle SRQ$?

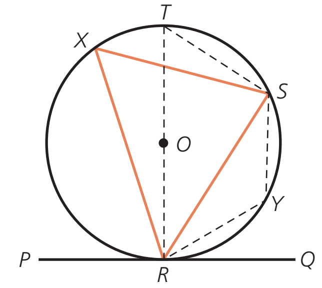


THEOREM The angle formed by a chord and a tangent is equal to the angle subtended by the chord in the segment on the other side of the chord.

Proof

Let PQ be tangent at R to the circle with centre O. Let RS be any chord, let X be any point in the major arc created by RS, and let Y be any point in the minor arc.

We will prove that $\angle RXS = \angle SRQ$
and $\angle RYS = \angle SRP$.



Join RO and extend the line to meet the circle at T.

Then $\angle RXS = \angle RTS$ (Angles on chord RS)

Also $\angle TSR = 90^\circ$ (Angle in a semicircle)

Then $\angle RTS + \angle TRS = 90^\circ$ (Angles in a triangle)

But $\angle SRQ + \angle TRS = 90^\circ$ (Tangent Radius Property)

Therefore $\angle SRQ = \angle RTS$

Therefore $\angle SRQ = \angle RXS$

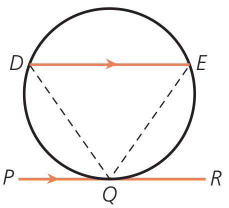
Similarly, $\angle RYS = \angle SRP$

Tangent Chord Property
The angle formed by a tangent and a chord is equal to the angle subtended by the chord in the segment on the other side of the chord.

EXAMPLE 1 PQR is tangent to a circle at Q. DE is a chord of the circle parallel to PQR. Prove that ΔQED is isosceles.

Proof

Join DQ and QE.



Then $\angle PQD = \angle DEQ$ (Tangent Chord Property)
$\angle EQR = \angle EDQ$ (Same)
$\angle DEQ = \angle EQR$ (Parallel lines)

Therefore $\angle DEQ = \angle EDQ$

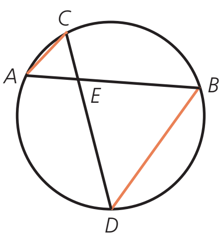
Then ΔDEQ is isosceles (equal angles).

When two non-parallel chords are drawn in a circle, they may intersect inside the circle. If not, they will certainly intersect outside the circle if they are extended. These extended chords are secants. From these intersecting lines, we obtain a useful theorem.

THEOREM If two chords intersect, the product of the two parts of one is equal to the product of the two parts of the other.

Proof

Let AB and CD be chords intersecting at E in a circle.
We prove that $AE \bullet EB = CE \bullet ED$.



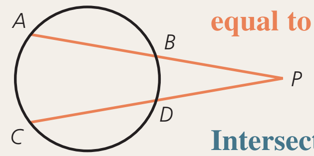
Join AC and BD.

In ΔACE and ΔDBE,
$\angle AEC = \angle DEB$ (Vertically opposite)
$\angle ACE = \angle EBD$ (Angles on arc AD)

Then $\Delta ACE \sim \Delta DBE$. (Angles equal)

Therefore $\frac{AE}{CE} = \frac{ED}{EB}$

or $AE \bullet EB = CE \bullet ED$



Intersecting Chords Property

If two chords in a circle intersect, the product of the two parts of one is equal to the product of the two parts of the other.

Intersecting Secants Property

If two secants AB and CD intersect at point P, then $PA \cdot PB = PC \cdot PD$.

You are asked to prove the Intersecting Secants Property in Exercise 3.5.

DEMONSTRATION Draw the diagram for secant intersection. Now let the secant PCD move so that, while P stays fixed, the chord CD gets shorter. When C and D become the same point, we have a tangent; if we call the common point T, then instead of $PC \cdot PD$, we have PT^2.

This indicates the following corollary to the Secant Intersection Property.

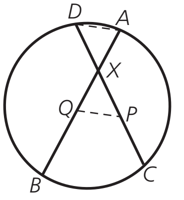
Corollary

If a tangent PT is drawn from a point on a secant AB, then $PA \cdot PB = PT^2$.

You are asked to prove this corollary in Exercise 3.5.

EXAMPLE 2 In a circle, two chords AB and CD intersect at X. P and Q are the midpoints of XC and XB respectively. Prove that $ADQP$ is a cyclic quadrilateral.

Proof



$BX \cdot XA = CX \cdot XD$ (Intersecting chords)

Then $\frac{1}{2}BX \cdot XA = \frac{1}{2}CX \cdot XD$

or $QX \cdot XA = PX \cdot XD$

or $\frac{PX}{QX} = \frac{XA}{XD}$

Also $\angle PXQ = \angle AXD$ (Opposite angles)

Therefore $\Delta PXQ \sim \Delta AXD$ (Similar Triangle Property)

Then $\angle PQX = \angle ADX$

or $\angle PQA = \angle ADP$

These angles are both subtended by AP.

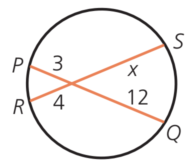
Therefore, $ADQP$ is a cyclic quadrilateral.

Exercise 3.5

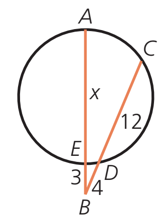
Part A

1. Find the value of the indicated variable(s) in each of the following.

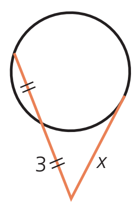
a.



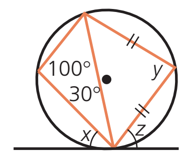
b.



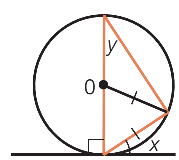
c.



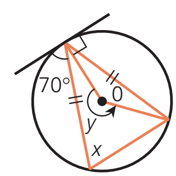
d.



e.



f.



Knowledge/ Understanding

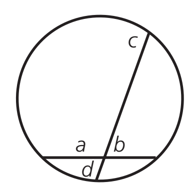
2. The sides BC, CA, and AB of ΔABC touch a circle at D, E, and F, respectively. If $\angle A = 48°$ and $\angle B = 80°$, calculate each angle of ΔDEF.

Part B

3. The tangent to a circle at C is parallel to a chord AB of the circle. Prove that ΔABC is isosceles.

Communication

4. For the given diagram, if

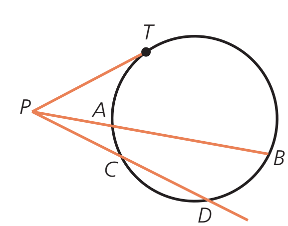


a. $a = 6$, $b = 4$, and $c = 8$, find d

b. $a = b$, $c = 12$, and $d = 3$, find a

c. $a = x$, $b = x + 5$, $c = x + 11$ and $d = x - 4$, find x

5. For the given diagram, if



a. $PA = 6$, $AB = 10$, and $PC = 8$, find CD

b. $PT = 18$ and $PC = 12$, find PD

c. $PT = 15$ and $AB = 16$, find PA

d. $PA = x$, $AB = 2x + 6$, $PC = x + 4$, and $CD = x - 3$, find x

6. AB is a chord of a circle, and AC is a diameter. The tangent to the circle at B contains a point D such that $AD \perp BD$. Prove that AB bisects $\angle DAC$.

7. Two circles intersect at P and Q. A line through P meets the circles at A and B. The tangents at A and B meet at C. Prove that $AQBC$ is a cyclic quadrilateral.

8. Two circles intersect at P and Q. A common tangent touches the circles at X and Y. Prove that $\angle XPY + \angle XQY = 180°$.

Application

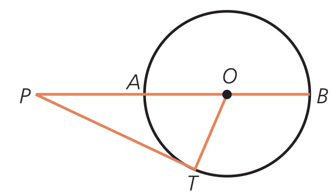
9. a. Prove the Intersecting Secants Property.

b. Prove the corollary to the Intersecting Secants Property.

Part C

Thinking/Inquiry/ Problem Solving

10. AB is a diameter of a circle with centre O. P is on BA extended, and PT is tangent to the circle. Use the corollary to the Intersecting Secants Property to prove that $PT \perp OT$.



Key Concepts Review

In this chapter, you have studied many of the properties of circles, chords, and angles in a circle. You should be familiar with the following:

1. circle chord properties and conditions for equal chords
2. the equality of angles in the same segment
3. properties of cyclic quadrilaterals
4. the tangent radius property
5. the tangent chord property

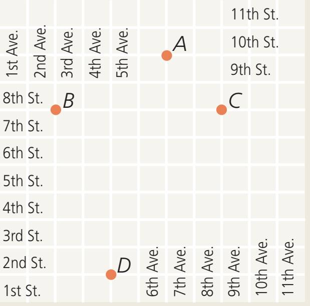
Rich Learning Link wrap-up investigate and apply

CHAPTER 3: GEOGRAPHIC PROFILING

Geographic profiling is a means of linking a series of crime sites to find where the criminal lives. It was created by Dr. Kim Rossmo, an adjunct professor at Simon Fraser University's School of Criminology, who is also a former Detective Inspector with the Vancouver Police Department. He says, "Seventy-five percent of serial killers hunt in their own community ... Geographic profiling is designed to track these people down." The software he has developed considers many geographic aspects of crime sites, including the location of central points.

Investigate and Apply

The map shows a section of a city where four similar crimes have been committed. The crimes were committed on four consecutive days in the order *A*, *B*, *C*, *D*.



1. Find the centres of four circles, each of which passes through three of the crime locations.
2. Is *ABDC* a cyclic quadrilateral? Provide as many different arguments as you can to justify your answer.
3. Does the interior of any of the four circles contain all four points? If so, where is its centre?
4. Speculate on the most likely location of the perpetrator's home and justify your conclusion with reasoning that includes mathematical reasoning.

INDEPENDENT STUDY

In addition to the location of central points, what other geographic factors should be taken into account when using crime locations to predict a criminal's place of residence? Do you think that they can all be accounted for using circle geometry?

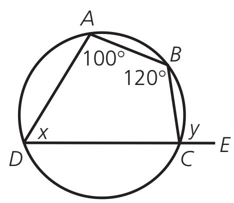
The RCMP's Violent Crime Linkage Analysis System relates dozens of separate factors to find patterns among crimes. What are some other quantifiable aspects of random crimes?

How is geographic profiling used in a court of law? Can it be used as proof of guilt or innocence?

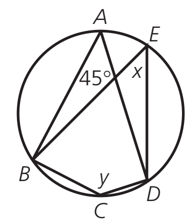
Review Exercise

1. Determine the value of the indicated variables in diagrams **a** and **b**.

a.

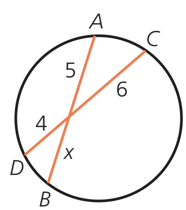


b.

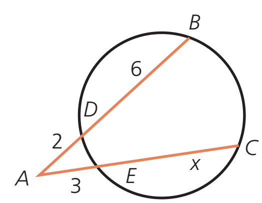


2. Calculate the value of the unknown quantities in each of the following diagrams. (Where it is used, *O* is the centre of the circle.)

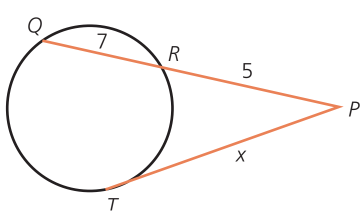
a.



b.

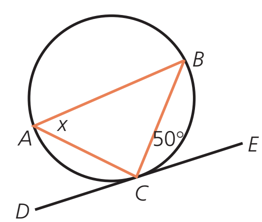


c.



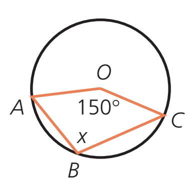
PT is a tangent to the circle.

d.

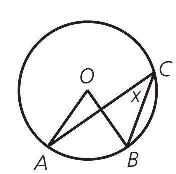


DE is a tangent to the circle at *C*.

e.



f.



$\angle AOB = 76°$

3. A series of triangles is drawn on the same side of a common base *AB* with each angle opposite *AB* of the same size. What can be said about the set of third vertices?

4. *ABCD* is a cyclic quadrilateral. If the diagonals *AC* and *BD* intersect at the centre *O*, prove that *ABCD* is a rectangle.

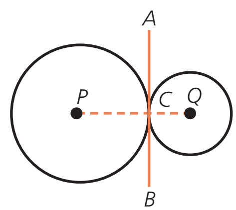
5. You are given a cyclic quadrilateral *ABCD* with the diagonals *AC* and *BD* intersecting at *E*. Side *BC* is extended to point *F*. If exterior angle $BCF = 100°$, $\angle AED = 85°$, and $\angle ABD = 60°$, find the size of each of the remaining angles in the quadrilateral.

6. A point is taken on the bisector of a given angle. Prove that a circle may be drawn with centre *P* that has the arms of the angle as tangents.

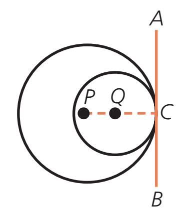
7. How would you draw a circle that touches two given straight lines if
   a. the two given straight lines are parallel?
   b. the two given straight lines are not parallel?

8. Two circles with centres *P* and *Q* touch a straight line and each other at the point *C*. Prove that *P*, *C*, and *Q* lie on the same straight line.

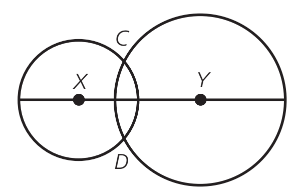
a.



b.

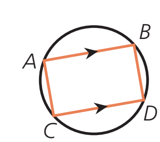


9. The chord *CD* has length 10. If the radius of the smaller circle is 8 and the radius of the larger circle is 13, determine the distance, *XY*, between the centres of the two circles.

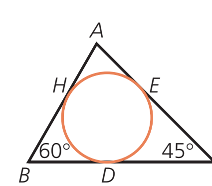


10. Suppose you are given three points *A*, *B*, and *C* and are told that they are on the arc of a circle. How would you locate the centre of this circle?

11. If *AB* and *CD* are two parallel chords, prove that $AC = BD$.

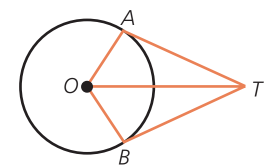


12. The inscribed circle of ΔABC touches its sides at *D*, *E*, and *H*. If $\angle ABC = 60°$ and $\angle ACB = 45°$, determine the measure of the angles of ΔDEH.

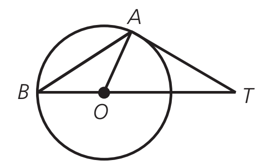


13. A circle is drawn so that it has as its diameter one of the equal sides of an isosceles triangle. How do we know that this circle passes through the middle of the base?

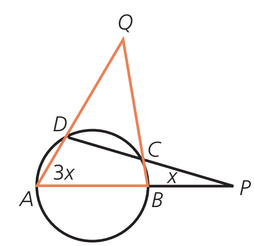
14. TA and TB are tangents to a circle with centre O and radius 10. If $\angle AOB = 120°$, determine the length of OT.



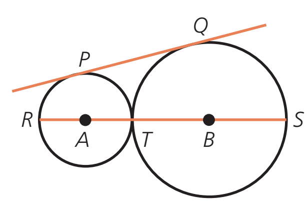
15. AT is the tangent at A to a circle with centre O. If $\angle ATB = x$, determine the size of $\angle OAB$ in terms of x.



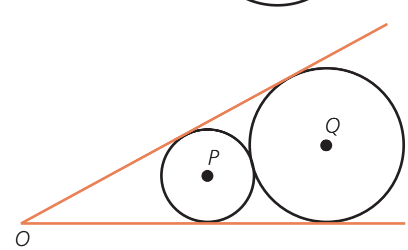
16. In the given diagram, $ABCD$ is a cyclic quadrilateral with $\angle DAB = 3x$. If AB and DC are extended to meet at P, then $\angle APD = x$. If AD and BC are extended to meet at Q, then $\angle AQB = 2x$. Find the value of x.



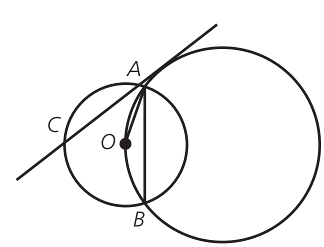
17. In the diagram, circles with centres A and B are tangent externally at T. PQ is a common tangent line. The line of centres intersects the circles at R and S as shown. RP and SQ meet at X when extended. Prove that $PXQT$ is a rectangle.



18. A circle with centre P and radius 10 is tangent to the sides of an angle of 60°. A larger circle with centre Q is tangent to the sides of the angle and to the first circle. What is the radius of the larger circle?



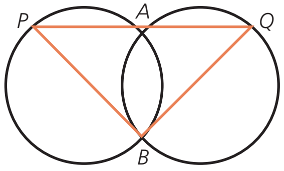
19. Two circles intersect at the points A and B, and one of them passes through O, which is the centre of the other circle. A tangent is drawn to the larger circle at A. Prove that OA bisects the angle between the common chord AB and the tangent to the circle at point A.



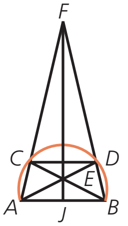
20. *ABCD* is a cyclic quadrilateral. *AB* and *DC* are extended to meet at point *P*. The lines *CB* and *DA* are extended to meet at *Q*. Circles are drawn around ΔQAB and ΔPBC. These circles intersect again at *R*. Prove that $\angle QRP = 180^\circ$, and the points *Q*, *R*, and *P* are on the same straight line.

21. ΔABC is a triangle in which $\angle A = 60^\circ$, $\angle B = 50^\circ$, and $\angle C = 70^\circ$. A circle is drawn that passes through the vertices *A*, *B*, and *C*. The bisectors of angles *A*, *B*, and *C* meet the circumference of the circle at *X*, *Y*, and *Z*, respectively. Determine the size of each of the angles in ΔXYZ.

22. Two circles have the same radius and intersect at the points *A* and *B*. A line is drawn through *A* that meets one circle at *P* and the other at *Q*. Prove that $PB = BQ$.



23. *AB* is the diameter of a semicircle. *C* and *D* are any two points on its arc. *AC* and *BD* are extended to meet at *F*. From *F*, a line through the intersection of *AD* and *BC* meets *AB* at *J*. Prove that *FJ* is perpendicular to *AB*.

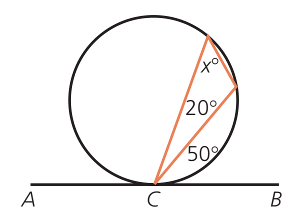


Chapter 3 Test

Achievement Category	Questions
Knowledge/Understanding	All
Thinking/Inquiry/Problem Solving	5, 6, 7
Communication	2
Application	All

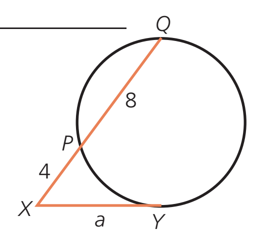
1. Determine the measure for each of the indicated values in the following diagrams. It is not necessary to show your work; only the answer is required.

   a. AB is tangent to the circle at C.



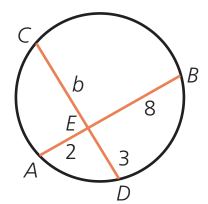
$x =$ ___________

   b. XY is tangent to the circle at Y.



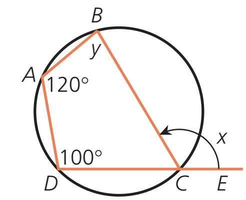
$a =$ ___________

   c. Two chords AB and CD intersect at E.



$b =$ ___________

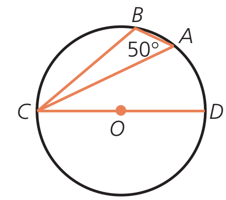
d. *ABCD* is a cyclic quadrilateral.



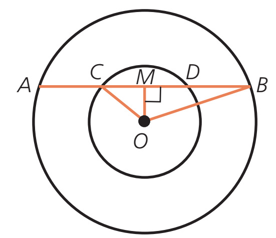
$x =$ __________

$y =$ __________

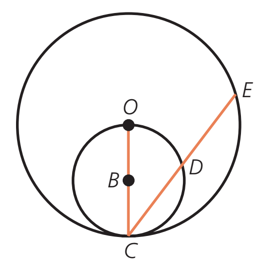
2. In the given diagram, *O* is the centre of the circle. If $\angle BAC = 50°$, determine the size of $\angle BCD$. Explain your reasoning.



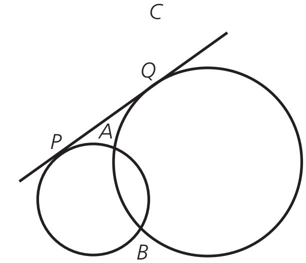
3. The two given circles are concentric with centre *O*. $OB = 17$, $OC = 10$, and $OM = 8$. Determine the length of *AC*.



4. In the diagram, *OC* is a radius of the larger circle and a diameter of the smaller circle, which has *B* as centre. Prove that $CD = DE$.



5. Two circles intersect at *A* and *B*, and *PQ* is a tangent to both circles. Prove that when *BA* is extended, it bisects *PQ* at the point *R*.



6. a. In a given circle, several equal chords are drawn. Prove that their midpoints lie on a circle concentric with the given circle.

   b. Prove that all such chords are tangents to this concentric circle.

7. *AB* is diameter of a circle, and *AP* is a chord. *AT* is another chord bisecting $\angle BAP$. Prove that the tangent at *T* cuts *AP* extended at right angles.

Extending and Investigating

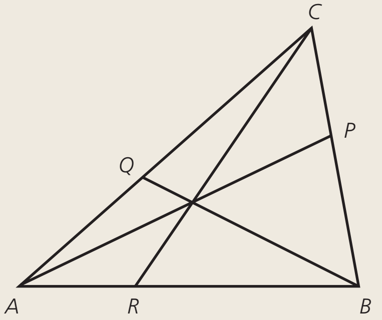
COMPUTER INVESTIGATIONS

1. A *Cevian* is a line drawn from a vertex of a triangle to the opposite side of the triangle. Some examples of Cevians are medians, altitudes, and angle bisectors.

   A well known theorem involving Cevians is Ceva's Theorem.

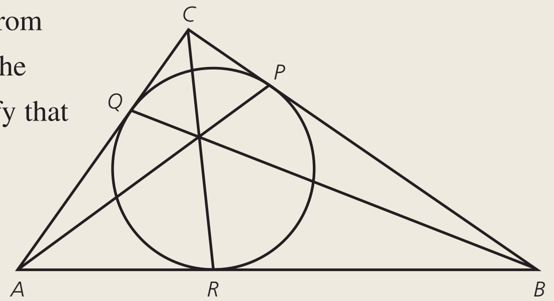
   Ceva's Theorem

   The cevians of ΔABC, AP, BQ, and CR are concurrent if and only if $\frac{AR}{RB} \bullet \frac{CQ}{QA} \bullet \frac{BP}{PC} = 1$.



   a. Draw a variety of different triangles and Cevians, and convince yourself that Ceva's Theorem is correct. (Note that Ceva's Theorem is written in *if and only if* form and must thus be verified in both directions.)

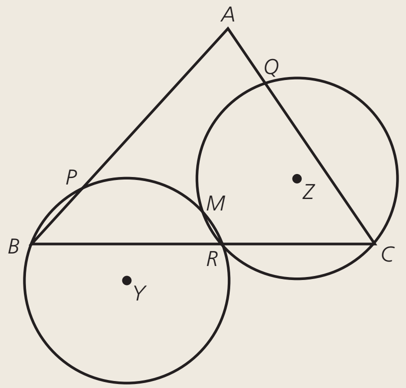
   b. Draw ΔABC and construct the incentre and incircle. (Recall that the incentre, I, is the point of intersection of the bisectors of the angles and that the incircle has centre I and is tangent to the three sides of the triangles.) From each vertex, draw a line to the point of contact on the opposite side. Using Geometer's Sketchpad®, verify that the three Cevians concur.



   c. Prove that the observation you made in **b** is correct by using Ceva's Theorem and the properties of tangents drawn to circles.

2. Draw any triangle ABC and on the three sides AB, AC, and BC, select three points P, Q, and R, respectively. Construct three circles according to the following instructions. The first circle passes through vertex C and the points R and Q. The second circle passes through B, P, and R and the third through A, P, and Q. (In our diagram we have drawn two of the circles.)

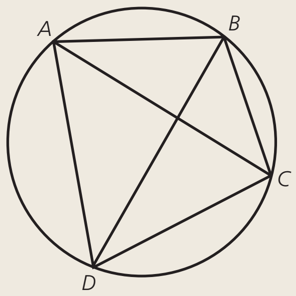
a. Using Geometer's Sketchpad®, construct all three circles and show that all three points have a point, call it M, in common.



b. We have drawn two of the circles and we have labelled the point M as the point of intersection. Prove that M lies on the circle passing through A, P and Q.
(*Hint:* Join P to M, M to Q, and M to R and then use properties of cyclic quadrilaterals to show that A, P, M, and Q are concyclic points.)

c. Label the centres of the three circles as X, Y, and Z and prove that $\Delta XYZ \sim \Delta ABC$.

3. Draw a circle and locate any four points A, B, C, and D on the circumference of the circle.



a. Using Geometer's Sketchpad®, show that $(AB)(DC) + (AD)(BC) = (AC)(BD)$. This is Ptolemy's Theorem.

b. Draw an equilateral triangle ABC and construct a circle that passes through A, B, and C.

(i) Using Geometer's Sketchpad®, show that if the point P is on the arc BC, $PA = PB + PC$.

(ii) Prove, using Ptolemy's Theorem, that this result is correct.

c. Draw a regular hexagon $ABCDEF$ in a circle and place a point on the arc BC.

(i) Find an analogous relationship for the hexagon to that found in **b** for the equilateral triangle.

(ii) Prove that this relationship is true using Ptolemy's Theorem.

Cumulative Review

CHAPTERS 1–3

1. On a map, 1 cm represents 1 km. A rectangular area measures 2.7 cm by 3.5 cm on the map. What is its actual area?

2. AB and CD are two chords of a circle that, when extended, meet outside the circle at a point O. Prove that $\Delta AOD \sim \Delta BOC$.

3. The lengths of the sides of a right-angled triangle are $2x - 5$, $2x + 2$, and $2x + 3$. Write an algebraic expression for the area of this triangle.

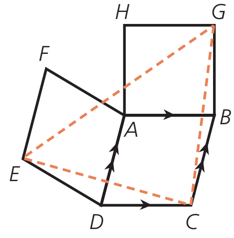
4. A and B are 8 cm apart. Find two points, each of which is 5 cm from A and 6 cm from B.

5. The medians BD and CE of ΔABC intersect at F. The areas of ΔDFC and ΔFBC are 8 and 16, respectively. Find the area of
   a. ΔEBF
   b. quadrilateral $AEFD$

6. The vertex A of ΔABC assumes various positions on the same side of BC, while BC remains fixed in length and position. If the sum of the angles B and C remains constant, what is the path traced out by the vertex A?

7. AB and CD are two perpendicular chords of a circle with centre O. Prove that $\angle AOD + \angle BOC = 180°$.

8. $ABCD$ is a parallelogram. $ABGH$ and $ADEF$ are squares. Prove that ΔCGE is isosceles.



9. Determine which of the following statements are true.

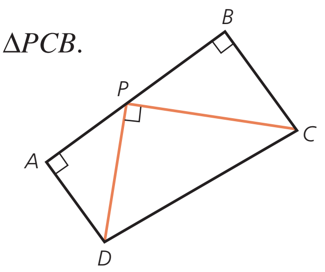
   a. A quadrilateral is a rhombus if and only if it is a parallelogram.

   b. Quadrilaterals have equal areas if and only if they are congruent.

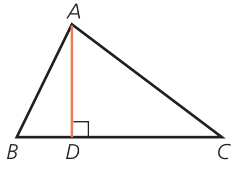
   c. Two lines L_1 and L_2 are parallel if and only if a line crossing L_1 and L_2 makes alternate angles equal.

10. If a quadrilateral has its area bisected by the line joining the midpoints of a pair of opposite sides, prove that these sides are parallel.

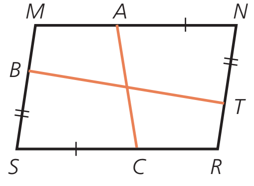
11. In the diagram, $\angle PDC = 45°$. Prove that $\Delta PDA \cong \Delta PCB$.



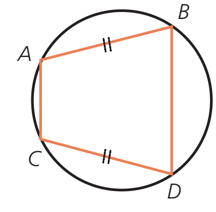
12. In ΔABC, a line is drawn from A perpendicular to BC so that it meets BC at D. Prove that $AD^2 = (BD)(DC)$ if and only if $\angle BAC = 90°$.



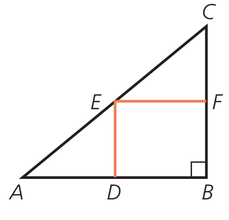
13. In the given figure, $SRNM$ is a parallelogram with $AN = SC$ and $NT = SB$. Prove that $AT = BC$ and AC bisects BT.



14. In a circle, two chords are drawn such that $AB = CD$. Prove that $AC \parallel BD$.

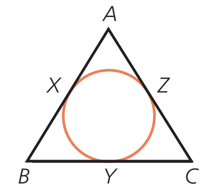


15. ABC is a right-angled triangle with $AB = 2BC$. Find the ratio of the area of the inscribed square $DEFB$ to the area of the triangle ABC.



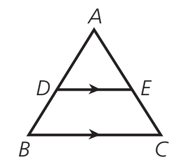
16. ΔABC is an isosceles triangle in which $\angle B = \angle C = 2\angle A$. BX bisects $\angle B$ and meets AC in X. Prove that BX and XA are consecutive sides of a regular pentagon.

17. A circle is inscribed in ΔABC as shown. Prove that $(AX)(BY)(CZ) = (BX)(YC)(ZA)$.

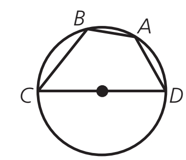


18. Two angles of a convex polygon are 100º and 140º. The others, which are equal, are angles of 120º. How many sides does the polygon have?

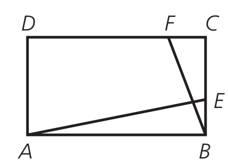
19. ABC is an isosceles triangle in which $AB = AC$. The line DE is a straight line parallel to BC. Prove that $DBCE$ is a cyclic quadrilateral.



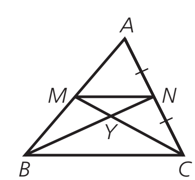
20. In the diagram, O is the centre of the circle. If $\angle DBA = 30º$, determine the measure of $\angle ADC$.



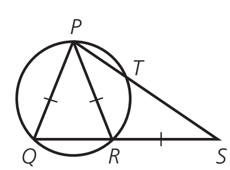
21. $ABCD$ is a square. E is a point on BC. Prove that $AE = BF$ if and only if BF is perpendicular to AE.



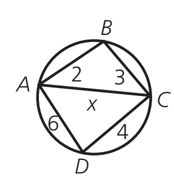
22. M and N are the midpoints of AB and AC, respectively. Prove that ΔBYC = quad $AMYN$.



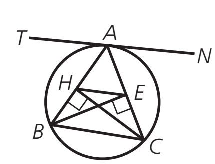
23. In the diagram, QR is extended to S, making $RS = PR$. PS intersects the circle at T. Prove that QT is the bisector of $\angle PQR$.



24. $ABCD$ is a cyclic quadrilateral with $AB = 2$, $BC = 3$, $CD = 4$ and $DA = 6$. Determine the value of x, the length of AC.



25. BE and CH are altitudes of ΔABC. TAN is the tangent at A to the circle shown. Prove that $HE \parallel TN$.



Chapter 4

VECTORS

Have you ever tried to swim across a river with a strong current? Have you sailed a boat, or run into a head wind? If your answer is yes, then you have experienced the effect of vector quantities. Vectors were developed in the middle of the nineteenth century as mathematical tools for studying physics. In the following century, vectors became an essential tool of navigators, engineers, and physicists. In order to navigate, pilots need to know what effect a crosswind will have on the direction in which they intend to fly. In order to build bridges, engineers need to know what load a particular design will support. Physicists use vectors in determining the thrust required to move a space shuttle in a certain direction. You will learn more about vectors in this chapter, and how vectors represent quantities possessing both magnitude and direction.

CHAPTER EXPECTATIONS In this chapter, you will

- represent vectors as directed line segments, **Section 4.1**
- determine the components and projection of a geometric vector, **Section 4.1**
- perform mathematical operations on geometric vectors, **Section 4.2**
- model and solve problems involving velocity and force, **Section 4.3, 4.4**

Review of Prerequisite Skills

A **vector** is a quantity, an inseparable part of which is a direction. Pause for a moment and think about physical quantities that have a direction. Force is an example. The force of gravity acts only downward, never sideways. Wind is another example. A wind from the north and a wind from the south have different physical consequences, even if the wind speeds are the same. Temperature, on the other hand, is not a vector quantity. Temperature does not *go* in any direction. Temperature is referred to as a **scalar** quantity.

We need both scalar and vector quantities to model complex physical systems. Meteorologists, for example, need data on air temperature and wind velocity, among other things, to make weather forecasts.

The object of this chapter is to introduce the mathematical properties of vectors and then show how vectors and scalars are used to describe many features of the physical world.

In this chapter, we introduce the concept of a vector, a mathematical object representing a physical quantity that has both magnitude and direction. We will concentrate on geometric representations of vectors, so that most of our discussion will be of two-dimensional vectors. In later chapters we will introduce algebraic representations of vectors, which will be more easily extended to higher dimensions.

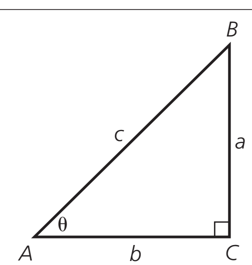
Before we begin this chapter, we will review some basic facts of trigonometry.

TRIGONOMETRIC RATIOS

In a right-angled triangle, as shown,

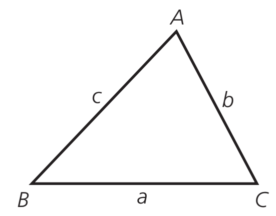
$$\sin\theta = \frac{a}{c} \qquad \cos\theta = \frac{b}{c}$$
$$\tan\theta = \frac{a}{b}$$

Note: The ratios depend on which angle is θ and which angle is 90°.



THE SINE LAW

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



THE COSINE LAW

$$a^2 = b^2 + c^2 - 2bc \cos A \text{ or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Exercise

1. State the exact value of each of the following.
   a. sin 60º
   b. cos 60º
   c. sin 135º
   d. tan 120º
   e. cos 30º
   f. tan 45º

2. A triangle ABC has $AB = 6$, $\angle B = 90º$, and $AC = 10$. State the exact value of tan A.

3. In ΔXYZ, $XY = 6$, $\angle X = 60º$, and $\angle Y = 70º$. Determine the values of XZ, YZ, and $\angle Z$ to two-decimal accuracy.

4. In ΔPQR, $PQ = 4$, $PR = 7$, and $QR = 5$. Determine the measures of the angles to the nearest degree.

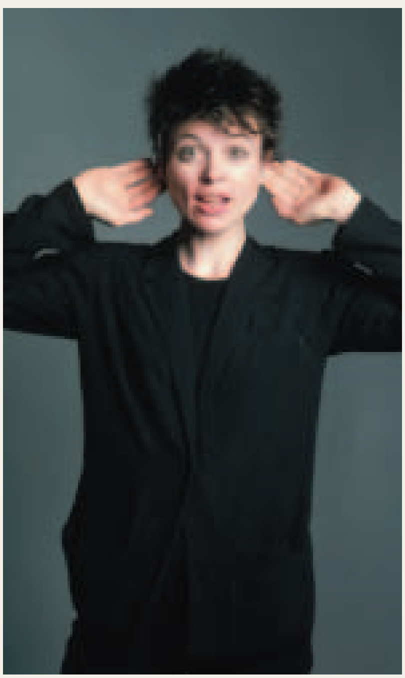
5. An aircraft control tower T is tracking two planes at points A, 3.5 km from T, and B, 6 km from T. If $\angle ATB = 70º$, determine the distance between the planes.

6. Three ships are at points A, B, and C such that $AB = 2$ km, $AC = 7$ km, and $\angle BAC = 142º$. What is the distance between B and C?

Rich Learning Link investigate

CHAPTER 4: VECTORS AND THE SUPERIOR COLLICULUS

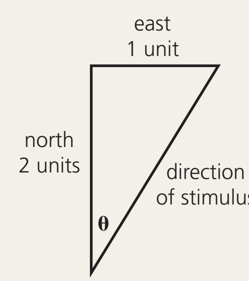
Neuroscientists have found cells in a deep layer of a part of the brain called the superior colliculus. These cells are tuned to the directions of distant visual and auditory stimuli. Each cell responds only to stimuli from a specific direction. Different cells are tuned to different directions. The tuning is broad, and the regions to which different cells are tuned overlap considerably. Neuroscientists have asked what it is about the activity in a group of cells with overlapping tuning regions that specifies the actual direction of a stimulus? For example, how is it that we can point accurately in the direction of a distant sound without seeing its source? One answer is that a cell responds more vigorously when the distance stimulus is in its direction. The direction is determined not by which cell fires most vigorously, but by a type of addition of the degrees to which the various cells have responded to the stimulus.



Investigate and Inquire

The type of addition performed in the brain can be illustrated by a simple case involving only two brain cells. Suppose that one of these cells responds to stimuli that are approximately north, while the other responds to stimuli that are approximately east. If the north cell responds twice as vigorously as the east cell, what is the direction of the stimulus? We can use vector addition to find out.

The answer is found by forming a triangle with a side pointing east and a side pointing north. The side pointing north is twice as long as the side pointing east. The third side is the actual direction of the stimulus. From the diagram, we see $\tan\theta = \frac{1}{2}$.



Solving, we find $\theta = \tan^{-1}\left(\frac{1}{2}\right) \cong 26.6°$.

So $\theta = 26.6°$.

Thus, the stimulus is 26.6° east of north.

What direction would be represented by a north-east cell responding three times as vigorously as an east cell?

DISCUSSION QUESTIONS

1. How many cells would be needed to represent all the directions in the plane?
2. Why do you think the direction is not just taken to be the one corresponding to the cell that fires most vigorously?

Section 4.1 — Vector Concepts

Vectors are a part of everyone's common experience. Consider a typical winter weather report that you might hear on the nightly news: *The temperature is presently –11 ºC, with a wind from the northwest at 22 km/h*. This weather report contains two different types of quantities. One quantity (the temperature) is expressed as a single numerical value. The other quantity (the wind velocity) has a numerical value (its magnitude) and also a direction associated with it.

These quantities are typical of the kinds encountered in science. They are classified as follows:

> **Quantities having magnitude only are called scalars.**
> **Quantities having both magnitude and direction are called vectors.**

There seems to be some overlap here. For example, the temperature could be thought of as having magnitude (11º) and direction (negative); in that sense, it could be considered as a one-dimensional vector. There is no problem with this interpretation; sometimes it is a useful way to look at such quantities. However, in most situations we find it easier to use positive and negative numbers as scalars, and restrict the term *vectors* to quantities that require (at least) two properties to define them.

Some examples of vector quantities are

Force — The force of gravity has a well defined magnitude and acts in a specific direction (down). The force of gravity is measured when you step on a scale. Force is a vector quantity.

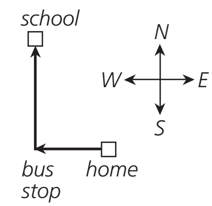
Displacement — When you walk from point *A* to point *B*, you travel a certain distance in a certain direction. Displacement is a vector quantity.

Magnetic Field — Some magnets are strong; others are weak. All cause a compass needle to swing around and point in a particular direction. A magnetic field is a vector quantity.

In a diagram, a vector is represented by an arrow: $\nearrow$. The length of the arrow is a positive real number and represents the magnitude of the vector. The direction in which the arrow points is the direction of the vector. For now we will restrict our discussion to vectors in two dimensions or to situations that can be expressed in two dimensions. Our definitions and conclusions are easily extended to three dimensions (or more).

EXAMPLE 1

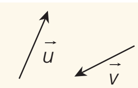
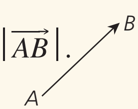
A student travels to school by bus, first riding 2 km west, then changing buses and riding a further 3 km north. Represent these displacements on a vector diagram.



Solution

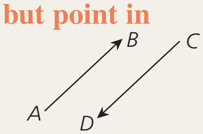
Suppose you represent a 1-km distance by a 1-cm line segment. Then, a 2-cm arrow pointing left represents the first leg of the bus trip. A 3-cm arrow pointing up represents the second leg. The total trip is represented by a diagram combining these vectors.

The notation used to describe vector quantities is as follows:

The algebraic symbol used in this text for a vector is a letter with an arrow on top. Some texts use boldface letters for vectors.	$\vec{u}$, $\vec{v}$ are vectors  **u**, **v** are also vectors
Scalar quantities are written as usual.	x, y, a, b are scalars
The magnitude of a vector is expressed by placing the vector symbol in absolute value brackets. The magnitude of a vector is a positive scalar.	$\lvert\vec{u}\rvert$, $\lvert\vec{v}\rvert$ are the magnitudes of the vectors $\vec{u}$, $\vec{v}$
Often it is necessary to explicitly state the initial point and the end point of a vector. Then, two capital letters are used. Such vectors are referred to as **point-to-point vectors**.	$\overrightarrow{AB}$ is the vector that starts at point A and ends at point B. Its magnitude is $\lvert\overrightarrow{AB}\rvert$. 

Certain other terms are used in connection with vectors.

Two vectors are *equal* if and only if their magnitudes and their directions are the same.
Two vectors are *opposite* if they have the same magnitude but point in opposite directions.
When two vectors are opposite, such as $\overrightarrow{AB}$ and $\overrightarrow{CD}$, one is the *negative* of the other: $\overrightarrow{AB} = -\overrightarrow{CD}$.
Two vectors are *parallel* if their directions are either the same or opposite.

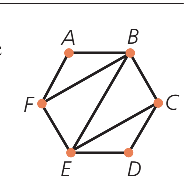


EXAMPLE 2

ABCDEF is a regular hexagon. Give examples of vectors which are

a. equal

b. parallel but having different magnitudes



c. equal in magnitude but opposite in direction

d. equal in magnitude but not parallel

e. different in both magnitude and direction

Solution

a. $\overrightarrow{AB} = \overrightarrow{ED}$

b. $\overrightarrow{FA} \parallel \overrightarrow{EB}$, but $|\overrightarrow{FA}| \neq |\overrightarrow{EB}|$

c. $|\overrightarrow{FE}| = |\overrightarrow{CB}|$, but $\overrightarrow{FE} = -\overrightarrow{CB}$

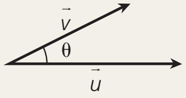
d. $|\overrightarrow{ED}| = |\overrightarrow{DC}|$, but $\overrightarrow{ED} \neq \overrightarrow{DC}$

e. $\overrightarrow{FB}, \overrightarrow{DC}$

There are other possible answers.

There is no special symbol for the direction of a vector. To specify the direction of a vector, we state the angle it makes with another vector or with some given direction such as a horizontal or vertical axis or a compass direction.

The angle between two vectors is the angle (≤180°) formed when the vectors are placed tail to tail; that is, starting at the same point.

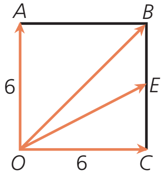


One way to determine the angle between two vectors is to examine geometrical relationships and use trigonometry.

EXAMPLE 3

OABC is a square with sides measuring 6 units. E is the midpoint of *BC*. Find the angle between the following vectors.

a. $\overrightarrow{OB}$ and $\overrightarrow{OC}$ b. $\overrightarrow{OE}$ and $\overrightarrow{OC}$ c. $\overrightarrow{OB}$ and $\overrightarrow{OE}$



Solution

a. The diagonal of the square bisects $\angle AOC$.
The angle between $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is 45°.

b. Using trigonometry, $\tan \angle EOC = \frac{3}{6}$, $\angle EOC \cong 26.6°$, so the angle between $\overrightarrow{OE}$ and $\overrightarrow{OC}$ is 26.6°.

c. The angle between $\overrightarrow{OB}$ and $\overrightarrow{OE}$ is the difference $45° - 26.6° = 18.4°$.

When two vectors are parallel, one of the vectors can be expressed in terms of the other using **scalar-multiplication**. Suppose, for example, M is the midpoint of the line segment AB. Since M is the midpoint, then $|\overrightarrow{AB}| = 2|\overrightarrow{AM}|$, and since the directions of $\overrightarrow{AB}$ and $\overrightarrow{AM}$ are the same, we write the **vector equations**

$$\overrightarrow{AB} = 2\overrightarrow{AM} \text{ or } \overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} \text{ or } \overrightarrow{BM} = -\frac{1}{2}\overrightarrow{AB}.$$

Thus, multiplication of a vector by a scalar k results in a new vector parallel to the original one but with a different magnitude. It is true in general that two vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} = k\vec{v}$.

A particularly useful type of vector is a vector with magnitude 1. Such vectors are called **unit vectors**. A unit vector is denoted by a *carat* (ˆ) placed over the symbol. When a vector and a unit vector are denoted by the same letter, for example $\vec{v}$ and $\hat{v}$, you should understand $\hat{v}$ to be a unit vector having the same direction as $\vec{v}$. Any vector can be expressed as a scalar multiple of a unit vector.

Unit Vectors

1. **A unit vector in the direction of any vector $\vec{v}$ can be found by dividing $\vec{v}$ by its magnitude $|\vec{v}|$:**
   $\hat{v} = \frac{1}{|\vec{v}|}\vec{v}$
2. **Any vector $\vec{v}$ can be expressed as the product of its magnitude $|\vec{v}|$ and a unit vector $\hat{v}$ in the direction of $\vec{v}$**
   $\vec{v} = |\vec{v}|\hat{v}$

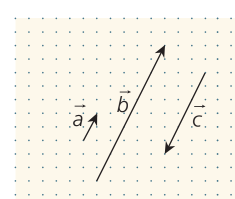
Another useful type of vector has magnitude 0. Such vectors are valuable even though their direction is undefined. The **zero vector** is denoted by $\vec{0}$.

EXAMPLE 4

Examine the vectors in the diagram.

a. Express $\vec{b}$ and $\vec{c}$ each as a scalar multiple of $\vec{a}$.

b. Express $\vec{a}$, $\vec{b}$, and $\vec{c}$ each in terms of the unit vector $\hat{a}$.



Solution

a. On the grid, each vector lies on the hypotenuse of a right-angled triangle with sides in the ratio 1:2, so the three vectors are parallel. The magnitudes of $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be found using the Pythagorean Theorem.

$|\vec{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$, $|\vec{b}| = \sqrt{5^2 + 10^2} = 5\sqrt{5}$,

and $|\vec{c}| = \sqrt{3^2 + 6^2} = 3\sqrt{5}$

Therefore $\vec{b} = 5\vec{a}$ and $\vec{c} = -3\vec{a}$.

b. The unit vector in the direction of $\vec{a}$ is $\hat{a} = \frac{1}{\sqrt{5}}\vec{a}$. Then $\vec{a} = \sqrt{5}\hat{a}$, $\vec{b} = 5\sqrt{5}\hat{a}$, and $\vec{c} = -3\sqrt{5}\hat{a}$.

Exercise 4.1

Part A

Communication

1. In your own words, explain the difference between a scalar and a vector.

Communication

2. Which of these physical quantities is a vector and which is a scalar?

a. the acceleration of a drag racer

b. the mass of the moon

c. the velocity of a wave at a beach

d. the frequency of a musical note

e. the speed of light

f. the age of a child

g. the friction on an ice surface

h. the volume of a box

i. the energy produced by an electric generator

j. the force of gravity

k. the speedometer reading in an automobile

l. the momentum of a curling stone

m. the time on a kitchen clock

n. the magnetic field of the earth

o. the density of a lead weigh

p. the pressure of the atmosphere

q. the area of a parallelogram

r. the temperature of a swimming pool

3. For each part of Example 2, state a second answer.

Part B

4. One car travelling 75 km/h passes another going 50 km/h. Draw vectors that represent the velocities of the two cars if they are going

a. in the same direction

b. in opposite directions

5. What is the angle between the following directions?

a. *N* and *NE*

b. *E* and *SW*

c. *S* and *W*

Knowledge/Understanding

6. Draw a vector to represent

a. the velocity of a fishing boat travelling at 8 knots on a heading of $S75°W$ (A knot is a speed of one nautical mile per hour.)

b. the position of a city intersection 7 blocks east and 3 blocks south of your present position

c. the displacement of a crate that moves 6 m up a conveyor belt inclined at an angle of 18°

d. the force exerted by a chain hoist carrying a load of 200 kg

Application

7. Radar in the control tower of an airport shows aircraft at directions of $N50°E$, $N70°W$, and $S20°W$, and distances of 5, 8, and 12 km, respectively.

a. In a diagram, draw vectors showing the positon of the three aircraft in relation to the tower.

b. The aircraft are travelling at velocities of 450 kph N, 550 kph $N70°W$, and 175 kph $N20°E$, respectively. At the positon of each aircraft in part **a**, draw small vectors to represent their velocities.

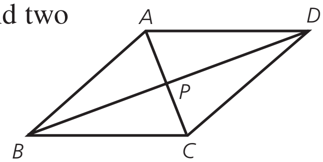
Knowledge/ Understanding

8. The points A, B, C, D, E, F, and G are equally spaced along a line. Name a vector which is equal to

a. $3\overrightarrow{BD}$ b. $\frac{1}{4}\overrightarrow{EA}$ c. $\frac{5}{2}\overrightarrow{DF}$ d. $\frac{2}{3}\overrightarrow{GC}$ e. $-2\overrightarrow{AD}$

Application

9. $ABCD$ is a rhombus. For each of the following, find two vectors $\vec{u}$ and $\vec{v}$ in this diagram (expressed as point-to-point vectors) such that

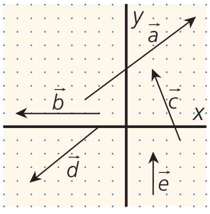


a. $\vec{u} = \vec{v}$ b. $\vec{u} = -\vec{v}$

c. $\vec{u} = 2\vec{v}$ d. $\vec{u} = \frac{1}{2}\vec{v}$

10. During takeoff, an aircraft rises 100 m for every 520 m of horizontal motion. Determine the direction of its velocity.

11. Determine the magnitude and the direction of each of the vectors in the given diagram. Express each direction as an angle measured counter-clockwise from a unit vector in the positive x direction.



12. A search and rescue aircraft, travelling at a speed of 240 km/h, starts out at a heading of N 20° W. After travelling for one hour and fifteen minutes, it turns to a heading of N 80° E and continues for another 2 hours before returning to base.

a. Determine the displacement vector for each leg of the trip.

b. Find the total distance the aircraft travelled and how long it took.

Part C

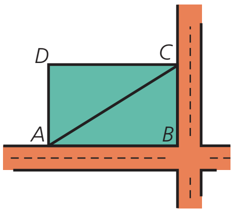
13. For what values of k is $|(k-2)\vec{v}| < |4\vec{v}|$, $(\vec{v} \neq \vec{0})$?

14. Prove that two vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} = k\vec{v}$.

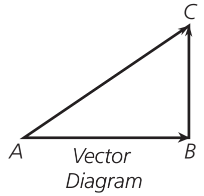
Section 4.2 — Vector Laws

In many applications of vectors to physical problems, we must find the combined effect or *sum* of two or more vectors. What, for example, is the combined effect of two or more forces acting on an object? How does wind velocity affect the velocity of an aircraft?

To determine what the sum of two vectors is, let us look first for a geometrical answer. Suppose the rectangle $ABCD$ is a park at the corner of an intersection. To get from A to C, some people will walk along the sidewalk from A to B and then from B to C. They follow a route described by the sum of two displacement vectors: $\overrightarrow{AB} + \overrightarrow{BC}$. Others may follow a shortcut through the park directly from A to C. This route is described by the displacement vector $\overrightarrow{AC}$.



Whichever route is followed, the displacement is the same; both get from A to C. Therefore $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.



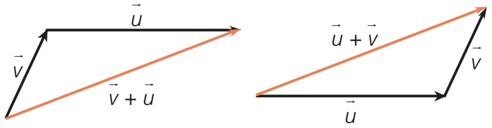
Vector Diagram

This model for vector addition is valid for all vectors, because, in general, vectors can be represented geometrically by a directed line segment.

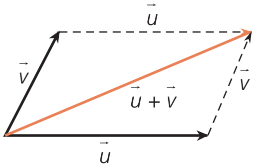
Triangle Law of Vector Addition

To find the sum of two vectors $\vec{u}$ and $\vec{v}$ using the triangle law of vector addition, draw the two vectors head to tail. The sum $\vec{u} + \vec{v}$, or *resultant*, is the vector from the tail of the first to the head of the second.

The order in which we add the vectors is unimportant. If the vectors are added in the opposite order, the result is the same. This demonstrates that vectors satisfy the **commutative law of addition:** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.



By combining the two triangles of the triangle law in one diagram, a parallelogram is formed.



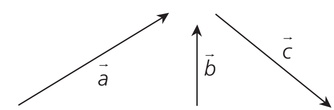
Parallelogram Law of Vector Addition

To find the sum of two vectors using the parallelogram law of vector addition, draw the two vectors tail to tail. Complete the parallelogram with these vectors as sides. The sum $\vec{u} + \vec{v}$ is the diagonal of the parallelogram from the point where the tails are joined to the point where the heads meet.

These two laws of addition are equivalent. The method we use depends on which is the most convenient for the problem at hand. When you set out to solve a problem involving vectors, start by drawing vector diagrams such as those on page 129.

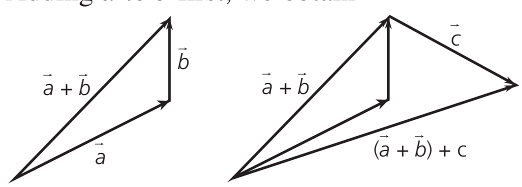
EXAMPLE 1

Given the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, sketch the sums $\vec{a} + \vec{b}$ and $(\vec{a} + \vec{b}) + \vec{c}$, $\vec{b} + \vec{c}$, $\vec{a} + (\vec{b} + \vec{c})$.

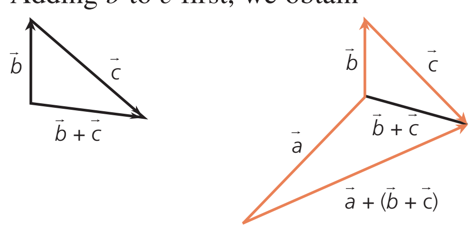


Solution

Adding $\vec{a}$ to $\vec{b}$ first, we obtain



Adding $\vec{b}$ to $\vec{c}$ first, we obtain



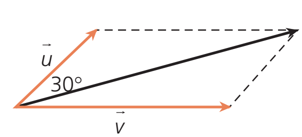
This example illustrates that vectors satisfy the *associative law of addition*: $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$. It means that we can omit the brackets and write simply $\vec{a} + \vec{b} + \vec{c}$.

EXAMPLE 2

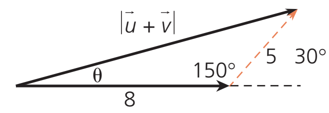
Find the magnitude and direction of the sum of two vectors $\vec{u}$ and $\vec{v}$, if their magnitudes are 5 and 8 units, respectively, and the angle between them is 30º.

Solution

Make a vector diagram showing the two vectors with an angle of 30º between them. Complete the parallelogram and draw the resultant.



The resultant is the third side of a triangle with sides 5 and 8. Observe that the angle between the vectors is *not* an angle in this triangle. The angle between the vectors is equal to an exterior angle of the triangle. (Why?) Use the angle of 150º and the cosine law to find the magnitude of the sum.



$$|\vec{u} + \vec{v}|^2 = 5^2 + 8^2 - 2(5)(8)\cos 150º$$
$$= 158.28$$
$$\text{Then } |\vec{u} + \vec{v}| \cong 12.6$$

The direction of $\vec{u} + \vec{v}$ is expressed as an angle measured relative to one of the given vectors, say $\vec{v}$. This is θ in the diagram. It can be found using the sine law.

$$\frac{\sin\theta}{5} = \frac{\sin 150º}{12.6}$$
$$\sin\theta = \frac{5\sin 150º}{12.6}$$
$$\theta \cong 11.4º$$

Therefore, the magnitude of $\vec{u} + \vec{v}$ is 12.6 units, and it makes an angle of approximately 11.4º with $\vec{v}$.

To subtract two vectors $\vec{a}$ and $\vec{b}$, we express the difference in terms of a sum. To find the vector $\vec{a} - \vec{b}$, use the opposite of $\vec{b}$ and add it to $\vec{a}$. Hence $\vec{a} - \vec{b}$ is equivalent to $\vec{a} + (-\vec{b})$.

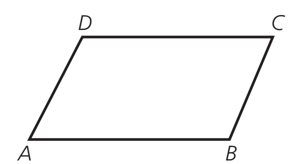
The difference of two equal vectors $\vec{a} - \vec{a}$ is the **zero vector**, denoted by $\vec{0}$. The zero vector has zero magnitude. Its direction is indeterminate.

EXAMPLE 3

In parallelogram *ABCD*, find the difference $\overrightarrow{AB} - \overrightarrow{AD}$

a. geometrically

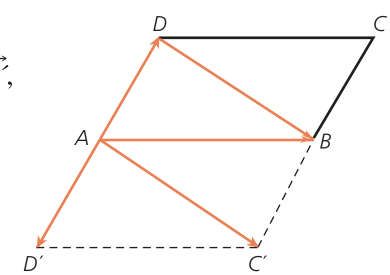
b. algebraically



Solution

a. Draw $\overrightarrow{AD'}$ opposite to $\overrightarrow{AD}$. Using the parallelogram law, draw the sum $\overrightarrow{AB} + \overrightarrow{AD'}$, which is $\overrightarrow{AC'}$ in the diagram.

But $\overrightarrow{AC'} = \overrightarrow{DB}$, so $\overrightarrow{AB} - \overrightarrow{AD} = \overrightarrow{DB}$



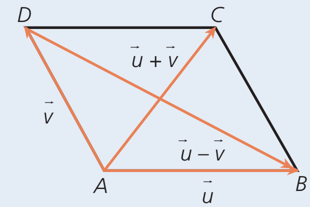
b. $\overrightarrow{AB} - \overrightarrow{AD} = \overrightarrow{AB} + (-\overrightarrow{AD})$

$= \overrightarrow{AB} + \overrightarrow{DA}$ ($\overrightarrow{DA}$ is the opposite of $\overrightarrow{AD}$)

$= \overrightarrow{DA} + \overrightarrow{AB}$ (Rearrange the order of the terms)

$= \overrightarrow{DB}$

In the parallelogram formed by two vectors $\vec{u}$ and $\vec{v}$



- the sum $\vec{u}+\vec{v}$ is the vector created by the diagonal from the tail of the two vectors
  $\vec{u}+\vec{v}=\overrightarrow{AC}$
- the difference $\vec{u}-\vec{v}$ is the vector created by the second diagonal
  $\vec{u}-\vec{v}=\overrightarrow{DB}$

Properties of Vector Addition

- $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ — **Commutative Law**
- $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ — **Associative Law**

Properties of Scalar Multiplication

- $(mn)\vec{a}=m(n\vec{a})$ — **Associative Law**
- $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$ — **Distributive Laws**
- $(m+n)\vec{a}=m\vec{a}+n\vec{a}$

Properties of the Zero Vector: $\vec{0}$

- $\vec{a}+\vec{0}=\vec{a}$

Each vector $\vec{a}$ has a negative $(-\vec{a})$ such that

- $\vec{a}+(-\vec{a})=\vec{0}$

These laws state that you may add vectors in any order you like and that you may expand and factor expressions in the usual way.

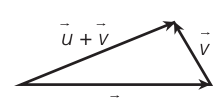
There are other basic vector relations that are universally true. We can demonstrate the validity of these relations by using vector diagrams. The following example illustrates this.

EXAMPLE 4

Show that $|\vec{u}+\vec{v}| \le |\vec{u}| + |\vec{v}|$. When does equality hold?

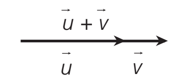
Solution

Make a diagram of two vectors $\vec{u}$ and $\vec{v}$, and their sum $\vec{u}+\vec{v}$. The three vectors form a triangle. The lengths of the sides of the triangle are the magnitudes of the vectors. From the diagram, the side $|\vec{u}+\vec{v}|$ must be less than the sum of the other two sides $|\vec{u}| + |\vec{v}|$. There is no triangle if it is greater.



Therefore $|\vec{u}+\vec{v}| < |\vec{u}| + |\vec{v}|$.

When $\vec{u}$ and $\vec{v}$ have the same direction, the triangle collapses to a single line, and $|\vec{u} + \vec{v}| = |\vec{u}| + |\vec{v}|$.



Triangle Inequality

For vectors $\vec{u}$ and $\vec{v}$, $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$.

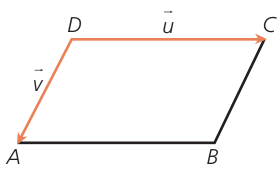
Exercise 4.2

Part A

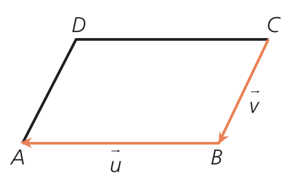
Communication

1. For each of the following, state the name of a vector equal to $\vec{u} + \vec{v}$ and equal to $\vec{u} - \vec{v}$.

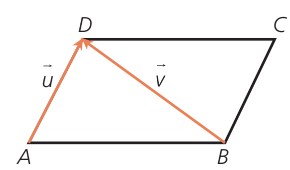
a.



b.



c.



Knowledge/Understanding

2. Seven points A, B, C, D, E, F, and G, are arranged in order from left to right on a single straight line. Express the vector $\overrightarrow{BE}$ as

a. the sum of two vectors, three vectors, and four vectors

b. the difference of two vectors in two different ways

3. What single vector is equivalent to each of these sums?

a. $\overrightarrow{PT} + \overrightarrow{TS} + \overrightarrow{SQ}$

b. $\overrightarrow{AC} - \overrightarrow{GE} + \overrightarrow{CE}$

c. $\overrightarrow{EA} - \overrightarrow{CB} + \overrightarrow{DB} + \overrightarrow{AD}$

d. $\overrightarrow{PT} - \overrightarrow{QT} + \overrightarrow{SR} - \overrightarrow{SQ}$

Part B

Knowledge/Understanding

4. Find the sum of the vectors $\vec{u}$ and $\vec{v}$ if θ is the angle between them.

a. $|\vec{u}| = 12$, $|\vec{v}| = 21$ and $\theta = 70°$

b. $|\vec{u}| = 3$, $|\vec{v}| = 10$, and $\theta = 115°$

Application

5. A tour boat travels 25 km due east and then 15 km $S50°E$. Represent these displacements in a vector diagram, then calculate the resultant displacement.

6. If $\hat{a}$ and $\hat{b}$ are unit vectors that make an angle of 60° with each other, calculate

a. $|3\hat{a} - 5\hat{b}|$

b. $|8\hat{a} + 3\hat{b}|$

7. What conditions must be satisfied by the vectors $\vec{u}$ and $\vec{v}$ for the following to be true?
   a. $|\vec{u} + \vec{v}| = |\vec{u} - \vec{v}|$ b. $|\vec{u} + \vec{v}| > |\vec{u} - \vec{v}|$ c. $|\vec{u} + \vec{v}| < |\vec{u} - \vec{v}|$

8. Under what conditions will three vectors having magnitudes of 7, 24, and 25, respectively, have the zero vector as a resultant?

9. Vectors $\vec{a}$ and $\vec{b}$ have magnitudes 2 and 3, respectively. If the angle between them is 50º, find the vector $5\vec{a} - 2\vec{b}$, and state its magnitude and direction.

10. Simplify the following expressions using the properties of vector operations.
    a. $4\vec{x} - 5\vec{y} - \vec{x} + 6\vec{y}$
    b. $2\vec{x} - 4(\vec{x} - \vec{y})$
    c. $8(3\vec{x} + 5\vec{y}) - 4(6\vec{x} - 9\vec{y})$
    d. $3\vec{x} - 6\vec{y} + 4(2\vec{y} - \vec{x}) - 6\vec{x}$

Application **11.** Let $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$, and $\vec{c} = 2\vec{i} - 3\vec{k}$. Find
    a. $\vec{a} + \vec{b} + \vec{c}$
    b. $\vec{a} + 2\vec{b} - 3\vec{c}$
    c. $-3\vec{b} + 4\vec{c}$

12. If $\vec{a} = 3\vec{x} + 2\vec{y}$ and $\vec{b} = 5\vec{x} - 4\vec{y}$, find $\vec{x}$ and $\vec{y}$ in terms of $\vec{a}$ and $\vec{b}$.

13. Check each identity algebraically, and illustrate with the use of a diagram.
    a. $\vec{x} + \dfrac{\vec{y} - \vec{x}}{2} = \dfrac{\vec{x} + \vec{y}}{2}$
    b. $\vec{x} - \dfrac{\vec{x} + \vec{y}}{2} = \dfrac{\vec{x} - \vec{y}}{2}$

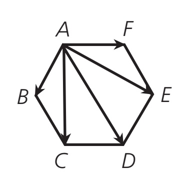
14. Illustrate for $k > 0$ that $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$.

15. Show geometrically that, for any scalar k and any vectors $\vec{u}$ and $\vec{v}$, $k(\vec{u} - \vec{v}) = \overrightarrow{ku} - \overrightarrow{kv}$.

16. By considering the angles between the vectors, show that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular when $|\vec{a}| = |\vec{b}|$.

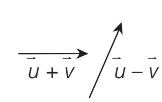
Part C

17. *ABCDEF* is a regular hexagon with sides of unit length. Find the magnitude and the direction of $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}$.



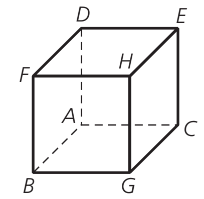
18. If $|\vec{x}| = 11$, $|\vec{y}| = 23$, and $|\vec{x} - \vec{y}| = 30$, find $|\vec{x} + \vec{y}|$.

19. The sum and the difference of two vectors $\vec{u}$ and $\vec{v}$ are given. Show how to find the vectors themselves.



Thinking/Inquiry/ Problem Solving

20. Represent by $\hat{i}$, $\hat{j}$, and $\hat{k}$, the three vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ that lie along adjacent edges of the cube in the given diagram. Express each of the following vectors in terms of $\hat{i}$, $\hat{j}$, and $\hat{k}$.



a. $\overrightarrow{FG}$, a diagonal of the front face of the cube

b. the other diagonals of the front, top and right faces of the cube

c. $\overrightarrow{BE}$, a body diagonal of the cube

d. the other body diagonals of the cube

e. What is the magnitude of a face diagonal? A body diagonal?

21. Prove that for any vectors $\vec{u}$ and $\vec{v}$, $|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2 = 2(|\vec{u}|^2 + |\vec{v}|^2)$.

Section 4.3 — Force as a Vector

A force on any object causes that object to undergo an acceleration. You can feel a force pushing you back into your seat whenever the car you are riding in accelerates from a stop light. You no longer feel any force once the car has reached a steady speed, but that does not mean that the force that set the car in motion has ceased to exist. Instead that force is now balanced by other forces such as air resistance and road friction. A steady speed is an example of a **state of equilibrium** in which the net force is zero.

It was Newton who first clarified these concepts and formulated the law that bears his name.

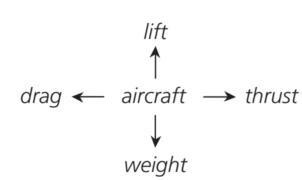
Newton's First Law of Motion
An object will remain in a state of equilibrium (which is a state of rest or a state of uniform motion) unless it is compelled to change that state by the action of an outside force.

The outside force mentioned in Newton's First Law refers to an unbalanced force. When you release a helium-filled balloon, it will rise into the air. It is attracted by the force of gravity like everything else but upward forces are greater, so it accelerates into the sky. Eventually it reaches an altitude where the atmosphere is less dense, and the buoyant forces and the force of gravity balance. In this state of equilibrium, it can float for days, as weather balloons often do.

EXAMPLE 1 Describe the forces acting on an aircraft flying at constant velocity.

Solution

An aircraft flying at a constant velocity is in a state of equilibrium. The engines provide thrust, the force propelling the aircraft forward. The thrust is counter-balanced by a drag force coming from air resistance. The air rushing past the wings produces lift, a force which counterbalances the force of gravity and keeps the plane aloft.



The magnitude of a force is measured in newtons, which is abbreviated as N. At the earth's surface, gravity causes objects to accelerate at a rate of approximately 9.8 m/s^2 as they fall. The magnitude of the gravitational force is the product of an object's mass and this acceleration. The gravitational force on a 1-kg object at the earth's surface is approximately 9.8 N. In other words, a 1-kg object weighs approximately 9.8 N.

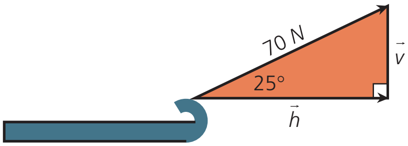
It is generally the case that several forces act on an object at once. It is important to know the net effect of all these forces, because an object's state of motion is determined by this net force. Since forces are vectors, the single force that has the same effect as all the forces acting together can be found by vector addition. This single force is the *resultant* of all the forces.

Sometimes a force acts on an object at an angle, so that only part of the force is affecting the motion of the object.

EXAMPLE 2 Jake is towing his friend on a toboggan, using a rope which makes an angle of 25º with the ground. If Jake is pulling with a force of 70 N, what horizontal force is he exerting on the toboggan?

Solution

First draw a diagram showing the force and its direction. Now consider that this force is the resultant of a horizontal force $\vec{h}$ and a vertical force $\vec{v}$. We show this by forming a triangle, with the original 70 N force as the resultant; $\vec{h}$ and $\vec{v}$ are perpendicular.



$$\text{Now } |\vec{h}| = 70 \cos 25º \\ \cong 63.4$$

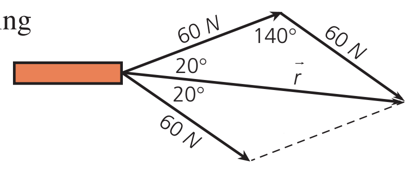
So the horizontal force is about 63.4 N.

We refer to the quantities $|\vec{h}|$ and $|\vec{v}|$ as the horizontal and vertical components of the original force.

EXAMPLE 3 Jake and Maria are towing their friends on a toboggan. Each is exerting a horizontal force of 60 N. Since they are walking side by side, the ropes pull one to each side; they each make an angle of 20º with the line of motion. Find the force pulling the toboggan forward.

Solution

Make a diagram showing the forces. By completing the parallelogram, we show the resultant $\vec{r}$, the diagonal of the parallelogram.



$$|\vec{r}|^2 = 60^2 + 60^2 - 2(60)(60)\cos 140^\circ$$
$$|\vec{r}| \cong 112.8$$

The towing force is about 113 N.

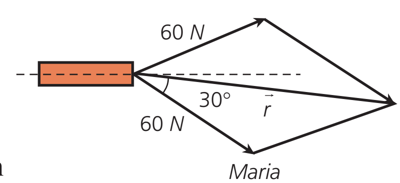
1. We could have solved this question by finding the component of each force along the direction of travel and adding the results.
2. If the forces had not been equal, the angles made with the direction of travel would not have been equal.

In Example 3, the toboggan is (probably) travelling at a constant speed, indicating that there is no unbalanced force on it. This is because there is a frictional force that is equal and opposite to the towing force.

The force that is equal in magnitude but opposite in direction to the resultant is called the **equilibrant**. It exactly counterbalances the resultant. In Example 2, the force of friction is the equilibrant, which keeps the towing force from accelerating the toboggan.

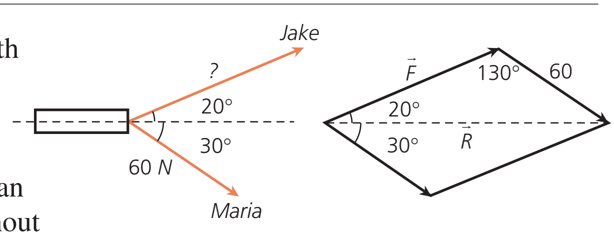
EXAMPLE 4

In Example 2, what if Maria starts pulling at an angle of 30º instead of 20º? As the diagram shows, the direction of the resultant will be a little to the right of the axis of the toboggan. This means that the toboggan will not travel forward in a straight line but will veer continually to the right. If these conditions remain unchanged, the toboggan will travel in a circle.



EXAMPLE 5

In Example 2, if Maria pulls with a force of 60 N at an angle of 30º, what should the magnitude of the force exerted by Jake at an angle of 20º be if the toboggan is to move straight forward without turning? According to the sine law,



$$\frac{\sin 30^\circ}{|\vec{F}|} = \frac{\sin 20^\circ}{60}$$
$$|\vec{F}| \cong 88 \text{ N}$$

Jake must pull with a force of 88 N. Since Jake is pulling harder than before, the resultant will be greater than before:

$$\frac{\sin 130^\circ}{R} = \frac{\sin 20^\circ}{60}$$
$$R \cong 134 \text{ N}$$

As in Example 2 and the subsequent discussion, make it a practice with force problems to look for ways to justify your numerical results and make them physically meaningful.

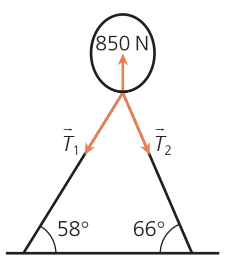
EXAMPLE 6

A large promotional balloon is tethered to the top of a building by two guy wires attached at points 20 m apart. If the buoyant force on the balloon is 850 N, and the two guy wires make angles of 58º and 66º with the horizontal, find the tension in each of the wires.

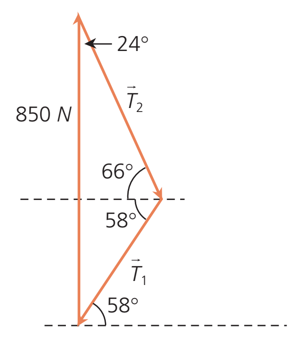
Solution

First draw the position diagram showing where the forces act. In this problem, the resultant of the two tensions must be 850 N to counterbalance the buoyant force of the balloon, which is the equilibrant. In making the force diagram, draw the tension vectors parallel to the corresponding lines in the position diagram.

In the diagrams, observe step by step how the angles in the position diagram are first translated into the force diagram, and then how these angles are used to determine the angles inside the force triangle.



Position Diagram



Force Diagram

Since all three angles in the force triangle are known, the magnitudes of the tension vectors $\vec{T}_1$ and $\vec{T}_2$ can be calculated using the sine law,

$$\frac{|\vec{T}_1|}{\sin 24^\circ} = \frac{850}{\sin 124^\circ} \quad \text{and} \quad \frac{|\vec{T}_2|}{\sin 32^\circ} = \frac{850}{\sin 124^\circ}.$$

Therefore $$|\vec{T}_1| = \frac{850 \sin 24^\circ}{\sin 124^\circ} \quad \text{and} \quad |\vec{T}_2| = \frac{850 \sin 32^\circ}{\sin 124^\circ}$$
$$\cong 417 \text{ N} \qquad\qquad \cong 543 \text{ N}$$

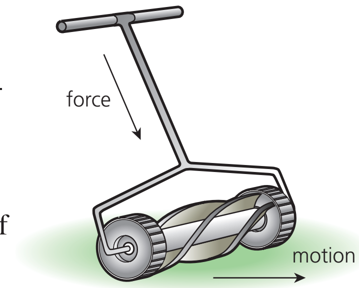
The tensions in the guy wires are approximately 417 N and 543 N, with the guy wire at the steeper angle having the greater tension.

EXAMPLE 7 Is it possible for an object to be in a state of equilibrium when forces of 10 N, 20 N, and 40 N act on it?

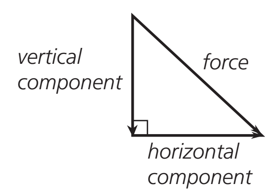
Solution

An object will be in a state of equilibrium when the resultant of all the forces acting on it is zero. This means that the three given force vectors must form a triangle. By the triangle inequality theorem, the sum of any two sides must be greater than the third, but in this case the magnitudes of the forces are such that $10 + 20 < 40$. Therefore, an object cannot be in a state of equilibrium with the three given forces acting on it.

In the discussion of forces in the previous examples, we assumed that an object is free to move in the direction of the force acting on it. Often, however, that is not the case. For example, when you push a lawn mower, you exert a force along the handle, but the mower does not move into the ground along the line of the force. It moves horizontally. So, how much of the force that you exert actually contributes to the motion?



To answer this question, we must resolve the force into horizontal and vertical components. The components are the magnitudes of forces acting horizontally and vertically, whose sum, by vector addition, is the original force.



EXAMPLE 8 A lawn mower is pushed with a force of 90 N directed along the handle, which makes an angle of 36° with the ground.

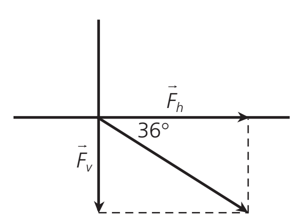
a. Determine the horizontal and vertical components of the force on the mower.

b. Describe the physical consequences of each component of the pushing force.

Solution

a. The force diagram is a right triangle.
The components are

$$\left|\vec{F}_h\right| = 90\cos(36°) \cong 72.8 \text{ N} \quad \text{and} \quad \left|\vec{F}_v\right| = 90\sin(36)° \cong 52.9 \text{ N}$$



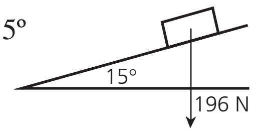
b. The horizontal component of the force, 72.8 N, moves the lawnmower forward across the grass. The vertical component of the force, 52.9 N, is in the same direction (down) as the force of gravity.

EXAMPLE 9 A 20-kg trunk is resting on a ramp inclined at an angle of 15º. Calculate the components of the force of gravity on the trunk that are parallel and perpendicular to the ramp. Describe the physical consequences of each.

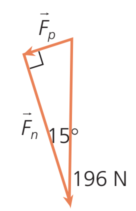
Solution

The force of gravity on the trunk is (20 kg) $\times$ (9.8 m/s^2) = 196 N acting down. The parallel and perpendicular components are

$$|\vec{F}_p| = 196 \sin 15^\circ \cong 51 \text{ N} \quad \text{and} \quad |\vec{F}_n| = 196 \cos 15^\circ \cong 189 \text{ N}$$



position diagram



force diagram

The parallel component points down the slope of the ramp. It tends to cause the trunk to slide down the slope. It is opposed by the force of friction acting up the slope. The perpendicular component presses the trunk against the ramp. The magnitude of the force of friction is proportional to this component.

Exercise 4.3

Part A

Communication

1. Name some common household objects on which the force of gravity is approximately 2 N; 20 N; 200 N. What is your weight in newtons?

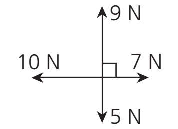
Knowledge/Understanding

2. Find the horizontal and vertical components of each of the following forces.
   a. 200 N acting at an angle of 30º to the horizontal
   b. 160 N acting at an angle of 71º to the horizontal
   c. 75 N acting at an angle of 51º to the vertical
   d. 36 N acting vertically

3. Find the resultant of each pair of forces acting on an object.
   a. forces of 7 N east and 12 N west
   b. forces of 7 N east and 12 N north
   c. forces of 6 N southwest and 8 N northwest
   d. forces of 6 N southeast and 8 N northwest

Part B

4. Find the magnitude of the resultant of the four forces shown in the given diagram.



5. Two forces $\vec{F}_1$ and $\vec{F}_2$ act at right angles to each other. Express the magnitude and direction of $\vec{F}_1 + \vec{F}_2$ in terms of $|\vec{F}_1|$ and $|\vec{F}_2|$.

6. Find the magnitude and the direction (to the nearest degree) of the resultant of each of the following systems of forces.
   a. forces of 3 N and 8 N acting at an angle of 60º to each other
   b. forces of 15 N and 8 N acting at an angle of 130º to each other

7. Find the magnitude and direction of the equilibrant of each of the following systems of forces.
   a. forces of 32 N and 48 N acting at an angle of 90º to each other
   b. forces of 16 N and 10 N acting at an angle of 10º to each other

Communication

8. Is it easier to pull yourself up doing chin-ups when your hands are 60 cm apart or 120 cm apart? Explain your answer.

Knowledge/Understanding

9. A mass of 10 kg is suspended from a ceiling by two cords that make angles of 30º and 45º with the ceiling. Find the tension in each of the cords.

10. Two forces of equal magnitude act at 60º to each other. If their resultant has a magnitude of 30 N, find the magnitude of the equal forces.

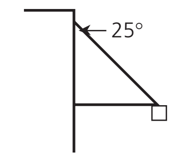
11. Which of the following sets of forces acting on an object could produce equilibrium?
   a. 5 N, 2 N, 13 N
   b. 7 N, 5 N, 5 N
   c. 13 N, 27 N, 14 N
   d. 12 N, 26 N, 13 N

12. Three forces of 5 N, 7 N, and 8 N are applied to an object. If the object is in a state of equilibrium
   a. show how the forces must be arranged
   b. calculate the angle between the lines of action of the 5 N and 7 N forces

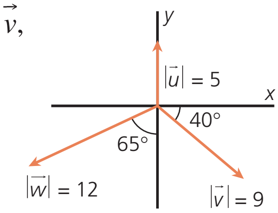
13. A man weighing 70 kg lies in a hammock whose ropes make angles of 20º and 25º with the horizontal. What is the tension in each rope?

14. A steel wire 40 m long is suspended between two fixed points 20 m apart. A force of 375 N pulls the wire down at a point 15 m from one end of the wire. State the tension in each part of the wire.

15. An advertising sign is supported by a horizontal steel brace extending at right angles from the side of a building, and by a wire attached to the building above the brace at an angle of 25º. If the force of gravity on the sign is 850 N, find the tension in the wire and the compression in the steel brace.



16. Find the x-and y-components of each of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.



17. A tractor is towing a log using a cable inclined at an angle of 15º to the horizontal. If the tension in the cable is 1470 N, what is the horizontal force moving the log?

18. A piece of luggage is on a conveyer belt that is inclined at an angle of 28º. If the luggage has a mass of 20 kg
   a. determine the components of the force of gravity parallel to and perpendicular to the conveyer belt
   b. explain the physical effect of each of these components

19. A child with a mass of 35 kg is sitting on a swing attached to a tree branch by a rope 5 m in length. The child is pulled back 1.5 m measured horizontally.
   a. What horizontal force will hold the child in this position?
   b. What is the tension in the rope?

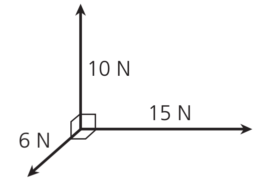
Application **20.** The main rotor of a helicopter produces a force of 55 kN. If the helicopter flies with the rotor revolving about an axis tilted at an angle of 8º to the vertical
   a. find the components of the rotor force parallel to and perpendicular to the ground
   b. explain the physical effect on the helicopter of each component of the rotor force

21. In order to keep a 250-kg crate from sliding down a ramp inclined at 25º, the force of friction that acts parallel to and up the ramp must have a magnitude of at least how many newtons?

22. A lawn roller with a mass of 50 kg is being pulled with a force of 320 N. If the handle of the roller makes an angle of 42º with the ground, what horizontal component of the force is causing the roller to move?

Part C

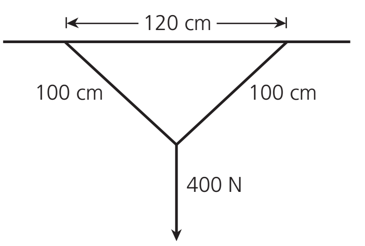
23. Three forces, each of which is perpendicular to the other two, act on an object. If the magnitudes of these forces are 6 N, 15 N, and 10 N, respectively, find the magnitude and direction of the resultant. (State the angles that the resultant makes with the two larger forces.)



24. Two tugs are towing a ship. The smaller tug is 10º off the port bow and the larger tug is 20º off the starboard bow. The larger tug pulls twice as hard as the smaller tug. In what direction will the ship move?

25. Braided cotton string will break when the tension exceeds 300 N. Suppose that a weight of 400 N is suspended from a 200-cm length of string, the upper ends of which are tied to a horizontal rod at points 120 cm apart.

    a. Show that the string will support the weight, when the weight is hung at the centre of the string.



    b. Will the string break if the weight is 80 cm from one end of the string?

Section 4.4 — Velocity as a Vector

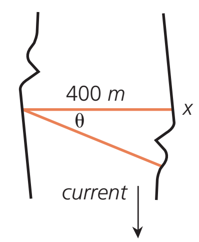
In elementary problems, the **speed** of a moving object is calculated by dividing the distance travelled by the travel time. In advanced work, speed is defined more carefully as the rate of change of distance with time. In any case, speed is a quantity having magnitude only, so it is classified as a scalar.

When the direction of motion as well as its magnitude is important, the correct term to use is **velocity**. Velocity is a vector quantity. Speed is the magnitude of a velocity.

Velocity vectors can be added. When you walk forward in the aisle of an aircraft in flight, the 2-km/hr velocity of your walk adds to the 500-km/hr velocity of the plane, making your total velocity 502 km/hr. When two velocities are not in the same direction, the resultant velocity determined from the addition of two velocity vectors is nevertheless a meaningful, physical quantity.

EXAMPLE 1

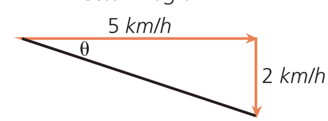
A canoeist who can paddle at a speed of 5 km/h in still water wishes to cross a river 400 m wide that has a current of 2 km/h. If he steers the canoe in a direction perpendicular to the current, determine the resultant velocity. Find the point on the opposite bank where the canoe touches.



Solution

As the canoe moves through the water, it is carried sideways by the current. So even though its heading is straight across the current, its actual direction of motion is along a line angling downstream determined by the sum of the velocity vectors.

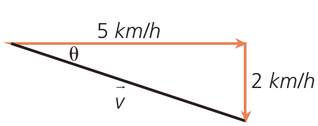
Vector Diagram

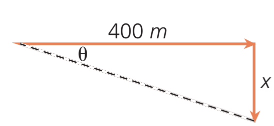


From the vector diagram,

$$|\vec{v}|^2 = (5)^2 + (2)^2 \qquad \text{and} \qquad \tan\theta = \frac{2}{5}$$
$$|\vec{v}| = \sqrt{29} \cong 5.4 \text{ km/h} \qquad\qquad \theta \cong 21.8^\circ$$

Therefore, the canoeist crosses the river at a speed of 5.4 km/h along a line at an angle of about 22º. The displacement triangle is similar to the vector triangle.



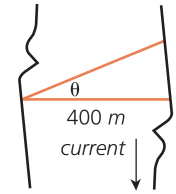


$$\frac{x}{2} = \frac{400}{5}$$
$$x = 160$$

He touches the opposite bank at a point 160 m downstream from the point directly opposite his starting point. We could also find x using the angle θ, but we must be careful *not* to round off in the process.

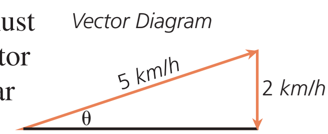
EXAMPLE 2

Suppose the canoeist of Example 1 had wished to travel straight across the river. Determine the direction he must head and the time it will take him to cross the river.



Solution

In order to travel directly across the river, the canoeist must steer the canoe slightly upstream. This time, it is the vector sum, not the heading of the canoe, which is perpendicular to the river bank. From the vector diagram,



$$|\vec{v}|^2 = (5)^2 - (2)^2 \qquad \text{and} \qquad \sin(\theta) = \frac{2}{5}$$

$$|\vec{v}| = \sqrt{21} \cong 4.6 \text{ km/h} \qquad \theta \cong 23.6°$$

Therefore, to travel straight across the river, the canoeist must head upstream at an angle of about 24°. His crossing speed will be about 4.6 km/h.

The time it takes to cross the river is calculated from

$$t = \frac{\text{river width}}{\text{crossing speed}} \qquad \text{(where the width is 0.4 km)}$$

$$= \frac{0.4}{\sqrt{21}} \qquad \text{(we avoid using rounded values if possible)}$$

$$\cong 0.087 \text{ h or } 5.2 \text{ min}$$

It takes the canoeist approximately 5.2 minutes to cross the river.

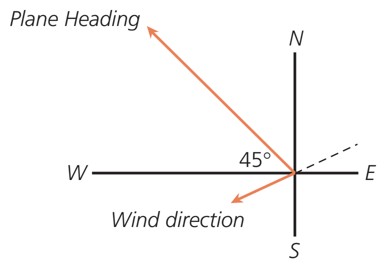
Wind affects a plane's speed and direction much the same way that current affects a boat's. The airspeed of a plane is the plane's speed relative to the mass of air it is flying in. This may be different in both magnitude and direction from the plane's ground speed, depending on the strength and direction of the wind.

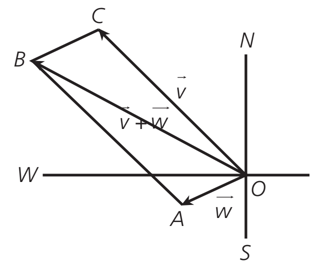
EXAMPLE 3

An airplane heading northwest at 500 km/h encounters a wind of 120 km/h from 25° north of east. Determine the resultant ground velocity of the plane.

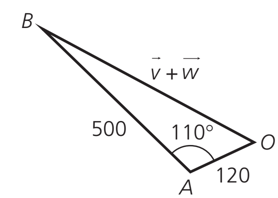
Solution

Since the wind is blowing from 25° north of east, it can be represented by a vector whose direction is west 25° south. This wind will blow the plane off its course, changing both its ground speed and its heading. Let $|\vec{v}|$ be the airspeed of the plane and $|\vec{w}|$ be the wind speed. On a set of directional axes, draw the two velocity vectors. Then draw the resultant velocity using the parallelogram law of vector addition.





In parallelogram $OCBA$, $\angle COA = 45º + 25º = 70º$, so $\angle OAB = 110º$. Then, in ΔOAB, two sides and the included angle are known, so the magnitude of the resultant velocity can be calculated using the cosine law.



$$|\vec{v}+\vec{w}|^2 = 500^2 + 120^2 - 2(500)(120)\cos 110º$$
$$|\vec{v}+\vec{w}| \cong 552.7$$

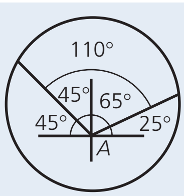
Store this answer in your calculator memory.
Next, $\angle AOB$ can be calculated from the sine law.

$$\frac{\sin \angle AOB}{500} = \frac{\sin 110º}{|\vec{v}+\vec{w}|}$$ (use the value of $|\vec{v}+\vec{w}|$ calculated above)

$$\angle AOB \cong 58.2º$$

$$\angle WOB = 58.2º - 25º = 33.2º$$

The resultant velocity has direction 33º north of west and a magnitude of 553 km/h.



A key step in solving problems such as that in Example 3 is to find an angle in the triangle formed by the vectors. Here is a helpful hint: identify which angle is formed by vectors whose directions are given, and draw small axes at the vertex of that angle. The diagram shows this alternate way to calculate that $\angle OAB = 110º$ in Example 3.

Vectors are needed to describe situations where two objects are moving relative to one another. When astronauts want to dock the space shuttle with the international space station, they must match the velocities of the two craft. As they approach, astronauts on each spacecraft can picture themselves to be stationary and the other craft to be moving. When they finally dock, even though the two spacecraft are orbiting the earth at thousands of miles per hour, their **relative velocity** is zero.

Relative velocity is the difference of two velocities. It is what an observer measures, when he perceives himself to be stationary. The principle that *all* velocities are relative was originally formulated by Einstein and became a cornerstone of his Theory of Relativity.

> **When two objects A and B have velocities $\vec{v}_A$ and $\vec{v}_B$, respectively, the velocity of B relative to A is**
> **$\vec{v}_{rel} = \vec{v}_B - \vec{v}_A$**

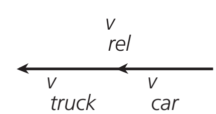
EXAMPLE 4

A car travelling east at 110 km/h passes a truck going in the opposite direction at 96 km/h.

a. What is the velocity of the truck relative to the car?

b. The truck turns onto a side road and heads northwest at the same speed. Now what is the velocity of the truck relative to the car?

Solution

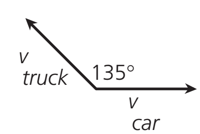


The vector diagram shows the velocity vectors of the car and the truck. These velocities are relative to someone standing by the side of the road, watching the two vehicles pass by. Since the car is going east, let its velocity be $\vec{v}_{car} = 110$. Then the truck's velocity is $\vec{v}_{truck} = -96$.

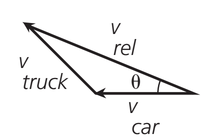
$$\begin{aligned}\vec{v}_{rel} &= \vec{v}_{truck} - \vec{v}_{car}\\ &= (-96) - (110)\\ &= -206 \text{ km/h or } 206 \text{ km/h west}\end{aligned}$$

This is the velocity that the truck appears to have, according to the driver of the car.

b. After the truck turns, the angle between the car and the truck velocities is 135º. The magnitude of the sum is found using the cosine law.



$$\begin{aligned}|\vec{v}_{rel}|^2 &= (96)^2 + (110)^2 - 2(96)(110)\cos 135^\circ\\ |\vec{v}_{rel}| &\cong 190.4 \text{ km/h}\end{aligned}$$



(Store this in your calculator.)

The angle of the relative velocity vector can be calculated from the sine law.

$$\frac{\sin \theta}{96} = \frac{\sin 135^\circ}{190.4}$$
$$\theta \cong 20.9^\circ$$

After the truck turns, its velocity is 190 km/h in a direction *W* 21º *N* relative to the car. Note that the relative velocity of the two vehicles does not depend on their position. It remains the same as long as the two vehicles continue to travel in the same directions without any changes in their velocities.

Exercise 4.4

Part A

Communication

1. A plane is heading due east. Will its ground speed be greater than or less than its airspeed, and will its flight path be north or south of east when the wind is from

a. *N* b. *S* 80º *W* c. *S* 30º *E* d. *N* 80º *E*

Knowledge/Understanding

2. A man can swim 2 km/h in still water. Find at what angle to the bank he must head if he wishes to swim directly across a river flowing at a speed of

a. 1 km/h b. 4 km/h

Knowledge/Understanding

3. A streetcar, a bus, and a taxi are travelling along a city street at speeds of 35, 42, and 50 km/h, respectively. The streetcar and the taxi are travelling north; the bus is travelling south. Find

a. the velocity of the streetcar relative to the taxi

b. the velocity of the streetcar relative to the bus

c. the velocity of the taxi relative to the bus

d. the velocity of the bus relative to the streetcar

Part B

4. A river is 2 km wide and flows at 6 km/h. A motor boat that has a speed of 20 km/h in still water heads out from one bank perpendicular to the current. A marina lies directly across the river on the opposite bank.

a. How far downstream from the marina will the boat reach the other bank?

b. How long will it take?

5. An airplane is headed north with a constant velocity of 450 km/h. The plane encounters a west wind blowing at 100 km/h.

a. How far will the plane travel in 3 h?

b. What is the direction of the plane?

Application

6. A light plane is travelling at 175 km/h on a heading of *N*8º *E* in a 40-km/h wind from *N*80º *E*. Determine the plane's ground velocity.

Application

7. A boat heads 15º west of north with a water speed of 3 m/s. Determine its velocity relative to the ground when there is a 2 m/s current from 40º east of north.

8. A plane is steering east at a speed of 240 km/h. What is the ground speed of the plane if the wind is from the northwest at 65 km/h? What is the plane's actual direction?

9. Upon reaching a speed of 215 km/h on the runway, a jet raises its nose to an angle of 18º with the horizontal and begins to lift off the ground.
   a. Calculate the horizontal and vertical components of its velocity at this moment.
   b. What is the physical interpretation of each of these components of the jet's velocity?

10. A pilot wishes to fly to an airfield $S20^{\circ}E$ of his present position. If the average airspeed of the plane is 520 km/h and the wind is from $N80^{\circ}E$ at 46 km/h,
    a. in what direction should the pilot steer?
    b. what will the plane's ground speed be?

11. A destroyer detects a submarine 8 nautical miles due east travelling northeast at 20 knots. If the destroyer has a top speed of 30 knots, at what heading should it travel to intercept the submarine?

Part C

12. An airplane flies from to Toronto to Vancouver and back. Determine which time is shorter.
    a. The time for the round trip when there is a constant wind blowing from Vancouver to Toronto.
    b. The time for the round trip when there is no wind.

13. A sailor climbs a mast at 0.5 m/s on a ship travelling north at 12 m/s, while the current flows east at 3 m/s. What is the speed of the sailor relative to the ocean floor?

14. A car is 260 m north and a truck 170 m west of an intersection. They are both approaching the intersection, the car from the north at 80 km/h, and the truck from the west at 50 km/h. Determine the velocity of the truck relative to the car.

Key Concepts Review

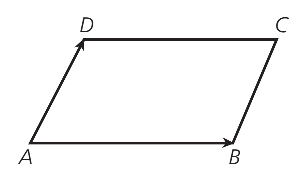
In this chapter, you have been introduced to the concept of a vector and have seen some applications of vectors. Perhaps the most important mathematical skill to develop from this chapter is that of combining vectors through vector addition, both graphically and algebraically.

Diagrams drawn free hand are sufficient, but try to make them realistic. It is not difficult to draw angles that are correct to within about 10º and to make lengths roughly proportional to the magnitudes of the vectors in a problem.

Once you have calculated answers, ask yourself if the calculated angles and magnitudes are consistent with your diagram, and if they are physically reasonable.

SUMS

Speaking informally, if you want to go from A to C you can travel directly along the vector $\overrightarrow{AC}$, or you can detour through B, travelling first along $\overrightarrow{AB}$, and then along $\overrightarrow{BC}$. This means that $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$, but observe how the detour point fits into the equation: it is the second letter of the first vector and the first letter of the second vector.



DIFFERENCES

Using the same diagram, if you want to go from D to B, you can travel directly along $\overrightarrow{DB}$, or you can detour through A, travelling first backwards along $\overrightarrow{AD}$, and then forwards along $\overrightarrow{AB}$. This translates into the equation $\overrightarrow{DB} = -\overrightarrow{AD} + \overrightarrow{AB}$, which of course is just the difference $\overrightarrow{DB} = \overrightarrow{AB} - \overrightarrow{AD}$. Note carefully that, on the right hand side of the equation, the order of the initial point D and the end point B are reversed, and the detour point is the initial letter of the two vectors.

Pay attention to and become familiar with details such as these. You will be able to draw and interpret vector diagrams and handle vector equations more quickly and correctly if you do.

Rich Learning Link wrap-up investigate and apply

CHAPTER 4: VECTORS AND THE SUPERIOR COLLICULUS

Brain cells in the superior colliculus are tuned to the directions of distant visual and auditory stimuli. Each cell responds only to stimuli located within a cone of directions. The vigour of a cell's response can be regarded as specifying the magnitude of a vector in the direction the cell represents. The resultant vector formed by summing the vectors represented by the individual cells points in the direction of the stimulus.

Dr. Randy Gallistel, a professor in the Department of Psychology at UCLA, whose research focus is in the cognitive neurosciences, has suggested that these neurological resultant vectors are "the first new idea about how the nervous system represents the value of a variable since the beginning of the [twentieth] century (from *Conservations in the Cognitive Neurosciences*, Ed. Michalels Gazzaniga, MA: Bradford Books/MIT Press, 1997)."

Investigate and Apply

1. What direction would be represented by a north cell responding three times as vigorously as a north-east cell, which, in turn, is responding twice as vigorously as an east cell?
2. Consider an ensemble of 36 cells, representing directions evenly distributed around a circle, with one cell representing north. One cell will represent 10° east of north, the next will represent 20° east of north, and so on. A cell always responds to some extent whenever a stimulus is within 20° of the cell's direction.
   a) Which cells will respond to a stimulus whose direction is north-east?
   b) A *response pattern* is a description of the relative proportions of the vigour of the various cells' responses. Give two possible response patterns for the cells found in part **a**.
3. How do you think the brain deals with the fact that several different response patterns can represent the same direction?

INDEPENDENT STUDY

Investigate the field of neuroscience.

What other things can be represented in the brain using resultant vectors formed from cells representing individual vectors?

What are some other questions to which neuroscientists are seeking answers?

What role does mathematics play in the search for answers to these questions?

Review Exercise

Communication

1. a. If $\vec{v} + \vec{t} = \vec{v}$, what is $\vec{t}$?

b. If $t\vec{v} = \vec{v}$, what is t?

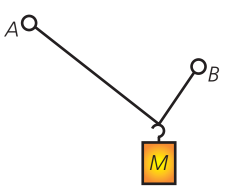
c. If $s\vec{v} = t\vec{u}$, and $\vec{u}$ is not parallel to $\vec{v}$, what are s and t?

2. Using vector diagrams, show that

a. $(a + b)\vec{u} = a\vec{u} + b\vec{u}$ b. $(ab)\vec{u} = a(b\vec{u})$

Communication

3. A mass M is hung on a line between two supports A and B.



a. Which part of the line supporting the mass has the greater tension? Explain.

b. The supports A and B are not at the same level. What effect does this have on the tension in the line? Explain.

4. Explain these properties of the zero vector:

a. $0\vec{v} = \vec{0}$ b. $\vec{v} + \vec{0} = \vec{v}$ c. if $\vec{u} + \vec{v} = \vec{0}$, then $\vec{u} = -\vec{v}$

Knowledge/ Understanding

5. If $\hat{i}$ and $\hat{j}$ are perpendicular unit vectors, what is the magnitude of

a. $3\hat{i} + 4\hat{j}$? *b.* $24\hat{i} - 7\hat{j}$? c. $a\hat{i} + b\hat{j}$?

6. Show that $|\vec{a}| + |\vec{b}| = |\vec{a} - \vec{b}|$, if $\vec{a}$ and $\vec{b}$ have opposite directions.

7. A 3-kg mass is hanging from the end of a string. If a horizontal force of 12 N pulls the mass to the side

a. find the tension in the string

b. find the angle the string makes with the vertical

Knowledge/ Understanding

8. Two forces $\vec{F}_1$ and $\vec{F}_2$ act on an object. Determine the magnitude of the resultant if

a. $|\vec{F}_1| = 54\ N$, $|\vec{F}_2| = 34\ N$, and the angle between them is 55º

b. $|\vec{F}_1| = 21\ N$, $|\vec{F}_2| = 45\ N$, and the angle between them is 140º

9. Two forces at an angle of 130º to each other act on an object. Determine their magnitudes if the resultant has a magnitude of 480 N and makes an angle of 55º with one of the forces.

10. Forces of 5 N, 2 N, and 12 N, all lying in the same plane, act on an object. The 5 N and 2 N forces lie on opposite sides of the 12 N force at angles of 40º and 20º, respectively. Find the magnitude and direction of the resultant.

Application

11. A 10-kg mass is supported by two strings of length 5 m and 7 m attached to two points in the ceiling 10 m apart. Find the tension in each string.

12. The pilot of an airplane that flies at 800 km/h wishes to travel to a city 800 km due east. There is a 80 km/h wind from the northeast.

a. What should the plane's heading be?

b. How long will the trip take?

Thinking/Inquiry/Problem Solving

13. An airplane heads due south with an air speed of 480 km/h. Measurements made from the ground indicate that the plane's ground speed is 528 km/h at 15º east of south. Calculate the wind speed.

14. A camp counsellor leaves a dock paddling a canoe at 3 m/s. She heads downstream at 30º to the current, which is flowing at 4 m/s.

a. How far downstream does she travel in 10 s?

b. What is the length of time required to cross the river if its width is 150 m?

15. A pilot wishes to reach an airport 350 km from his present position at a heading of *N* 60º *E*. If the wind is from *S* 25º *E* with a speed of 73 km/h, and the plane has an airspeed of 450 km/h, find

a. what heading the pilot should steer

b. what the ground speed of the plane will be

c. how many minutes it will take for the plane to reach its destination

Thinking/Inquiry/Problem Solving

16. A coast guard cutter is steering west at 12 knots, when its radar detects a tanker ahead at a distance of 9 nautical miles travelling with a relative velocity of 19 knots, on a heading of *E* 14º *N*. What is the actual velocity of the tanker?

Application

17. Twice a week, a cruise ship carries vacationers from Miami, Florida, to Freeport in the Bahamas, and then on to Nassau before returning to Miami. The distance from Miami to Freeport is 173 km on a heading of *E* 20º *N*. The distance from Freeport to Nassau is 217 km on a heading of *E* 50º *S*. Once a week the ship travels directly from Miami to Nassau. Determine the displacement vector from Miami to Nassau.

18. If $a\vec{u} + b\vec{v} = \vec{0}$ and $\vec{u}$ and $\vec{v}$ have different directions, what must a and b equal?

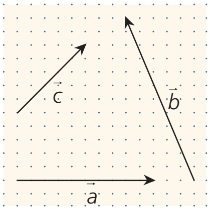
19. Show geometrically that $\left| |\vec{u}| - |\vec{v}| \right| \leq |\vec{u} + \vec{v}|$. Under what conditions does equality hold?

Chapter 4 Test

Achievement Category	Questions
Knowledge/Understanding	2, 4, 5
Thinking/Inquiry/Problem Solving	8
Communication	1
Application	3, 6, 7

1. Under what conditions is $|\vec{u} + \vec{v}| = |\vec{u}| + |\vec{v}|$?

2. Copy the three given vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ onto graph paper, then accurately draw the following three vectors.



   a. $\vec{u} = \vec{a} + 3\vec{c}$
   b. $\vec{v} = \vec{b} - \vec{a}$
   c. $\vec{w} = \frac{2}{3}\vec{b} - 5\vec{c} + \vec{a}$

3. Simplify $3(4\vec{u} + \vec{v}) - 2\vec{u} - 3(\vec{u} - \vec{v})$.

4. Illustrate in a diagram the vector property $4(\vec{a} + \vec{b}) = 4\vec{a} + 4\vec{b}$. What is this property called?

5. Forces of 15 N and 11 N act a point at 125° to each other. Find the magnitude of the resultant.

6. A steel cable 14 m long is suspended between two fixed points 10 m apart horizontally. The cable supports a mass of 50 kg at a point 6 m from one end. Determine the tension in each part of the cable.

7. A ferry boat crosses a river and arrives at a point on the opposite bank directly across from its starting point. The boat can travel at 4 m/s and the current is 1.5 m/s. If the river is 650 m wide at the crossing point, in what direction must the boat steer and how long will it take to cross?

8. What is the relative velocity of an airplane travelling at a speed of 735 knots on a heading of $E\ 70°\ S$ with respect to an aircraft at the same height steering $W\ 50°\ S$ at a speed of 300 knots?

Extending and Investigating

NON-EUCLIDEAN GEOMETRY

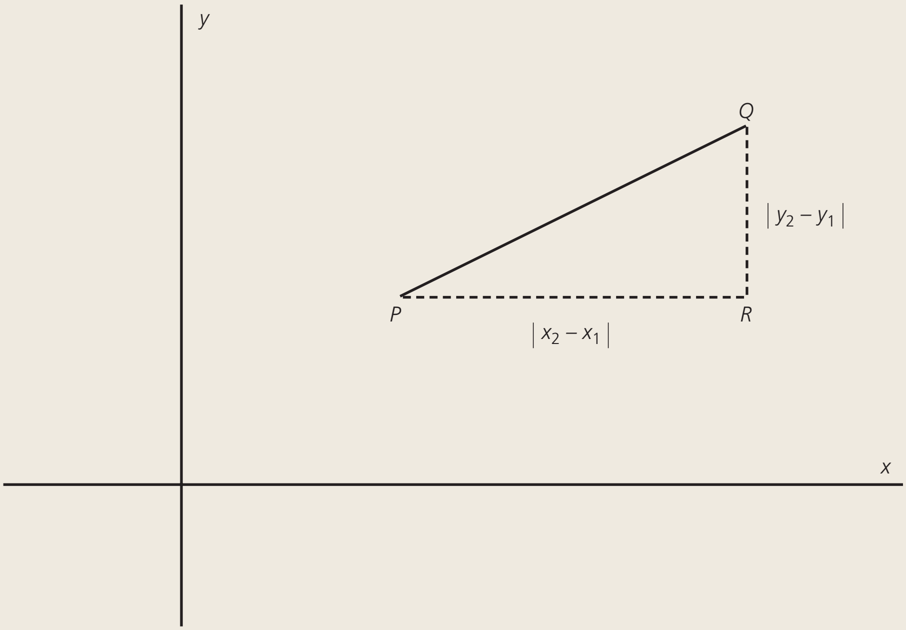
The word *geometry* comes from the Greek words for *earth* and *measure*. When we solve geometrical problems, the rules or assumptions we make are chosen to match our experience with the world we live in. For example, since locally the earth looks flat, it makes sense to talk about planar figures such as triangles, circles, and so on. But what happens if we change the rules? For example, we normally define distance in Euclidean terms. When we represent points and figures in terms of coordinates on the Cartesian plane, then the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

If we ask for the locus of all points that are a constant distance, say 1, from the given point (0, 0), we get the circle with equation $x^2 + y^2 = 1$.

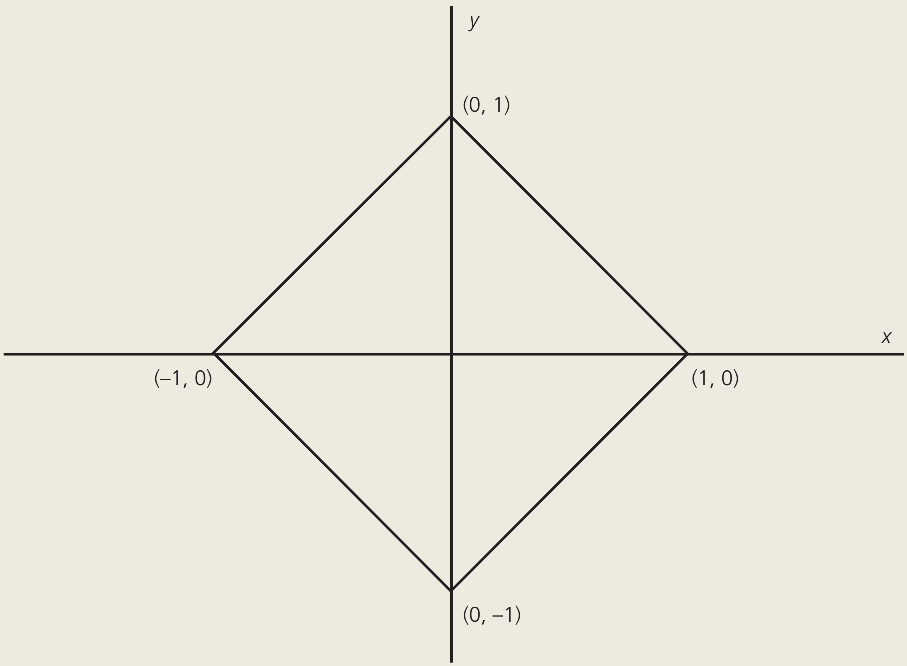
One way to create a whole new geometry is to change the way we measure distance. For example, we can use the so-called taxi-cab distance given by

$$t(P, Q) = |x_1 - x_2| + |y_1 - y_2|$$

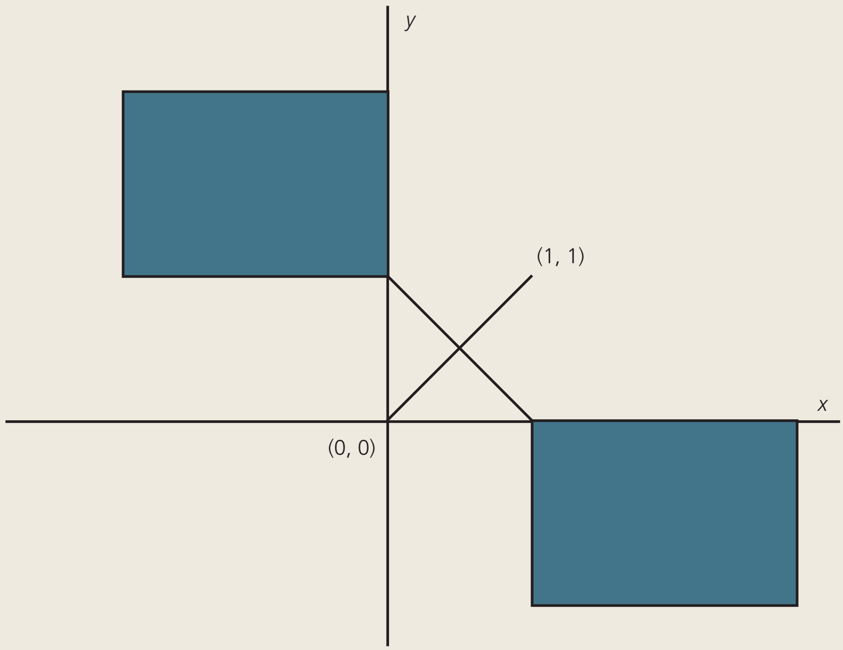


The taxi-cab distance between P and Q is the sum of the lengths PR and RQ. The reason for the colourful name is that it is the actual distance driven if a cab is restricted to a rectangular grid of streets. Note that $t(P, Q) \leq d(p, Q)$ for any pair of points P and Q.

With this definition of distance, we can ask the same locus question. What is the set of all points a taxi-cab distance of 1 from the origin? If $P(x, y)$ is any point on the locus, then the equation of the locus is $|x - 0| + |y - 0| = 1$ or $|x| + |y| = 1$. The locus is plotted below, and turns out to be a square. The graph can be produced by a graphing calculator or by hand. In this case, it is easiest to break the problem into four cases depending on x and y being positive or negative.



You can investigate many other locus problems in this new geometry. For example, find the set of points that are equidistant from (0, 0) and (1, 1). If we use Euclidean distance, we get a straight line, the right bisector of the line segment joining the two points. The following diagram shows what happens with taxi-cab distance.



For $0 \le x \le 1$, the right bisector is the line, as with Euclidean distance. However, for $x \ge 1$, $y \le 0$ and $x \le 0$, $y \ge 1$, all points are equidistant from (0, 0) and (1, 1).

There are many other ways to generate non-Euclidean geometries. Another example is to look at geometry on the surface of a sphere. In this geometry, straight lines (the shortest path between two points) become arcs of circles.

For fun, try the following with taxi-cab distance:

1. Find an equilateral triangle with taxi-cab side length 1. Are all angles equal?

2. Sketch the locus of all points that are equidistant from (0, 0) and (1, 2).

3. The line segment joining (0, 0) to (1, 0) is rotated about the origin. What happens to its length?

Chapter 5

ALGEBRAIC VECTORS AND APPLICATIONS

For quantities that have both magnitude and direction, the directed line segment or arrow is an excellent introductory method. But what about a quantity that has more than three dimensions? In such cases, an algebraic vector model is required. A vector model allows you to add, subtract, and multiply by a scalar vector. We can also use this model to multiply one vector by another vector. The development of the vector model was made possible because, thanks to Descartes and analytic geometry, many geometric ideas already had an algebraic counterpart. For example, a line could be represented by a picture or by an equation. We will see the real power of vectors in this chapter, when we will use them to solve problems in the third dimension and beyond.

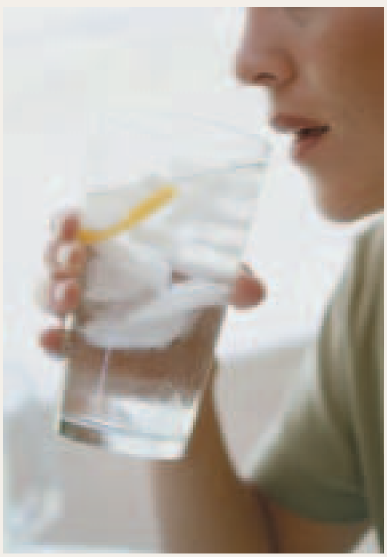
CHAPTER EXPECTATIONS In this chapter, you will

- determine equations of lines in two- and three-dimensional space, **Section 5.1**
- determine the intersection of a line and a plane in three-dimensional space, **Section 5.1**
- represent Cartesian vectors, **Section 5.1, 5.2**
- determine and interpret dot and cross products of geometric vectors, **Section 5.3, 5.4, 5.5**
- perform mathematical operations on Cartesian vectors, **Section 5.3, 5.4, 5.5**

Rich Learning Link investigate

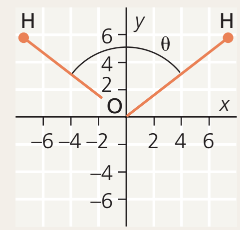
CHAPTER 5: MOLECULAR BOND ANGLES

Atoms bond together to form the molecules that make up the substances around us. The geometry of molecules is a factor in determining many of the chemical properties of these substances. Ethyl alcohol and dimethyl ether are both formed from two carbon atoms, six hydrogen atoms, and one oxygen atom (C_2H_6O), but they have very different chemical and physical attributes. The properties of enzymes, protein molecules that speed up biochemical reactions, depend upon precise fits between molecules with specific shapes. One aspect of molecular geometry that interests chemists is called the bond angle. It is the angle between two bonds in a molecule. For example, the angle formed where two hydrogen atoms link to an oxygen atom to form water (H_2O) is about 104.5°.



Investigate

A water molecule can be studied in a Cartesian plane. If we allow each unit on the plane to represent 10^{-11} metres and place the oxygen atom at the origin, then the hydrogen atoms are located symmetrically at about (7.59, 5.88) and (–7.59, 5.88). The bond angle formed at the oxygen atom is



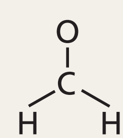
$\theta = 180 - 2 \times \tan^{-1}\left(\frac{5.88}{7.59}\right) = 104.5°$.

Can you explain why this calculation is correct?

Nitrogen trioxide (NO_3^-) is an example of a trigonal planar molecule. It consists of four atoms in a plane: three oxygen atoms surrounding and individually bonding to a single nitrogen atom. Because there are three identical atoms surrounding the nitrogen atom, the three are evenly spaced around a circle. The bond angle for each of the three bonds is, therefore, 360 ÷ 3 = 120°.

DISCUSSION QUESTIONS

1. If the distance between the nitrogen atom and each oxygen atom in NO_3^- is 1.22×10^{-10} metres, what is one way to assign planar coordinates to the atoms?
2. Formaldehyde (H_2CO) is a trigonal planar molecule with the carbon in the centre. The bond between the carbon and the oxygen is shorter than the bond between the carbon and either one of the hydrogen atoms. Which is likely to be smaller, the O-C-H bond angle or the H-C-H bond angle?



Formaldehyde

3. Can three-atom molecules always be studied in a plane? Can four-atom molecules always be studied in a plane? What about molecules with more than four atoms?

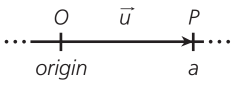
Section 5.1 — Algebraic Vectors

In this chapter, we establish principles that allow the use of algebraic methods in the study of vectors. The application of algebra to problems in geometry first became possible in 1637, when Descartes introduced the concept of a **coordinate system.**

A line is a geometrical object. How is a coordinate system for a line constructed? First, choose an arbitrary point on the line as a reference point, or origin. Next, associate with each point P on the line a real number a. How? Let the sign of a indicate which side of the origin P is on, and let the magnitude of a represent the distance from the origin to P. The result is known as the real number line, and a is called the coordinate of P.

The correspondence between points on the line and real numbers is complete in this sense: each point on the line has a different real number as its coordinate, and every real number corresponds to one and only one point on the line.

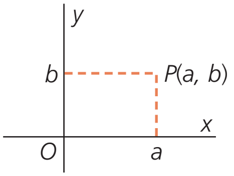
Now let $\vec{u}$ be a vector on this line. Move the vector until its initial point is at the origin. Its endpoint will fall on some point P with coordinate a. The coordinate a contains everything you need to describe the vector $\vec{u}$.



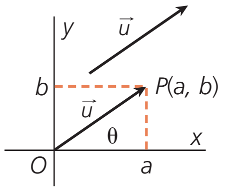
The absolute value $|a|$ is the magnitude of $\vec{u}$, and the sign of a tells you its direction.

We have now established the connection between the coordinates of a point and a geometrical vector on a line. This amounts to an algebraic representation of a geometrical vector. It is the first step in the development of algebraic methods to handle vector problems.

A line is one-dimensional. A plane has two dimensions. But the same process leads to an algebraic representation of a vector in a plane. The Cartesian coordinate system for a plane is constructed from two real number lines—the x-axis and the y-axis—placed at right angles in the plane. The axes are oriented so that a counter-clockwise rotation about the origin carries the positive x-axis into the positive y-axis. Any point P in the plane is identified by an ordered pair of real numbers (a, b), which are its coordinates.



Let $\vec{u}$ be a vector in the plane. Move $\vec{u}$ until its initial point is at the origin. Its endpoint will fall on some point P with coordinates (a, b). The magnitude of $\vec{u}$ can be determined from (a, b) using the Pythagorean Theorem. The direction



of $\vec{u}$ can be expressed in terms of the angle θ between $\vec{u}$ and the positive x-axis.

We can observe that, just as in the case of a line, the magnitude and direction of $\vec{u}$ are determined entirely by the coordinates of P. Nothing else is needed. Therefore, the ordered pair (a, b) is a valid representation of the vector $\vec{u}$.

Any vector $\vec{u}$ in a plane can be written as an ordered pair (a, b), where its magnitude $|\vec{u}|$ and direction θ are given by the equations

$$|\vec{u}| = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

with θ measured counter-clockwise from the positive x-axis to the line of the vector. The formula above gives two values of θ, $0 \leq \theta < 360°$. The actual value depends on the quadrant in which $P(a, b)$ lies.

The ordered pair (a, b) is referred to as an algebraic vector. The values of a and b are the x- and y-components of the vector.

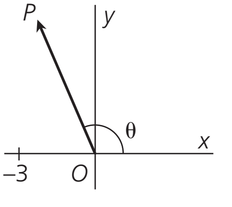
It is important to remember that the ordered pair (a, b) can be interpreted in two different ways: it can represent either a point with coordinates a and b, or a vector with components a and b. The context of a problem will tell you whether (a, b) represents a point or a vector.

EXAMPLE 1

The position vector of a point P is the vector $\overrightarrow{OP}$ from the origin to the point. Draw the position vector of the point $P(-3, 7)$, express it in ordered pair notation, and determine its magnitude and direction.

Solution

The point $P(-3, 7)$ is in the second quadrant. The position vector of P is $\overrightarrow{OP} = (-3, 7)$. The magnitude and direction of $\overrightarrow{OP}$ are calculated as follows:

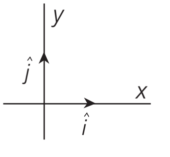


$$|\overrightarrow{OP}|^2 = (-3)^2 + (7)^2 \qquad \tan\theta = \frac{7}{-3}$$

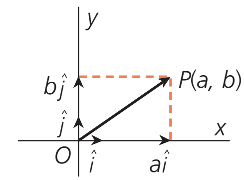
$$|\overrightarrow{OP}| = \sqrt{58} \qquad \theta \cong 113°$$

Thus, the magnitude of $\overrightarrow{OP}$ is $\sqrt{58}$. Its direction makes an angle of approximately 113° with the positive x-axis.

Another notation commonly used to describe algebraic vectors in a plane employs unit vectors. Define the vectors $\hat{i} = (1, 0)$ and $\hat{j} = (0, 1)$. These are unit vectors that point in the direction of the positive x-axis and positive y-axis, respectively.



As you can see in the diagram, the position vector of point $P(a, b)$, and thus any vector $\vec{u}$ in the plane, can be expressed as the vector sum of scalar multiples of $\hat{i}$ and $\hat{j}$.

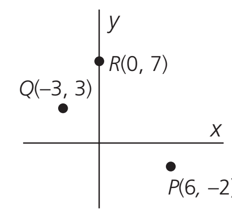


Ordered pair notation and unit vector notation are equivalent. Any algebraic vector can be written in either form:

$$\vec{u} = \overrightarrow{OP} = (a, b) \quad \text{or} \quad \vec{u} = \overrightarrow{OP} = a\hat{i} + b\hat{j}$$

EXAMPLE 2

Express the position vector of each of the points shown in the diagram as an ordered pair and in unit vector notation.



Solution

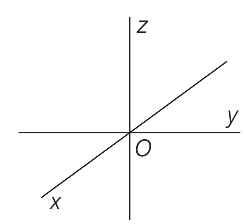
$\overrightarrow{OP} = (6, -2)$
$\quad = 6\hat{i} - 2\hat{j}$

$\overrightarrow{OQ} = (-3, 3)$
$\quad = -3\hat{i} + 3\hat{j}$

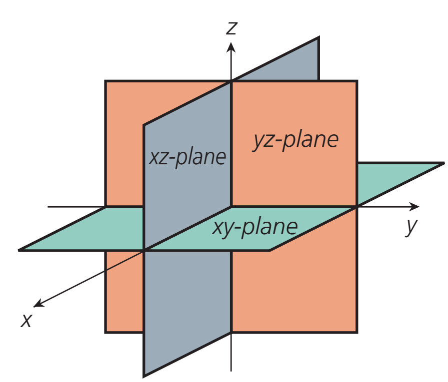
$\overrightarrow{OR} = (0, 7)$
$\quad = 7\hat{j}$

A coordinate system for three-dimensional space is formed in much the same way as a coordinate system for a two-dimensional plane. Some point in space is chosen as the origin. Through the origin, three mutually perpendicular number lines are drawn, called the x-axis, the y-axis, and the z-axis. Each point in space corresponds to an ordered triple of real numbers (a, b, c), which are its coordinates on the three axes.

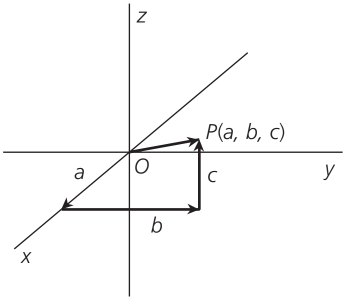
There are two different ways to choose the positive directions of the axes. As a rule, mathematicians use a right-handed coordinate system. If you could grasp the z-axis of a right-handed system with your right hand, pointing your thumb in the direction of the positive z-axis, your fingers should curl from the positive x-axis toward the positive y-axis. A left-handed system would have the positive y-axis oriented in the opposite direction.

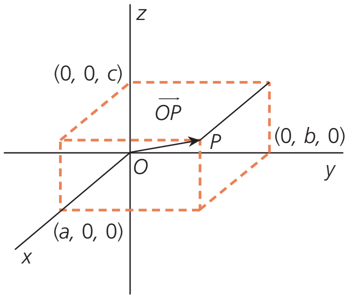


A plane in space that contains two of the coordinate axes is known as a coordinate plane. The plane containing the x- and y-axes, for instance, is called the xy-plane. The other two coordinate planes are named similarly. A point such as $(-4, 0, 1)$, which has a y-coordinate of 0, lies in the xz-plane.



To plot a point $P(a, b, c)$ in space, move a units from the origin in the x direction, then b units in the y direction, and then c units in the z direction. Be sure each move is made along a line parallel to the corresponding axis. Drawing a rectangular box will help you to see the three-dimensional aspect of such diagrams.





Just as in two dimensions, any vector in space can be placed with its initial point at the origin. Its tip will then fall on some point P with coordinates (a, b, c), from which its magnitude and direction can be determined. The ordered triple (a, b, c), therefore, represents an algebraic vector in three dimensions. Alternatively, this vector could be expressed in terms of unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, where $\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, and $\hat{k} = (0, 0, 1)$.

Any vector $\vec{u}$ in three-dimensional space can be written as an ordered triple, $\vec{u} = \overrightarrow{OP} = (a, b, c)$, or in terms of unit vectors, $\vec{u} = \overrightarrow{OP} = a\hat{i} + b\hat{j} + c\hat{k}$. Its magnitude is given by $|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$.

EXAMPLE 3

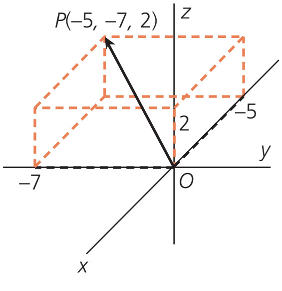
Locate the point P, sketch the position vector $\overrightarrow{OP}$ in three dimensions, and calculate its magnitude.

a. $P(-5, -7, 2)$

b. $\overrightarrow{OP} = 3\hat{i} + 5\hat{j} - 4\hat{k}$

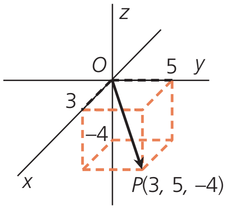
Solution

a.



$$|\overrightarrow{OP}| = \sqrt{(-5)^2 + (-7)^2 + (2)^2} = \sqrt{78}$$

b.



$$|\overrightarrow{OP}| = \sqrt{(3)^2 + (5)^2 + (-4)^2} = \sqrt{50}$$

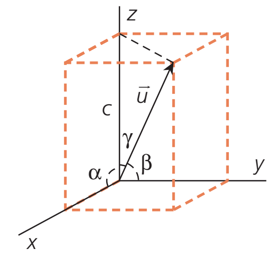
In two dimensions, we can describe the direction of a vector by a single angle. In three dimensions, we use three angles, called direction angles.

The direction angles of a vector (a, b, c) are the angles α, β, and γ that the vector makes with the positive x-, y-, and z-axes, respectively, where $0° \le \alpha, \beta, \gamma \le 180°$.

In this context, the components a, b, and c of the vector $\vec{u}$ are referred to as direction numbers.

In the given diagram, the direction angles are all acute angles. We can see the right triangle that relates $\vec{u}$, the direction number c, and the direction angle γ, from which it follows that $\cos \gamma = \frac{c}{|\vec{u}|}$.

The other direction numbers and angles are related in the same way.



The direction cosines of a vector are the cosines of the direction angles α, β and γ, where

$$\cos \alpha = \frac{a}{|\vec{u}|}, \cos \beta = \frac{b}{|\vec{u}|}, \text{ and } \cos \gamma = \frac{c}{|\vec{u}|}.$$

Note that if you divide a vector (a, b, c) by its magnitude $\vec{u}$, you create a unit vector with components $\left(\frac{a}{|\vec{u}|}, \frac{b}{|\vec{u}|}, \frac{c}{|\vec{u}|}\right)$, which is exactly $(\cos \alpha, \cos \beta, \cos \gamma)$. Thus, the direction cosines are the components of a unit vector. Consequently,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

It follows from this that the direction cosines, and hence the direction angles, are not all independent. From any two of them you can find the third.

EXAMPLE 4 Find the direction cosines and the direction angles of the vector $\vec{u} = (0, 5, -3)$.

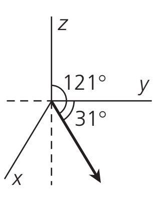
Solution

The magnitude of $\vec{u}$ is $\sqrt{(0)^2 + (5)^2 + (-3)^2} = \sqrt{34}$.
The direction cosines and angles are therefore

$\cos \alpha = \frac{0}{\sqrt{34}}$, $\alpha = 90°$

$\cos \beta = \frac{5}{\sqrt{34}}$, $\beta \cong 31°$

$\cos \gamma = \frac{-3}{\sqrt{34}}$, $\gamma \cong 121°$



This vector is perpendicular to the x-axis, and is, therefore, parallel to the yz-plane.

EXAMPLE 5

A vector $\vec{u}$ makes angles of 60º and 105º, respectively, with the x- and y-axes. What is the angle between $\vec{u}$ and the z-axis?

Solution

$\cos^2 60° + \cos^2 105° + \cos^2 \gamma = 1$

$\cos \gamma = \pm \sqrt{1 - \cos^2 60° - \cos^2 105°}$

$\gamma \cong 34°$ or $146°$

The angle between $\vec{u}$ and the z-axis is 34º or 146º, so there are two possible vectors.

Exercise 5.1

Part A

Communication

1. What is the difference between an algebraic vector and a geometric vector?

2. Rewrite each of the following vectors in the form $a\hat{i} + b\hat{j}$.
   a. $(-5, 2)$ b. $(0, 6)$ c. $(-1, 6)$

3. Rewrite each of the following vectors as an ordered pair.
   a. $2\hat{i} + \hat{j}$ b. $-3\hat{i}$ c. $5\hat{i} - 5\hat{j}$

Knowledge/Understanding

4. Rewrite each of the following vectors in the form $a\hat{i} + b\hat{j} + c\hat{k}$.
   a. $(-2, 1, 1)$ b. $(3, 4, -3)$ c. $(0, 4, -1)$ d. $(-2, 0, 7)$

Knowledge/Understanding

5. Rewrite each of the following vectors as an ordered triple.
   a. $3\hat{i} - 8\hat{j} + \hat{k}$ b. $-2\hat{i} - 2\hat{j} - 5\hat{k}$
   c. $2\hat{j} + 6\hat{k}$ d. $-4\hat{i} + 9\hat{j}$

6. Express each of the following vectors as an algebraic vector in the form (a, b).
   a. $|\vec{u}| = 12, \theta = 135°$ b. $|\vec{v}| = 36, \theta = 330°$
   c. $|\vec{w}| = 16, \theta = 190°$ d. $|\vec{x}| = 13, \theta = 270°$

7. Express each of the following vectors as a geometric vector by stating its magnitude and direction.

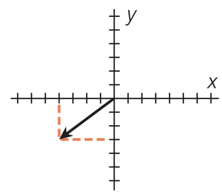
a. $\vec{u} = (-6\sqrt{3}, 6)$ b. $\vec{v} = (-4\sqrt{3}, -12)$

c. $\vec{w} = (4, 3)$ d. $\vec{x} = (0, 8)$

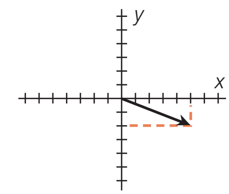
Knowledge/Understanding

8. What vector is represented in each of the following diagrams?

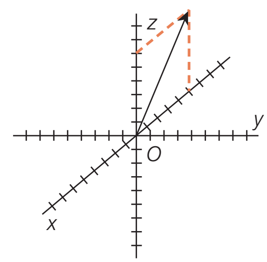
a.



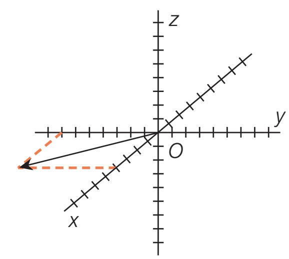
b.



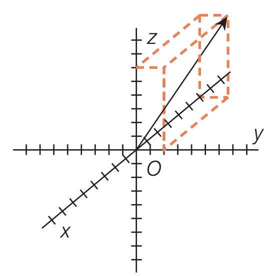
c.



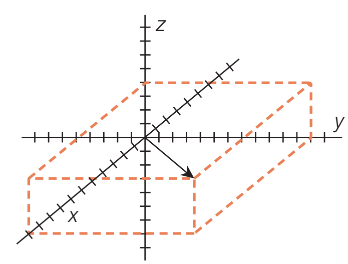
d.



e.



f.



9. For each of the following, draw the *x*-axis, *y*-axis, and *z*-axis, and accurately plot the points.

$A(-3, 0, 0)$ $B(0, 2, 0)$ $C(0, 0, -2)$ $D(-3, 2, 0)$

$E(3, 0, -2)$ $F(0, 2, 3)$ $G(-2, 0, 3)$ $H(0, 3, -2)$

Part B

Communication

10. Describe where each of the following sets of points is located.

a. (0, 0, 6), (0, 0, −3), (0, 0, 4) b. (0, 2, 8), (0, −8, 2), (0, −2, 2)

c. (3, 0, 3), (3, 0, −5), (−3, 0, 5) d. (−1, 2, 0), (0, 4, 0), (5, −6, 0)

e. (1, 3, −2), (1, 3, 6), (1, 3, 11) f. (2, 2, 2), (−3, −3, −3), (8, 8, 8)

11. Where are the following general points located?

a. $A(x, y, 0)$ b. $B(x, 0, 0)$
c. $C(0, y, z)$ d. $D(0, 0, z)$
e. $E(x, 0, z)$ f. $F(0, y, 0)$

12. For each of the following, draw the x-axis, y-axis, and z-axis and accurately draw the position vectors.

a. $M(6, -4, 2)$ b. $N(-3, 5, 3)$
c. $P(2, 3, -7)$ d. $Q(-4, -9, 5)$
e. $R(5, -5, -1)$ f. $T(-6, 1, -8)$

13. Find the magnitude and the direction of the following vectors.

a. $\vec{OE} = (1, 7)$ b. $\vec{OF} = (0, -6)$
c. $\vec{OG} = (-9, 12)$ d. $\vec{OH} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
e. $\vec{OJ} = \left(\frac{2}{\sqrt{5}}, -\frac{\sqrt{6}}{\sqrt{5}}\right)$ f. $\vec{OK} = (-\sqrt{6}, 0)$

14. Find the magnitude of the following vectors.

a. $(-12, -4, 6)$ b. $(8, -27, 21)$
c. $\left(\frac{14}{27}, \frac{-22}{27}, \frac{-7}{27}\right)$ d. $(-\sqrt{2}, 2\sqrt{3}, \sqrt{2})$

Communication **15.** Can the sum of two unit vectors be a unit vector? Explain. Can the difference?

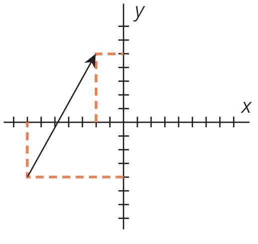
16. a. Calculate $|\vec{a}|$ when $\vec{a} = (2, 3, -2)$.
b. Find $\frac{1}{|\vec{a}|}\vec{a}$. Is it a unit vector?

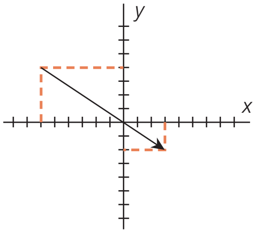
17. a. Find the magnitude of the vector $\vec{v} = 2\hat{i} - 3\hat{j} - 6\hat{k}$.
b. Find a unit vector in the direction of $\vec{v}$.

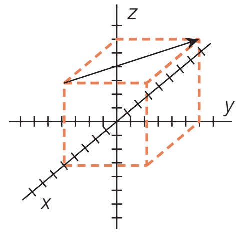
18. If $\vec{v} = (3, 4, 12)$, find a unit vector in the direction opposite to $\vec{v}$.

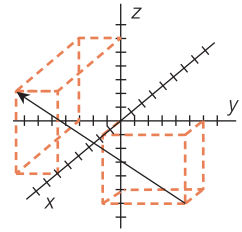
19. Show that any unit vector in two dimensions can be written as $(\cos \theta, \sin \theta)$, where θ is the angle between the vector and the x-axis.

Application **20.** Reposition each of the following vectors so that its initial point is at the origin, and determine its components.

a.


b.


c. 

d. 

Thinking/Inquiry/ Problem Solving

21. Draw a diagram of a vector $\vec{u} = (a, b, c)$ that illustrates the relationship between

a. $\vec{u}$, a, and $\cos \alpha$ (α acute)

b. $\vec{u}$, b, and $\cos \beta$ (β obtuse)

22. The direction angles of a vector are all equal. Find the direction angles to the nearest degree.

Part C

23. Prove that the magnitude of the vector $\overrightarrow{OP} = (a, b, c)$ is given by $\left|\overrightarrow{OP}\right| = \sqrt{a^2 + b^2 + c^2}$.

Thinking/Inquiry/ Problem Solving

24. Give a geometrical interpretation of the vector $\vec{u} = (4, 2, -5, 2)$. Make a reasonable conjecture about its magnitude.

Section 5.2 — Operations with Algebraic Vectors

As we saw in Section 5.1, all vectors can be expressed in terms of the unit vectors $\hat{i}$ and $\hat{j}$ in two dimensions, or $\hat{i}$, $\hat{j}$, and $\hat{k}$ in three dimensions, or, equivalently, in terms of ordered pairs or triplets. Vectors such as $\hat{i}$, $\hat{j}$, and $\hat{k}$, which have been chosen to play this special role, are termed **basis vectors.** They form a basis for the two- or three-dimensional spaces in which vectors exist. In Example 1, we establish the uniqueness of the algebraic representation of a vector in terms of these basis vectors.

EXAMPLE 1

Prove that the representation of a two-dimensional algebraic vector in terms of its x- and y-components is unique.

Solution

Using the method of proof by contradiction, we begin by assuming that the vector $\vec{u}$ can be written in terms of components in two different ways:

$$\vec{u} = a_1\hat{i} + b_1\hat{j} \quad \text{and} \quad \vec{u} = a_2\hat{i} + b_2\hat{j}$$

Since they represent the same vector, these expressions must be equal.

$$a_1\hat{i} + b_1\hat{j} = a_2\hat{i} + b_2 j$$

Some rearrangement produces the equations

$$a_1\hat{i} - a_2\hat{i} = -b_1\hat{j} + b_2\hat{j}$$

$$(a_1 - a_2)\hat{i} = (-b_1 + b_2)\hat{j}$$

The last equation states that a scalar multiple of $\hat{i}$ equals a scalar multiple of $\hat{j}$. But this cannot be true. The unit vectors $\hat{i}$ and $\hat{j}$ have different directions, and no multiplication by a scalar can make the vectors equal. The only possible way the equation can be valid is if the coefficients of $\hat{i}$ and $\hat{j}$ are zero, that is, $a_1 = a_2$ and $b_1 = b_2$, which means that the two representations of the vector $\vec{u}$ are not different after all.

A proof of uniqueness for vectors in three dimensions is more complicated and will be explored in Chapter 6.

The uniqueness of algebraic vectors leads to a fundamental statement about the equality of algebraic vectors.

Two algebraic vectors are equal if and only if their respective Cartesian components are equal.

Since all vectors can be expressed in terms of the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, all the rules of vector algebra discussed in Chapter 4 apply to algebraic vectors. In two dimensions, for instance, scalar multiplication of a vector and addition of two vectors, written in both unit vector and ordered pair notation, look like this:

scalar multiplication

$$k(a\hat{i} + b\hat{j}) = ka\hat{i} + kb\hat{j}$$
or
$$k(a, b) = (ka, kb)$$

vector addition

$$(a_1\vec{i} + b_1\vec{j}) + (a_2\vec{i} + b_2\vec{j}) = (a_1 + a_2)\vec{i} + (b_1 + b_2)\vec{j}$$
or
$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

EXAMPLE 2 If $\vec{u} = (5, -7)$ and $\vec{v} = (-2, 3)$, find $\vec{w} = 6\vec{u} - 4\vec{v}$.

Solution

In ordered pair notation

$$\begin{aligned}\vec{w} &= 6\vec{u} - 4\vec{v}\\ &= 6(5, -7) - 4(-2, 3)\\ &= (30, -42) + (8, -12)\\ &= (38, -54)\end{aligned}$$

In unit vector notation

$$\begin{aligned}\vec{w} &= 6\vec{u} - 4\vec{v}\\ &= 6(5\hat{i} - 7\hat{j}) - 4(-2\hat{i} + 3\hat{j})\\ &= 30\hat{i} - 42\hat{j} + 8\hat{i} - 12\hat{j}\\ &= 38\hat{i} - 54\hat{j}\end{aligned}$$

EXAMPLE 3 Using vectors, demonstrate that the three points $A(5, -1)$, $B(-3, 4)$, and $C(13, -6)$ are collinear.

Solution

The three points will be collinear if the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same direction, or the opposite direction.

$$\overrightarrow{AB} = (-8, 5)$$
$$\overrightarrow{BC} = (16, -10)$$

Then $\overrightarrow{BC} = -2\overrightarrow{AB}$

$\overrightarrow{AB}$ and $\overrightarrow{\text{BC}}$ have the opposite direction, so the points A, B, and C must be collinear.

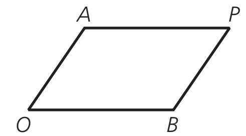
EXAMPLE 4 If $A(1, -5, 2)$ and $B(-3, 4, 4)$ are opposite vertices of parallelogram $OAPB$ and O is the origin, find the coordinates of P.

Solution

$$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$$

But $\overrightarrow{BP} = \overrightarrow{OA} = (1, -5, 2)$

Then $\overrightarrow{OP} = (-3, 4, 4) + (1, -5, 2)$
$= (-2, -1, 6)$



Therefore, point P has coordinates $(-2, -1, 6)$.

Exercise 5.2

Part A

Communication

1. In two dimensions, the unit vectors $\hat{i}$ and $\hat{j}$ have been chosen as the basis vectors in terms of which all other vectors in the plane are expressed.
   a. Consider the merits of this choice as opposed to using vectors that do not have unit magnitude.
   b. Consider the merits of this choice as opposed to using vectors that are not perpendicular.

Knowledge/Understanding

2. Find a single vector equivalent to each expression below.
   a. $(2, -4) + (1, 7)$
   b. $5(1, 4)$
   c. $0(4, -5)$
   d. $(-6, 0) + 7(1, -1)$
   e. $(2, -1, 3) + (-2, 1, 3)$
   f. $2(1, 1, -4)$
   g. $(4, -1, 3) - (-2, 1, 3)$
   h. $2(-1, 1, 3) + 3(-2, 3, -1)$
   i. $2(0, 1, 0) + 5(0, 0, 1)$
   j. $-\frac{1}{2}(4, -6, 8) + \frac{3}{2}(4, -6, 8)$
   k. $5(0, -2, -4) - 4(3, 8, 0)$
   l. $-2(-3, 2, 4) + 5(3, 2, 8)$

Knowledge/Understanding

3. Simplify each of the following expressions.
   a. $(2\hat{i} + 3\hat{j}) + 4(\hat{i} - \hat{j})$
   b. $3(\hat{i} - 2\hat{j} + 3\hat{k}) - 3(-\hat{i} + 4\hat{j} - 3\hat{k})$
   c. $-3(\hat{i} - \hat{k}) - (2\hat{i} + \hat{k})$
   d. $5(9\hat{i} - 7\hat{j}) - 5(-9\hat{i} + 7\hat{k})$

4. Given $\vec{a} = (2, -1, 4)$, $\vec{b} = (3, 8, -6)$, and $\vec{c} = (4, 2, 1)$, find a single vector equivalent to each of the following expressions.
   a. $2\vec{a} - \vec{b}$
   b. $\vec{a} - \vec{b}$
   c. $3\vec{a} - \vec{b} - 2\vec{c}$
   d. $\vec{a} + \vec{b} + 2\vec{c}$
   e. $-2\vec{a} + \vec{b} - \vec{c}$
   f. $4\vec{a} - 2\vec{b} + \vec{c}$

5. Given $\vec{x} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{y} = 2\hat{j} + 4\hat{k}$, express each quantity in terms of $\hat{i}$, $\hat{j}$, and $\hat{k}$.
   a. $3\vec{x} + \vec{y}$
   b. $\vec{x} + \vec{y}$
   c. $\vec{x} - \vec{y}$
   d. $\vec{y} - \vec{x}$

Application

6. If $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j}$, calculate each magnitude.

   a. $|\vec{a} + \vec{b}|$ b. $|\vec{a} - \vec{b}|$ c. $|2\vec{a} - 3\vec{b}|$

7. If $D(3, 4, 5)$ and $E(-2, 1, 5)$ are points in space, calculate each expression and state what it represents.

   a. $|\overrightarrow{OD}|$ b. $|\overrightarrow{OE}|$ c. $\overrightarrow{DE}$

   d. $|\overrightarrow{DE}|$ e. $\overrightarrow{ED}$ f. $|\overrightarrow{ED}|$

Part B

8. Using vectors, demonstrate that these points are collinear.

   a. $P(15, 10)$, $Q(6, 4)$, and $R(-12, -8)$

   b. $D(33, -5, 20)$, $E(6, 4, -16)$, and $F(9, 3, -12)$

9. For each set of points A, B, C, and D, determine whether $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and whether $|\overrightarrow{AB}| = |\overrightarrow{CD}|$.

   a. $A(2, 0)$, $B(3, 6)$, $C(4, 1)$ $D(5, -5)$

   b. $A(0, 1, 0)$, $B(4, 0, 1)$, $C(5, 1, 2)$, $D(2, 3, 5)$

   c. $A(2, 4, 6)$, $B(3, 4, 1)$, $C(4, 1, 3)$, $D(5, 1, -2)$

Application

10. If $PQRS$ is a parallelogram in a plane, where P is $(4, 2)$, Q is $(-6, 1)$, and S is $(-3, -4)$, find the coordinates of R.

11. If three vertices of a parallelogram in a plane are $(-5, 3)$, $(5, 2)$, and $(7, -8)$, determine all the possible coordinates of the fourth vertex.

Thinking/Inquiry/Problem Solving

12. If $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are three edges of a **parallelepiped** where O is $(0, 0, 0)$, A is $(2, 4, -2)$, B is $(3, 6, 1)$, and C is $(4, 0, -1)$, find the coordinates of the other vertices of the parallelepiped.

Application

13. A line segment has endpoints with position vectors $\overrightarrow{\mathrm{OA}_1}$ and $\overrightarrow{\mathrm{OA}_2}$. The midpoint of the line segment is the point with position vector
$\overrightarrow{OM} = \dfrac{\overrightarrow{OA_1} + \overrightarrow{OA_2}}{2}$.
Find the position vector of the midpoint of the line segment from

   a. $A(-5, 2)$ to $B(13, 4)$ b. $C(3, 0)$ to $D(0, -7)$

   c. $E(6, 4, 2)$ to $F(-2, 8, -2)$ d. $G(0, 16, -5)$ to $H(9, -7, -1)$

14. a. Find x and y if $3(x, 1) - 2(2, y) = (2, 1)$.

   b. Find x, y, and z if $2(x, -1, 4) - 3(-4, y, 6) - \frac{1}{2}(4, -2, z) = (0, 0, 0)$.

Part C

15. Find the components of the unit vector with direction opposite to that of the vector from $X(7, 4, -2)$ to $Y(1, 2, 1)$.

16. a. Find the point on the y-axis that is equidistant from the points $(2, -1, 1)$ and $(0, 1, 3)$.

    b. Find a point not on the y-axis that is equidistant from the points $(2, -1, 1)$ and $(0, 1, 3)$.

Thinking/Inquiry/ Problem Solving

17. a. Find the length of the median AM in the triangle ABC, for the points $A(2, \frac{3}{2}, -4)$, $B(3, -4, 2)$, and $C(1, 3, -7)$.

    b. Find the distance from A to the **centroid** of the triangle.

18. The centroid of the n points with position vectors $OA_1, OA_2, \ldots, OA_n$ is the point with position vector

$$\overrightarrow{OC} = \frac{\overrightarrow{OA_1} + \overrightarrow{OA_2} + \ldots + \overrightarrow{OA_n}}{n}.$$

Find the centroid of each of the following sets of points.

    a. $A(1, 2)$, $B(4, -1)$, $C(-2, -2)$

    b. $I(1, 0, 0)$, $J(0, 1, 0)$, $K(0, 0, 1)$

    c. $A_1(3, -1)$, $A_2(1, 1)$, $A_3(7, 0)$, $A_4(4, 4)$

    d. $C(0, 0, 0)$, $I(1, 0, 0)$, $J(0, 1, 0)$, $K(0, 0, 1)$

19. The **centre of mass** of the masses $m_1, m_2, \ldots, m_n$ at the points with position vectors $\overrightarrow{OA_1}, \overrightarrow{OA_2}, \ldots, \overrightarrow{OA_n}$, respectively, is the point with position vector

$$\overrightarrow{OG} = \frac{m_1\overrightarrow{OA_1} + m_2\overrightarrow{OA_2} + \ldots + m_n\overrightarrow{OA_n}}{m_1 + m_2 + \ldots + m_n}.$$

In some kinds of problems, a collection of masses can be replaced by a single large mass $M = m_1 + m_2 + \ldots + m_n$ located at the centre of mass, for the purposes of calculation. Calculate the centre of mass in each case.

    a. A mass of 2 units at $(0, 0)$, a mass of 3 units at $(4, 1)$, a mass of 5 units at $(-1, -7)$, and a mass of 1 unit at $(11, -9)$.

    b. A mass of 1 unit at $(1, 4, -1)$, a mass of 3 units at $(-2, 0, 1)$, and a mass of 7 units at $(1, -3, 10)$.

Section 5.3 — The Dot Product of Two Vectors

technology APPENDIX P. 531

Certain applications of vectors in physics and geometry cannot be handled by the operatons of vector addition and scalar multiplication alone. Other, more sophisticated combinations of vectors are required. The **dot product** of two vectors is one of these combinations.

The dot product of two vectors $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \bullet \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta,$$

where θ is the angle between the two vectors.

Since the quantity $|\vec{u}|\,|\vec{v}| \cos\theta$ on the right is the product of three scalars, the dot product of two vectors is a scalar. For this reason, the dot product is also called the **scalar product.**

EXAMPLE 1

Find the dot product of $\vec{u}$ and $\vec{v}$ in each of the following cases, where θ is the angle between the vectors.

a. $|\vec{u}| = 7$, $|\vec{v}| = 12$, $\theta = 60°$

b. $|\vec{u}| = 20$, $|\vec{v}| = 3$, $\theta = \frac{5\pi}{6}$

c. $|\vec{u}| = 24$, $|\vec{v}| = 9$, $\theta = 34°$

Solution

a. $\vec{u} \bullet \vec{v} = |\vec{u}|\,|\vec{v}| \cos 60°$

$= (7)(12)(0.5)$

$= 42$

b. $\vec{u} \bullet \vec{v} = |\vec{u}|\,|\vec{v}| \cos \frac{5\pi}{6}$

$= (20)(3)\left(-\frac{\sqrt{3}}{2}\right)$

$= -30\sqrt{3}$

c. $\vec{u} \bullet \vec{v} = |\vec{u}|\,|\vec{v}| \cos 34°$

$= (24)(9)(0.8290)$

$\cong 179.1$

EXAMPLE 2

Prove that two non-zero vectors $\vec{u}$ and $\vec{v}$ are perpendicular, if and only if $\vec{u} \bullet \vec{v} = 0$.

Proof

The condition that $\vec{u} \bullet \vec{v} = 0$ is sufficient. Nothing else is needed to guarantee that the vectors are perpendicular, because

if $\quad \vec{u} \bullet \vec{v} = 0$

then $\quad |\vec{u}|\,|\vec{v}| \cos\theta = 0$

or $\quad \cos\theta = 0$, (since the vectors are non-zero)

Therefore, $\theta = \pm 90^\circ$,
which means that the vectors must be perpendicular.
The condition that $\vec{u} \bullet \vec{v} = 0$ is necessary, because

if $\vec{u} \bullet \vec{v} \neq 0$

then $|\vec{u}|\,|\vec{v}| \cos\theta \neq 0$,
which means that $\cos\theta$ cannot be zero.

Consequently, θ cannot be 90°.
So the vectors are perpendicular *only if* $\vec{u} \bullet \vec{v} = 0$.

The following properties of the dot product will be demonstrated in Exercise 5.3.

You can multiply by a scalar either before or after taking the dot product. $a(\vec{u} \bullet \vec{v}) = (a\vec{u}) \bullet \vec{v} = \vec{u} \bullet (a\vec{v})$

You can expand a dot product of a vector with the sum of two other vectors as you would in ordinary multiplication. $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

The dot product of a vector $\vec{u}$ with itself is the square of the magnitude of the vector. $\vec{u} \bullet \vec{u} = |\vec{u}|^2$

Dot products of the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ are of particular importance.

Because they are unit vectors,

$$\hat{i} \bullet \hat{i} = 1$$
$$\hat{j} \bullet \hat{j} = 1$$
$$\hat{k} \bullet \hat{k} = 1$$

Because they are perpendicular,

$$\hat{i} \bullet \hat{j} = \hat{j} \bullet \hat{i} = 0$$
$$\hat{j} \bullet \hat{k} = \hat{k} \bullet \hat{j} = 0$$
$$\hat{k} \bullet \hat{i} = \hat{i} \bullet \hat{k} = 0$$

The dot product is 1 if the vectors are the same, and 0 if they are different.

These results are used to work out the dot product of two algebraic vectors, which, for vectors in space, proceeds in this manner:

If $\vec{u} = u_x\hat{i} + u_y\hat{k} + u_z\hat{k}$ and $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

$$\begin{aligned}
\text{then } \vec{u} \bullet \vec{v} &= (u_x\hat{i} + u_y\hat{j} + u_z\hat{k}) \bullet (v_x\hat{i} + v_y\hat{j} + v_z\hat{k}) \\
&= u_xv_x(\hat{i} \bullet \hat{i}) + u_xv_y(\hat{i} \bullet \hat{j}) + u_xv_z(\hat{i} \bullet \hat{k}) \\
&\quad + u_yv_x(\hat{j} \bullet \hat{i}) + u_yv_y(\hat{j} \bullet \hat{j}) + u_yv_z(\hat{j} \bullet \hat{k}) \\
&\quad + u_zv_x(\hat{k} \bullet \hat{i}) + u_zv_y(\hat{k} \bullet \hat{j}) + u_zv_z(\hat{k} \bullet \hat{k}) \\
&= u_xv_x(1) + u_xv_y(0) + u_xv_z(0) \\
&\quad + u_yv_x(0) + u_yv_y(1) + u_yv_z(0) \\
&\quad + u_zv_x(0) + u_zv_y(0) + u_zv_z(1) \\
&= u_xv_x + u_yv_y + u_zv_z.
\end{aligned}$$

EXAMPLE 3

Find the dot product of $\vec{u}$ and $\vec{v}$, where

a. $\vec{u} = (-5, 2)$ and $\vec{v} = (3, 4)$

b. $\vec{u} = (1, 0, 4)$ and $\vec{v} = (-2, 5, 8)$

Solution

$$\begin{aligned}\vec{u} \bullet \vec{v} &= (-5, 2) \bullet (3, 4)\\ &= (-5)(3) + (2)(4)\\ &= -15 + 8\\ &= -7\end{aligned}$$

$$\begin{aligned}\vec{u} \bullet \vec{v} &= (1, 0, 4) \bullet (-2, 5, 8)\\ &= (1)(-2) + (0)(5) + (4)(8)\\ &= -2 + 0 + 32\\ &= 30\end{aligned}$$

EXAMPLE 4

Find the angle θ between each of the following pairs of vectors

a. $\vec{u} = (6, -5)$ and $\vec{v} = (-1, 3)$

b. $\vec{u} = (-3, 1, 2)$ and $\vec{v} = (5, -4, -1)$

Solution

Since $\vec{u} \bullet \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$,

then $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{|\vec{u}|\,|\vec{v}|}$.

$$\begin{aligned}\text{a. } \cos\theta &= \frac{(6, -5) \bullet (-1, -3)}{|(6, -5)|\,|(-1, -3)|}\\ &= \frac{(6)(-1) + (-5)(-3)}{\sqrt{(6)^2 + (-5)^2}\sqrt{(-1)^2 + (-3)^2}}\\ &= \frac{9}{\sqrt{61}\sqrt{10}} \cong 0.3644\\ \therefore\ \theta &\cong 69^{\circ}\end{aligned}$$

$$\begin{aligned}\text{b. } \cos\theta &= \frac{(-3\ , 1, 2) \bullet (5, -4, -1)}{|(-3, 1, 2)|\,|(5, -4, -1)|}\\ &= \frac{(-3)(5) + (1)(-4) + (2)(-1)}{\sqrt{(-3)^2 + (1)^2 + (2)^2}\sqrt{(5)^2 + (-4)^2 + (-1)^2}}\\ &= \frac{-21}{\sqrt{14}\sqrt{42}} = \frac{-\sqrt{3}}{2}\\ \therefore\ \theta &= 150^{\circ}\end{aligned}$$

technology

The dot product of two algebraic vectors is a relatively simple operation to perform by hand. Nevertheless, if you have a programmable calculator, you might like to write a short program that will do this calculation automatically. Instructions on how to prepare such a program are found in the Appendix.

Exercise 5.3

Part A

Communication

1. a. What is the dot product of two vectors, if the angle between them is 0º? 90º? 180º?

b. What is the angle between two vectors if their dot product is positive? negative? zero?

2. Calculate the dot product $\vec{u} \cdot \vec{v}$, given the magnitudes of the two vectors and the angle θ between them.

a. $|\vec{u}| = 3, |\vec{v}| = 4, \theta = 45º$

b. $|\vec{u}| = 6, |\vec{v}| = 5, \theta = 60º$

c. $|\vec{u}| = 9, |\vec{v}| = 3, \theta = \frac{3\pi}{4}$

d. $|\vec{u}| = \frac{2}{3}, |\vec{v}| = \frac{9}{8}, \theta = 90º$

3. Examine each of the following pairs of vectors. State whether or not the vectors are perpendicular, then sketch each pair, and find their dot product.

a. $\vec{a} = (4, 1), \vec{b} = (-1, 4)$

b. $\vec{c} = (5, 2), \vec{d} = (-5, -2)$

c. $\vec{p} = (1, 0), \vec{q} = (0, -1)$

d. $\vec{u} = (7, 8), \vec{v} = (4, -7)$

Knowledge/Understanding

4. Find the dot product of each of the following pairs of vectors and state which pairs are perpendicular.

a. $\vec{a} = (-1, 3, 4), \vec{b} = (1, 3, -2)$

b. $\vec{x} = (-2, 2, 4), \vec{y} = (4, 1, -2)$

c. $\vec{m} = (-5, 0, 0), \vec{n} = (0, -3, 0)$

d. $\vec{l} = (0, -3, 4), \vec{l} = (0, -3, 4)$

e. $\vec{u} = (0, 5, 6), \vec{v} = (7, 0, 1)$

f. $\vec{c} = (8, -11, -5)$, $\vec{d} = (-7, -11, -13)$

5. a. Find three vectors perpendicular to (2, −3).

b. How many unit vectors are perpendicular to a given vector in the *xy*-plane?

6. a. Find three non-collinear vectors perpendicular to (2, −3, 1).

b. How many unit vectors are perpendicular to a given vector in three dimensions?

7. Calculate, to four decimal places, the cosine of the angle between each of the following pairs of vectors.

a. $\vec{a} = (8, 9), \vec{b} = (9, 8)$

b. $\vec{c} = (1, -2, 3), \vec{d} = (4, 2, -1)$

Part B

Knowledge/Understanding

8. Determine the angle between the following vectors.

a. $\vec{a} = (3, 5), \vec{b} = (-4, 1)$

b. $\vec{c} = (5, 6, -7), \vec{d} = (-2, 3, 1)$

c. $\hat{i} = (1, 0, 0), \vec{m} = (1, 1, 1)$

d. $\vec{p} = (2, -4, 5), \vec{q} = (0, 2, 3)$

Application

9. Given $\vec{a} = (2, 3, 7)$ and $\vec{b} = (-4, y, -14)$,

a. for what value of y are the vectors collinear?

b. for what value of y are the vectors perpendicular?

10. Find any vector $\vec{w}$ that is perpendicular to both $\vec{u} = 3\hat{j} + 4\hat{k}$ and $\vec{v} = 2\hat{i}$.

11. If the vectors $\vec{a} = (2, 3, 4)$ and $\vec{b} = (10, y, z)$ are perpendicular, how must y and z be related?

12. For $\vec{u} = (1, 5, 8)$ and $\vec{v} = (-1, 3, -2)$, verify that

a. $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

b. $\vec{u} \bullet \vec{u} = |\vec{u}|^2$ and $\vec{v} \bullet \vec{v} = |\vec{v}|^2$

c. $(\vec{u} + \vec{v}) \bullet (\vec{u} - \vec{v}) = |\vec{u}|^2 - |\vec{v}|^2$

d. $(\vec{u} + \vec{v}) \bullet (\vec{u} + \vec{v}) = |\vec{u}|^2 + 2\vec{u} \bullet \vec{v} + |\vec{v}|^2$

e. $(2\vec{u}) \bullet \vec{v} = \vec{u} \bullet (2\vec{v}) = 2(\vec{u} \bullet \vec{v})$

13. If $\vec{u} = (2, 2, -1)$, $\vec{v} = (3, -1, 0)$, and $\vec{w} = (1, 7, 8)$, verify that $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$.

14. Expand and simplify.

a. $(4\hat{i} - \hat{j}) \bullet \hat{j}$

b. $\hat{k} \bullet (\hat{j} - 3\hat{k})$

c. $(\hat{i} - 4\hat{k}) \bullet (\hat{i} - 4\hat{k})$

15. Expand and simplify.

a. $(3\vec{a} + 4\vec{b}) \bullet (5\vec{a} + 6\vec{b})$

b. $(2\vec{a} - \vec{b}) \bullet (2\vec{a} + \vec{b})$

16. Find $(3\vec{a} + \vec{b}) \bullet (2\vec{b} - 4\vec{a})$, if $\vec{a} = -\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$.

17. Two vectors $2\vec{a} + \vec{b}$ and $\vec{a} - 3\vec{b}$ are perpendicular. Find the angle between $\vec{a}$ and $\vec{b}$, if $|\vec{a}| = 2|\vec{b}|$.

18. Given $\hat{a}$ and $\hat{b}$ unit vectors,

a. if the angle between them is 60º, calculate $(6\hat{a} + \hat{b}) \bullet (\hat{a} - 2\hat{b})$

b. if $|\hat{a} + \hat{b}| = \sqrt{3}$, determine $(2\hat{a} - 5\hat{b}) \bullet (\hat{b} + 3\hat{a})$

Application

19. The vectors $\vec{a} = 3\hat{i} - 4\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ are the diagonals of a parallelogram. Show that this parallelogram is a rhombus, and determine the lengths of the sides and the angles between the sides.

20. a. If $\vec{a}$ and $\vec{b}$ are perpendicular, show that $|\vec{a}|^2 + |\vec{b}|^2 = |\vec{a} + \vec{b}|^2$. What is the usual name of this result?

b. If $\vec{a}$ and $\vec{b}$ are not perpendicular, and $\vec{a} - \vec{b} = \vec{c}$, express $|\vec{c}|^2$ in terms of $\vec{a}$ and $\vec{b}$. What is the usual name of this result?

Part C

Communication

21. If the dot product of $\vec{a}$ and $\vec{b}$ is equal to the dot product of $\vec{a}$ and $\vec{c}$, this does not necessarily mean that $\vec{b}$ equals $\vec{c}$. Show why this is so

a. by making an algebraic argument

b. by drawing a geometrical diagram

22. Find a unit vector that is parallel to the *xy*-plane and perpendicular to the vector $4\hat{i} - 3\hat{j} + \hat{k}$.

23. Three vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ satisfy $\vec{x} + \vec{y} + \vec{z} = \vec{0}$. Calculate the value of $\vec{x} \bullet \vec{y} + \vec{y} \bullet \vec{z} + \vec{z} \bullet \vec{x}$, if $|\vec{x}| = 2$, $|\vec{y}| = 3$, and $|\vec{z}| = 4$.

24. A body diagonal of a cube is a line through the centre joining opposite vertices. Find the angles between the body diagonals of a cube.

Thinking/Inquiry/Problem Solving

25. a. Under what conditions is $(\vec{a} + \vec{b}) \bullet (\vec{a} - \vec{b}) = 0$? Give a geometrical interpretation of the vectors $\vec{a}$, $\vec{b}$, $\vec{a} + \vec{b}$, and $\vec{a} - \vec{b}$.

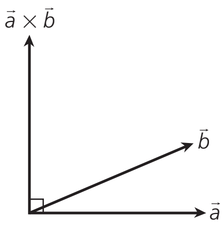
b. Use the dot product to show that two vectors, which satisfy the equation $|\vec{u} + \vec{v}| = |\vec{u} - \vec{v}|$, must be perpendicular. How is the figure defined by $\vec{u}$ and $\vec{v}$ related to the figure defined by $\vec{a}$ and $\vec{b}$ of part **a**?

26. Prove that $|\vec{a} \bullet \vec{b}| \leq |\vec{a}|\,|\vec{b}|$. When does equality hold? Express this inequality in terms of components for vectors in two dimensions and for vectors in three dimensions. (This is known as the Cauchy-Schwarz Inequality.)

Section 5.4 — The Cross Product of Two Vectors

We have already defined the dot product of two vectors, which gives a scalar quantity. In this section we introduce a new product called the **cross product** or **vector product.** The cross product of two vectors $\vec{a}$ and $\vec{b}$ is a vector that is perpendicular to both $\vec{a}$ and $\vec{b}$. Hence, this cross product is defined only in three-dimensional space. The cross product is useful in many geometric and physical problems in three-dimensional space; it is used to help define torque and angular velocity in statics and dynamics, and it is also used in electromagnetic theory. We will use it to find vectors perpendicular to two given vectors.

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be two vectors in three-dimensional space. Let us find all the vectors $\vec{v} = (x, y, z)$ that are perpendicular to both $\vec{a}$ and $\vec{b}$.



These vectors satisfy both
$\vec{a} \bullet \vec{v} = 0$ and $\vec{b} \bullet \vec{v} = 0$.
Hence,

$$a_1x + a_2y + a_3z = 0 \quad ①$$

$$b_1x + b_2y + b_3z = 0 \quad ②$$

We solve these two equations for x, y, and z. Multiply equation ① by b_3, and equation ② by a_3 to obtain

$$a_1b_3x + a_2b_3y + a_3b_3z = 0 \quad ③$$

$$a_3b_1x + a_3b_2y + a_3b_3z = 0 \quad ④$$

Now eliminate z by subtracting equation ③ from equation ④ to obtain

$$(a_3b_1 - a_1b_3)x + (a_3b_2 - a_2b_3)y = 0$$

This is equivalent to

$$\frac{x}{a_2b_3 - a_3b_2} = \frac{y}{a_3b_1 - a_1b_3}$$

Using a similar procedure, we eliminate x from the original equations to obtain

$$\frac{y}{a_3b_1 - a_1b_3} = \frac{z}{a_1b_2 - a_2b_1}$$

Let $y = k(a_3b_1 - a_1b_3)$, for some constant k.

Then $x = k(a_2b_3 - a_3b_2)$ and $z = k(a_1b_2 - a_2b_1)$.

Then the vector $\vec{v}$ perpendicular to both $\vec{a}$ and $\vec{b}$ is of the form
$\vec{v} = (x, y, z) = k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$.
The cross product of $\vec{a}$ and $\vec{b}$ is chosen to be the vector of this form that has $k = 1$.

The Cross Product or Vector Product of $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is the vector

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

This is a rather complicated expression to remember. It can be expressed as follows:

$$\vec{a} \times \vec{b} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

We showed above that any vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$ can be written as $k(\vec{a} \times \vec{b})$. This is one of the most useful properties of the cross product.

Finding a Vector Perpendicular to Two Vectors

If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors in three-dimensional space, then every vector perpendicular to both $\vec{a}$ and $\vec{b}$ is of the form $k(\vec{a} \times \vec{b})$, for $k \in R$.

EXAMPLE 1 Find a vector perpendicular to both (1, 3, 2) and (4, −6, 7).

Solution

The cross product will be one such vector. From the definition of the cross product,

$$\begin{aligned}(1, 3, 2) \times (4, -6, 7) &= (3(7) - 2(-6), 2(4) - 1(7), 1(-6) - 3(4)) \\ &= (33, 1, -18)\end{aligned}$$

Hence, one vector perpendicular to (1, 3, 2) and (4, −6, 7) is (33, 1, −18).

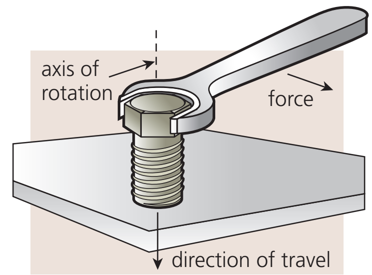
Hint: It is very easy to make errors in calculating a cross product. However, there is an easy check that should always be done after calculating any cross product. If $\vec{v} = \vec{a} \times \vec{b}$, you can always check that $\vec{a} \cdot \vec{v} = 0$ and $\vec{b} \cdot \vec{v} = 0$.

In our example,

$$\begin{aligned}(1, 3, 2) \cdot (33, 1, -18) &= 1(33) + 3(1) + 2(-18) \\ &= 0 \\ (4, -6, 7) \cdot (33, 1, -18) &= 4(33) - 6(1) + 7(-18) \\ &= 0\end{aligned}$$

Hence, (33, 1, −18) is perpendicular to both (1, 3, 2) and (4, −6, 7).

The definition of cross product is motivated by the mechanical act of turning a bolt with a wrench, a process which involves vectors pointing in three different directions.



If, for instance, a bolt with a right-hand thread is turned clockwise, it moves down along the axis of rotation in a direction perpendicular to both the wrench handle and the turning force. This gives a definition for the magnitude of the cross-product vector.

The magnitude of the cross product of two vectors $\vec{a}$ and $\vec{b}$ is

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$

where θ is the angle between the vectors, $0 \leq \theta \leq 180°$.

We will prove that $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$.

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ so that

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

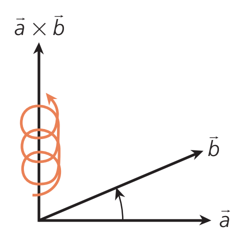
Then $|\vec{a} \times \vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$

By adding and subtracting $(a_1b_1)^2 + (a_2b_2)^2 + (a_3b_3)^2$ to the right side this can be rewritten as

$$\begin{aligned} |\vec{a} \times \vec{b}|^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 \\ &= |\vec{a}|^2|\vec{b}|^2 - (|\vec{a}|\,|\vec{b}| \cos\theta)^2 \\ &= |\vec{a}|^2|\vec{b}|^2 (1 - \cos^2\theta) \\ &= |\vec{a}|^2|\vec{b}|^2 \sin^2\theta \end{aligned}$$

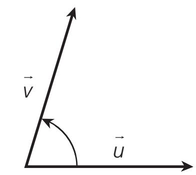
Since $0 \leq \theta \leq 180°$, $\sin\theta \geq 0$, and so $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$.

There are two vectors perpendicular to $\vec{a}$ and $\vec{b}$ with the same magnitude but opposite in direction. The choice of direction of the cross product is such that $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system. Hence, we have the following geometric description of the cross product.

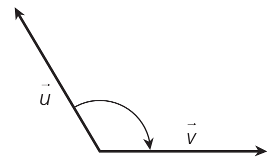


The cross product of the vectors $\vec{a}$ and $\vec{b}$ in three-dimensional space is the vector whose magnitude is $|\vec{a}|\,|\vec{b}| \sin\theta$ and whose direction is perpendicular to $\vec{a}$ and $\vec{b}$, such that $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system.

The direction of the cross product $\vec{u} \times \vec{v}$ of two vectors drawn in a plane can be found by placing your right hand on the diagram so that your fingers curl in the direction of rotation from $\vec{u}$ to $\vec{v}$, through an angle less than 180º. Your thumb points in the direction of $\vec{u} \times \vec{v}$. Try it on the two diagrams below.



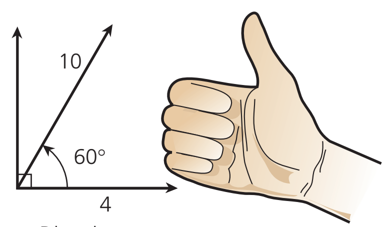
$\vec{u} \times \vec{v}$ points upward



$\vec{u} \times \vec{v}$ points downward

EXAMPLE 2

If $|\vec{u}| = 4$ and $|\vec{v}| = 10$ and the angle θ between $\vec{u}$ and $\vec{v}$ is 60º, find $|\vec{u} \times \vec{v}|$.



Direction up

Solution

$$\begin{aligned} |\vec{u} \times \vec{v}| &= |\vec{u}|\,|\vec{v}| \sin 60^\circ \\ &= (4)(10)\left(\frac{\sqrt{3}}{2}\right) \\ &\cong 34.6 \end{aligned}$$

Then $\vec{u} \times \vec{v}$ has magnitude 34.6 and a direction vertically up from the plane defined by $\vec{u}$ and $\vec{v}$.

EXAMPLE 3

Find the cross product of $\vec{u} = 6\hat{i} - 2\hat{j} - 3\hat{k}$ and $\vec{v} = 5\hat{i} + \hat{j} - 4\hat{k}$.

Solution

By direct substitution,

$$\begin{aligned} \vec{u} \times \vec{v} &= [(-2)(-4) - (-3)(1)]\hat{i} + [(-3)(5) - (6)(-4)]\hat{j} + [(6)(1) - (-2)(5)]\hat{k} \\ &= [(8) - (-3)]\hat{i} + [(-15) - (-24)]\hat{j} + [(6) - (-10)]\hat{k} \\ &= 11\hat{i} + 9\hat{j} + 16\hat{k} \end{aligned}$$

or

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} -2 & -3 \\ 1 & -4 \end{vmatrix}\hat{i} + \begin{vmatrix} -3 & 6 \\ -4 & 5 \end{vmatrix}\hat{j} + \begin{vmatrix} 6 & -2 \\ 5 & 1 \end{vmatrix}\hat{k} \\ &= 11\hat{i} + 9\hat{j} + 16\hat{k} \end{aligned}$$

Properties of the Cross Product

Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors in three-dimensional space and let $t \in R$.

$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ (Anti-commutative Law)

$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (Distributive Law)

$k(\vec{a} \times \vec{b}) = (k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b})$

These properties can be checked by using the definition of the cross product. Notice that the first property means that the cross product is not commutative. For example, $\vec{i} \times \vec{j} = \vec{k}$, but $\vec{j} \times \vec{i} = -\vec{k}$.

Since the result of a cross product is a vector, you may form the dot product or the cross product of this vector with a third vector. The quantity $(\vec{u} \times \vec{v}) \bullet \vec{w}$ is known as the triple scalar product of three vectors, because it is a scalar quantity. The brackets are not really needed to specify the order of operations, because $\vec{u} \times (\vec{v} \bullet \vec{w})$ is meaningless. (Why?) The quantity $(\vec{u} \times \vec{v}) \times \vec{w}$ is a vector and is called the triple vector product. Brackets are required in this expression to specify the order of operations. Both of these quantities arise in the application of vectors to physical and geometrical problems. Some of their properties are investigated in the exercises.

Exercise 5.4

Part A

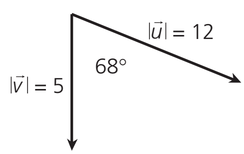
Communication

1. If $\vec{w} = \vec{u} \times \vec{v}$, explain why $\vec{w} \bullet \vec{u}$, $\vec{w} \bullet \vec{v}$, and $\vec{w} \bullet (a\vec{u} + b\vec{v})$ are all zero.

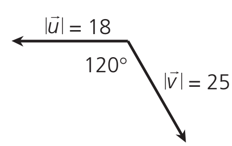
Knowledge/Understanding

2. Find $|\vec{u} \times \vec{v}|$ for each of the following pairs of vectors. State whether $\vec{u} \times \vec{v}$ is directed into or out of the page.

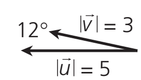
a.



b.



c.



Communication

3. State whether the following expressions are vectors, scalars, or meaningless.

a. $\vec{a} \bullet (\vec{b} \times \vec{c})$ b. $(\vec{a} \bullet \vec{b}) \times (\vec{b} \bullet \vec{c})$ c. $(\vec{a} + \vec{b}) \bullet \vec{c}$

d. $\vec{a} \times (\vec{b} \bullet \vec{c})$ e. $(\vec{a} \times \vec{b}) \bullet (\vec{b} \times \vec{c})$ f. $(\vec{a} + \vec{b}) \times \vec{c}$

g. $\vec{a} \bullet (\vec{b} \bullet \vec{c})$ h. $(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c})$ i. $(\vec{a} \times \vec{b}) - \vec{c}$

j. $\vec{a} \times (\vec{b} \times \vec{c})$ k. $(\vec{a} \bullet \vec{b}) + (\vec{b} \bullet \vec{c})$ l. $(\vec{a} \bullet \vec{b}) - \vec{c}$

Knowledge/Understanding

4. Use the cross product to find a vector perpendicular to each of the following pairs of vectors. Check your answer using the dot product.

a. (4, 0, 0) and (0, 0, 4) b. (1, 2, 1) and (6, 0, 6)

c. (2, −1, 3) and (1, 4, −2) d. (0, 2, −5) and (−4, 9, 0)

Part B

5. Find a unit vector perpendicular to $\vec{a} = (4, -3, 1)$ and $\vec{b} = (2, 3, -1)$.

6. Find two vectors perpendicular to both (3, −6, 3) and (−2, 4, 2).

7. Express the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ as ordered triples and show that
   a. $\hat{i} \times \hat{j} = \hat{k}$.
   b. $\hat{k} \times \hat{j} = -\hat{i}$.

8. Using components, show that
   a. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ for any vectors $\vec{u}$ and $\vec{v}$.
   b. $\vec{u} \times \vec{v} = \vec{0}$, if $\vec{u}$ and $\vec{v}$ are collinear.

9. Prove that $|\vec{a} \times \vec{b}| = \sqrt{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2}$.

10. Given $\vec{a} = (2, 1, 0)$, $\vec{b} = (-1, 0, 3)$, and $\vec{c} = (4, -1, 1)$, calculate the following triple scalar and triple vector products.
    a. $\vec{a} \times \vec{b} \cdot \vec{c}$
    b. $\vec{b} \times \vec{c} \cdot \vec{a}$
    c. $\vec{c} \times \vec{a} \cdot \vec{b}$
    d. $(\vec{a} \times \vec{b}) \times \vec{c}$
    e. $(\vec{b} \times \vec{c}) \times \vec{a}$
    f. $(\vec{c} \times \vec{a}) \times \vec{b}$

11. By choosing $\vec{u} = \vec{v}$, show that $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$. This means that, in general, the cross product is not associative.

Application

12. Given two non-collinear vectors $\vec{a}$ and $\vec{b}$, show that $\vec{a}$, $\vec{a} \times \vec{b}$, and $(\vec{a} \times \vec{b}) \times \vec{a}$ are mutually perpendicular.

Thinking/Inquiry
Problem Solving

13. Prove that the triple scalar product of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ has the property that $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$. Carry out the proof by expressing both sides of the equation in terms of components of the vectors.

Part C

Communication

14. If the cross product of $\vec{a}$ and $\vec{b}$ is equal to the cross product of $\vec{a}$ and $\vec{c}$, this does not necessarily mean that $\vec{b}$ equals $\vec{c}$. Show why this is so
    a. by making an algebraic argument.
    b. by drawing a geometrical diagram.

Thinking/Inquiry
Problem Solving

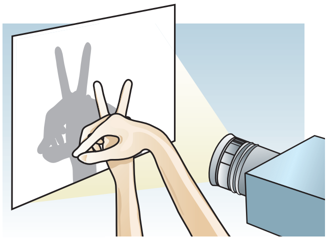
15. a. If $\vec{a} = (1, 3, -1)$, $\vec{b} = (2, 1, 5)$, $\vec{v} = (-3, y, z)$, and $\vec{a} \times \vec{v} = \vec{b}$, find y and z.
    b. Find *another* vector $\vec{v}$ for which $\vec{a} \times \vec{v} = \vec{b}$.
    c. Explain why there are infinitely many vectors $\vec{v}$ for which $\vec{a} \times \vec{v} = \vec{b}$.

Section 5.5 — Applications of Dot and Cross Products

In this section, we will apply the dot product and the cross product to problems in geometry and physics.

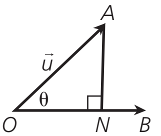
Projections

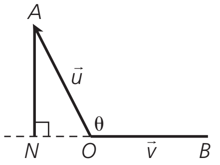
Mathematically, a **projection** is formed by *dropping* a perpendicular from each of the points of an object onto a line or plane. The shadow of an object, in certain circumstances, is a physical example of a projection.



The projection of one vector onto another can be pictured as follows. In the given diagram, where $\vec{u} = \overrightarrow{OA}$ and $\vec{v} = \overrightarrow{OB}$, the projection of $\vec{u}$ onto $\vec{v}$ is the vector $\overrightarrow{ON}$. There is no special symbol for a projection. In this text, we use the notation

$$\overrightarrow{ON} = \text{Proj}(\vec{u} \text{ onto } \vec{v}).$$





The magnitude of $\overrightarrow{ON}$ is given by

$$\left| |\vec{u}|\cos\theta \right| = \frac{\left| |\vec{u}|\cos\theta \right| |\vec{v}|}{|\vec{v}|} = \frac{\left| |\vec{u}||\vec{v}|\cos\theta \right|}{|\vec{v}|} = \frac{|\vec{u} \cdot \vec{v}|}{|\vec{v}|}$$

As you can see from the given diagrams, the direction of $\overrightarrow{ON}$ is the same as the direction of $\vec{v}$ when θ is acute, and opposite to $\vec{v}$ when θ is obtuse. The sign of cos θ in the dot product takes care of both possibilities. Therefore,

The projection of $\vec{u}$ onto $\vec{v}$ is

$\text{Proj}(\vec{u} \text{ onto } \vec{v}) = \dfrac{(\vec{u} \cdot \vec{v})\vec{v}}{|\vec{v}|^2}$.

Its magnitude is $\dfrac{|\vec{u} \cdot \vec{v}|}{|\vec{v}|}$.

EXAMPLE 1 Find the projection of $\vec{u} = (5, 6, -3)$ onto $\vec{v} = (1, 4, 5)$.

Solution

First, we calculate $\vec{u} \cdot \vec{v}$ and $|\vec{v}|^2$.

$$\vec{u} \cdot \vec{v} = (5, 6, -3) \cdot (1, 4, 5) = 14$$

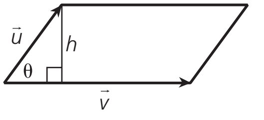
$$|\vec{v}|^2 = 1^2 + 4^2 + 5^2 = 42$$

Therefore, $\text{Proj}(\vec{u} \text{ onto } \vec{v}) = \dfrac{(\vec{u} \cdot \vec{v})\vec{v}}{|\vec{v}|^2}$

$$= \frac{14(1, 4, 5)}{42}$$

$$= \left(\frac{1}{3}, \frac{4}{3}, \frac{5}{3}\right)$$

Area of a Parallelogram



The area of a parallelogram is the product of its base and its height: $A = bh$. The base of the parallelogram in the given diagram is $|\vec{v}|$ and its height is equal to $|\vec{u}| \sin \theta$. Its area is therefore $A = |\vec{v}|\,|\vec{u}| \sin \theta$, which you will recognize to be the magnitude of the cross product of the two vectors $\vec{u}$ and $\vec{v}$ that make up the sides of the parallelogram.

The area of a parallelogram having $\vec{u}$ and $\vec{v}$ as sides is $A = |\vec{u} \times \vec{v}|$.

EXAMPLE 2 Find the area of the triangle with vertices $P(7, 2, -5)$, $Q(9, -1, -6)$, and $R(7, 3, -3)$.

Solution

Start by finding the vectors that form two sides of this triangle. The area of the triangle is half the area of the parallelogram having these vectors as sides.

Vectors $\overrightarrow{PQ} = (2, -3, -1)$

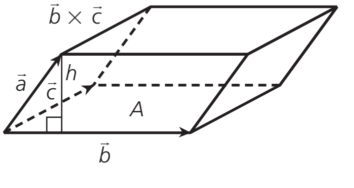
$\overrightarrow{PR} = (0, 1, 2)$

Cross product $\overrightarrow{PQ} \times \overrightarrow{PR} = (2, -3, -1) \times (0, 1, 2) = (-5, -4, 2)$

Magnitude $|(-5, -4, 2)| = \sqrt{(-5)^2 + (-4)^2 + (2)^2} = \sqrt{45}$

The area of the triangle is therefore $\dfrac{\sqrt{45}}{2}$ or $\dfrac{3\sqrt{5}}{2}$.

Volume of a Parallelepiped



A **parallelepiped** is a box-like solid, the opposite faces of which are parallel and congruent parallelograms. Its edges are three non-coplanar vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

The volume V of a parallelepiped, like that of a cylinder, is the area of the base A times the height h, which is measured along a line perpendicular to the base. The area of the base is the area of the parallelogram determined by the vectors $\vec{b}$ and $\vec{c}$:

$$A = |\vec{b} \times \vec{c}|$$

The height is the magnitude of the projection of $\vec{a}$ onto the normal to the base, which is in the direction of $\vec{b} \times \vec{c}$:

$$\begin{aligned} h &= |\text{Proj}(\vec{a} \text{ onto } \vec{b} \times \vec{c})| \\ &= \frac{|\vec{a} \bullet (\vec{b} \times \vec{c})|}{|\vec{b} \times \vec{c}|} \end{aligned}$$

The volume is therefore

$$\begin{aligned} V &= Ah \\ &= |\vec{b} \times \vec{c}| \frac{|\vec{a} \bullet (\vec{b} \times \vec{c})|}{|\vec{b} \times \vec{c}|} \\ &= |\vec{a} \bullet (\vec{b} \times \vec{c})| \end{aligned}$$

In other words, the volume of the parallelepiped is the magnitude of the triple scalar product of the three vectors that make up its edges. Since the volume is a constant, independent of which face is chosen as the base, this result illustrates an important property of the triple scalar product. If $\vec{a} \bullet \vec{b} \times \vec{c} = t$, and t is a constant, then $\vec{b} \bullet \vec{c} \times \vec{a} = \vec{c} \bullet \vec{a} \times \vec{b} = t$,
and $\vec{a} \bullet \vec{c} \times \vec{b} = \vec{c} \bullet \vec{b} \times \vec{a} = \vec{b} \bullet \vec{a} \times \vec{c} = -t$.

Work

In everyday life, the word *work* is applied to any form of activity that requires physical exertion or mental effort. In physics, the word *work* has a much narrower meaning: **work** is done whenever a force acting on an object causes a displacement of the object from one position to another. For instance, work is done by the force of gravity when an object falls, because the force displaces the object from a higher to a lower position. While it might seem like *hard work* to hold a heavy object, if you do not move, you are doing no work.

The work done by a force is defined as the dot product

$$\begin{aligned} W &= \vec{F} \bullet \vec{d} \\ &= |\vec{F}|\ |\vec{d}| \cos \theta \end{aligned}$$

where $\vec{F}$ is the force acting on an object
$\vec{d}$ is the displacement caused by the force
and θ is the angle between the force and displacement vectors.

Work is a scalar quantity. The unit of work is a *joule* (J).

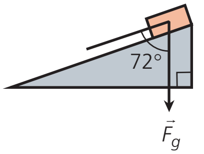
EXAMPLE 3

a. A 25-kg box is located 8 m up a ramp inclined at an angle of 18º to the horizontal. Determine the work done by the force of gravity as the box slides to the bottom of the ramp.

b. Determine the minimum force, acting at an angle of 40º to the horizontal, required to slide the box back up the ramp. (Ignore friction.)

Solution

a. The angle between the displacement down the ramp and the force of gravity is the difference $90º - 18º = 72º$. The force of gravity is $\vec{F}_g = (25 \text{ kg})(9.8 \text{ m/s}^2) = 245\ N$. Therefore, the work done by gravity would be



$$\begin{aligned} W &= \vec{F}_g \cdot \vec{d} \\ &= (245)(8) \cos 72º \\ &\cong 606 \text{ J} \end{aligned}$$

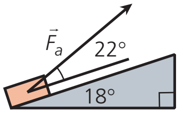
b. The gravitational force acting down the ramp is $245 \cos 72º$.

The applied force acting up the ramp is $|\vec{F}_a| \cos 22º$.

Then $|\vec{F}_a| = \dfrac{245 \cos 72º}{\cos 22º}$

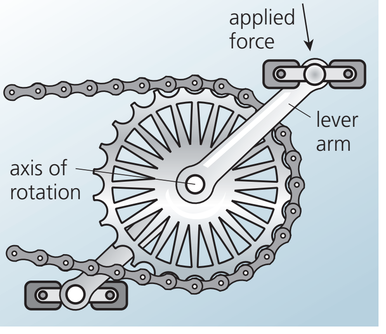
$\doteq 81.7$

The force must exceed 81.7 N.



Torque

Sometimes, instead of causing a change in position, a force causes an object to turn; that is, the force causes an angular rather than a linear displacement. This turning effect of a force is called **torque.**



The force exerted by a cyclist on a bicycle pedal, for example, turns the pedal about an axis. The distance along the shaft of the pedal from the axis of rotation to the point at which the force is applied is known as the **lever arm.** The maximum turning effect occurs when the force is perpendicular to the lever arm.

The torque caused by a force is defined as the cross product

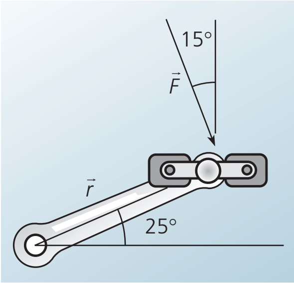
$$\vec{T} = \vec{r} \times \vec{F}$$
$$= |\vec{r}|\,|\vec{F}| \sin\theta\,\hat{n}$$

where $\vec{F}$ **is the applied force**
$\vec{r}$ **is the vector determined by the lever arm acting from the axis of rotation**
θ **is the angle between the force and the lever arm**
and $\hat{n}$ **is a unit vector perpendicular to both** $\vec{r}$ **and** $\vec{F}$

Torque is a vector quantity. It is measured in units of newton metres (N-m).

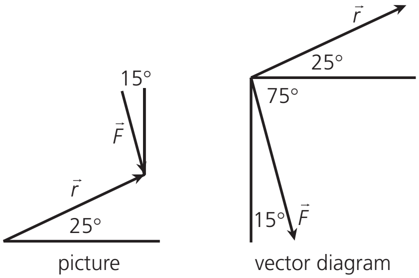
EXAMPLE 4

Find the torque produced by a cyclist exerting a force of 115 N on a pedal in the position shown in the diagram, if the shaft of the pedal is 16 cm long.



Solution

As with any problem involving forces, the first step is to change the picture showing where the forces act into a vector diagram. In this case, that means placing the vectors tail to tail and determining the angle between them. This angle, as you can see, is 100º. Therefore, the magnitude of the torque is



$$|\vec{T}| = |\vec{r}|\,|\vec{F}| \sin\theta$$
$$= (0.16)(115)\sin 100^\circ$$
$$\cong 18.1 \text{ N-m}$$

The direction of the torque vector is into the page, as determined by the right-hand rule.

Exercise 5.5

Part A

Knowledge/Understanding

1. For each of the following, find the projection of $\vec{u}$ onto $\vec{v}$ and calculate its magnitude.
   a. $\vec{u} = (2, 5), \vec{v} = (6, 4)$
   b. $\vec{u} = (-2, 4), \vec{v} = (-3, 2)$
   c. $\vec{u} = (3, 6, -2), \vec{v} = (-4, 3, 8)$
   d. $\vec{u} = (27, 11, -4), \vec{v} = (0, 0, 8)$

Communication

2. a. If $\vec{u}$ and $\vec{v}$ are non-zero vectors, but Proj($\vec{u}$ onto $\vec{v}$) $= \vec{0}$, what conclusion can be drawn?
   b. If Proj($\vec{u}$ onto $\vec{v}$) $= \vec{0}$, does it follow that Proj($\vec{v}$ onto $\vec{u}$) $= \vec{0}$? Explain.

3. Find the projection of $\vec{u} = (2, 3, -4)$ onto each of the coordinate axes.

4. Find the projection of $\overrightarrow{PQ}$ onto each of the coordinate axes, where P is the point (2, 3, 5) and Q is the point (−1, 2, 5).

Part B

Application

5. a. Find the projection of an edge of a unit cube onto one of its body diagonals.
   b. Find the projection of a body diagonal of a unit cube onto one of its edges.

6. Calculate the area of the parallelogram with sides consisting of the vectors
   a. $\vec{a} = (1, 2, -2)$ and $\vec{b} = (-1, 3, 0)$
   b. $\vec{c} = (-6, 4, -12)$ and $\vec{d} = (9, -6, 18)$

7. Find the area of the triangle with the given vertices.
   a. (7, 3, 4), (1, 0, 6), and (4, 5, −2)
   b. (1, 0, 0), (0, 1, 0), and (0, 0, 1)

8. Find the volume of the parallelepiped determined by the vectors $\vec{a} = (2, -5, -1)$, $\vec{b} = (4, 0, 1)$, and $\vec{c} = (3, -1, -1)$.

Knowledge/Understanding

9. For each of the following, calculate the work done by a force $\vec{F}$ that causes a displacement $\vec{d}$, if the angle between the force and the displacement is θ.
   a. $|\vec{F}| = 220\ N$, $|\vec{d}| = 15$ m, $\theta = 49°$
   b. $|\vec{F}| = 4.3\ N$, $|\vec{d}| = 2.6$ m, $\theta = 85°$
   c. $|\vec{F}| = 14\ N$, $|\vec{d}| = 6$ m, $\theta = 110°$
   d. $|\vec{F}| = 4000$ kN, $|\vec{d}| = 5$ km, $\theta = 90°$

10. How much work is done in sliding a refrigerator 1.5 m across a kitchen floor against a frictional force of 150 N?

Application **11.** How much work is done by gravity in causing a 30-kg rock to tumble 40 m down a slope at an angle of 52º to the vertical?

12. A pedicab is pulled a distance of 300 m by a force of 110 N applied at an angle of 6º to the roadway. Calculate the work done.

13. How much work is done against gravity by a workman carrying a 8-kg sheet of plywood up a 3-m ramp inclined at an angle of 20º to the horizontal?

14. A 35-kg trunk is dragged 10 m up a ramp inclined at an angle of 12º to the horizontal by a force of 90 N applied at an angle of 20º to the ramp. At the top of the ramp, the trunk is dragged horizontally another 15 m by the same force. Find the total work done.

15. For each of the following, find the work done by a force $\vec{F}$ that causes a displacement $\vec{d}$.
   a. $\vec{F} = 2\hat{i}$, $\vec{d} = 5\hat{i} + 6\hat{j}$
   b. $\vec{F} = 4\hat{i} + \hat{j}$, $\vec{d} = 3\hat{i} + 10\hat{j}$
   c. $\vec{F} = (800, 600)$, $\vec{d} = (20, 50)$
   d. $\vec{F} = 12\hat{i} - 5\hat{j} + 6\hat{k}$, $\vec{d} = -2\hat{i} + 8\hat{j} - 4\hat{k}$

16. If a 10-N force, acting in the direction of the vector (1, 1), moves an object from $P(-2, 1)$ to $Q(5, 6)$, calculate the work done. The distance is in metres.

17. Find the work done by a 30-N force acting in the direction of the vector $(-2, 1, 5)$, which moves an object from $A(2, 1, 5)$ to $B(3, -1, 2)$. The distance is in metres.

Application **18.** A 50-N force is applied to the end of a 20-cm wrench and makes an angle of 30º with the handle of the wrench.
   a. What is the torque on a bolt at the other end of the wrench?
   b. What is the maximum torque that can be exerted by a 50-N force on this wrench and how can it be achieved?

Part C

19. Under what circumstances is
   a. $\text{Proj}(\vec{u} \text{ onto } \vec{v}) = \text{Proj}(\vec{v} \text{ onto } \vec{u})$?
   b. $|\text{Proj}(\vec{u} \text{ onto } \vec{v})| = |\text{Proj}(\vec{v} \text{ onto } \vec{u})|$?

Review Exercise

1. Express each vector in the form $a\hat{i} + b\hat{j} + c\hat{k}$.

   a. (1, 3, 2) b. (1, 0, 5) c. (−6, −8, 11) d. (9, −6, 2)

2. Express each vector in the form (a, b, c).

   a. $3\hat{i} - 2\hat{j} + 7\hat{k}$ b. $-9\hat{i} + 3\hat{j} + 14k$

   c. $\hat{i} + \hat{j}$ d. $2\hat{i} - 9\hat{k}$

3. a. Find the dot product of the two vectors $\vec{u}$ and $\vec{v}$ where $\vec{u} = (3, -4, 1), \vec{v} = (2, 1, -5)$.

   b. Find the angle between $\vec{u}$ and $\vec{v}$.

4. Find a vector that is perpendicular to both of the vectors $\vec{a} = (1, 2, 4)$ and $\vec{b} = (0, 3, -2)$.

5. Expand $(\vec{a} + \vec{b}) \bullet (\vec{a} - \vec{b})$. Write your answer in the simplest form.

6. Expand $(\vec{a} + \vec{b}) \bullet (\vec{c} + \vec{d})$.

7. The cosine of the angle between $\vec{a}$ and $\vec{b}$ is $\frac{4}{21}$. Find p, if $\vec{a} = 6\hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = -2\hat{i} + p\hat{j} - 4\,\hat{k}$.

8. Find λ so that the vectors $\hat{i} + \hat{j} + \hat{k}$ and $\lambda^2\hat{i} - 2\lambda\hat{j} + \hat{k}$ are perpendicular.

9. Calculate the dot product of $4\vec{x} - \vec{y}$ and $2\vec{x} + 3\vec{y}$, if $|\vec{x}| = 3$, $|\vec{y}| = 4$, and the angle between $\vec{x}$ and $\vec{y}$ is 60º.

10. A vector $\vec{u}$ with direction angles α_1, β_1, and γ_1 is perpendicular to a vector $\vec{v}$ with direction angles α_2, β_2, and γ_2. Prove that

    $\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0$.

11. Show that $\vec{x} \bullet \vec{y} = \frac{1}{2}(|\vec{x} + \vec{y}^2| - |\vec{x}|^2 - |\vec{y}|^2)$.

12. A triangle has vertices $A(-1, 3, 4)$, $B(3, -1, 1)$, and $C(5, 1, 1)$.

    a. Show that the triangle is right-angled.

    b. Calculate the area of triangle ABC.

c. Calculate the perimeter of triangle ABC.

d. Determine the fourth vertex needed to complete a rectangle.

13. Find the projection of $\vec{u} = (17, -3, 8)$

a. onto each of the coordinate axes

b. onto each of the coordinate planes

14. Use the cross product to find the area of the triangle whose vertices all lie in the xy-plane at coordinates $A(-7, 3, 0)$, $B(3, 1, 0)$ and $C(2, -6, 0)$.

15. A regular tetrahedron has one vertex at the origin, one vertex at (0, 1, 0), and one vertex, with a positive x-coordinate, on the xy-plane.

a. Find the coordinates of the four vertices.

b. Find the coordinates of the centroid of the tetrahedron.

c. How far is the centroid from each vertex?

16. For any vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, show that

a. $(\vec{a} \times \vec{b}) \times \vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$

b. $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \bullet \vec{c})\vec{b} - (\vec{b} \bullet \vec{c})\vec{a}$

17. Find the volume of the tetrahedron with vertices (1, 1, 2), (3, −4, 6), (−7, 0, −1), and (−1, 5, 8).

Rich Learning Link wrap-up

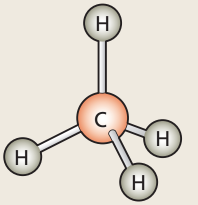
investigate and apply

CHAPTER 5: MOLECULAR BOND ANGLES

The geometry of a molecule is one factor in determining its properties. Bond angles are one quantitative aspect of molecular geometry. The term *valence electrons* refers to those electrons that are most weakly bound to an atom and are, therefore, involved in the formation of chemical bonds. A theory about the relationship between valence electrons and angles of chemical bonds was proposed in 1939 by N. V. Sidgwick and H. M. Powell. They theorized that bonds tend to keep as far apart as possible.

Investigate and Apply

Methane (CH_4) consists of four hydrogen atoms bonded to a single carbon atom. The hydrogen atoms are all 1.095×10^{-11} metres from the carbon atom, and they are distributed evenly in three dimensions to be as far apart as possible. The resulting shape for the four hydrogen atoms is called a regular tetrahedron. The carbon atom is located at the centre of the regular tetrahedron. Other regular tetrahedral molecules include SiH_4, GeH_4, and SnH_4. They are different sizes, but they all have the hydrogen atoms evenly distributed.



Methane

One way to define a regular tetrahedron in three-dimensional space is to connect the four vertices at (0, 0, 0), (1, 1, 0), (1, 0, 1), and (0, 1, 1).

1. Verify that the four points listed are all equidistant from each other.
2. Draw a three-dimensional coordinate system and then draw the vertices and edges of the given regular tetrahedron.
3. Find the centre of the given regular tetrahedron. *Hint:* Its three coordinates are all equal, and it is equidistant from each vertex.
4. Use dot product methods to find the angle formed between any two vectors extending from the centre of the regular tetrahedron to two of its vertices.
5. Why is your answer to question **4** the bond angle in CH_4, SiH_4, GeH_4, SnH_4, and any other regular tetrahedral molecule?

INDEPENDENT STUDY

What are the bond angles in tetrahedral molecules such as CH_3Cl, CH_3Br, and BrO_3F, whose shapes are not regular tetrahedrons? Explain the differences.

What other methods do chemists use to determine bond angles?

What are other quantitative aspects of molecules that chemists measure and use?

Chapter 5 Test

Achievement Category	Questions
Knowledge/Understanding	2, 3
Thinking/Inquiry/Problem Solving	7
Communication	1
Application	4–6

1. What can you conclude about the vectors $\vec{u}$ and $\vec{v}$ if

   a. $\vec{u} \bullet \vec{v} = 0$ b. $\vec{u} \bullet \vec{v} = |\vec{u}|\,|\vec{v}|$ c. $\vec{u} \times \vec{v} = \vec{0}$

   d. $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}|$ e. $(\vec{u} \times \vec{v}) \bullet \vec{u} = 0$ f. $(\vec{u} \times \vec{v}) \times \vec{u} = \vec{0}$

2. Given $\vec{u} = 6\hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{v} = -3\hat{i} + 4\hat{j} + \hat{k}$, find

   a. $4\vec{u} - 3\vec{v}$ b. $\vec{u} \bullet \vec{v}$

   c. $\vec{u} \times \vec{v}$ d. a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$

3. a. Draw x-, y-, and z-axes and make a sketch of

   i. the position vector $\overrightarrow{OP}$ of the point $P(3, -2, 5)$

   ii. the projection of $\overrightarrow{OP}$ onto the z-axis

   iii. the projection of $\overrightarrow{OP}$ onto the xy-plane

   b. Determine the magnitudes of the projections in **a**, parts **ii** and **iii.**

4. A parallelogram $ABCD$ has vertices $A(-1, 2, -1)$, $B(2, -1, 3)$, and $D(-3, 1, -3)$. Determine

   a. the coordinates of C b. the angle at A c. the area

5. A box is dragged 16 m across a level floor by a 75-N force at an angle of 35º to the floor. It is then dragged by the same force 8 m up a ramp inclined at an angle of 20º to the floor. Determine the total work done by the force.

6. A force of 50 N acts at the end of a wrench 18 cm long.

   a. In what direction should the force act to produce the maximum torque? (Draw a diagram.)

b. What is the maximum torque? (State both magnitude and direction.)

c. At what angle will the force produce half the maximum torque? Indicate this angle on your diagram.

7. Use the dot product to find an expression for the cosine of the acute angle between the diagonals of a rectangle with sides $\vec{a}$ and $\vec{b}$.

Extending and Investigating

VECTORS IN COMMUNICATION

You have seen examples of geometric and algebraic vectors. A third common type of vector is the 0 - 1 string vector. Such vectors are composed of 0s and 1s in a row. A vector of length four might be 0010, or 1010, or 1111, and so on. Such vectors, because they correspond to current on (1) or current off (0), are of great value in computer communication.

Using strings of length 5 allows for the creation of 32 strings:
00000, 00001, 00011, ..., 11111.

Satisfy yourself that there are 32 length-5 strings.

Using 26 of these we can define the alphabet as follows:

A: 00001	G: 00111	M: 01101	S: 10011	Y: 11001
B: 00010	H: 01000	N: 01110	T: 10100	Z: 11010
C: 00011	I: 01001	O: 01111	U: 10101	
D: 00100	J: 01010	P: 10000	V: 10110	
E: 00101	K: 01011	Q: 10001	W: 10111	
F: 00110	L: 01100	R: 10010	X: 11000	

If you are familiar with binary numbers, you will note that this is a simple assignment of the number 1 in a five-digit display to represent A, the number 2 to represent B, and so on. The first advantage gained is that this allows the simple transmission of messages.

Using this system, the message *This is clever* can be transmitted as
1010001000010011001101001100110001101100001011011000101100 10.

Note that it is up to the receiver to create words from a string of letters and that, unless one of the unused strings is designated for the purpose, there is no punctuation.

The weakness of this system is that messages such as this are easily intercepted. They are not very secure if privacy is desired. This problem is, surprisingly, easily overcome by defining an arithmetic of addition (and subtraction) which can effectively *hide* the message. This provides the foundation of *cryptography*, the art of secret messages.

We define addition as follows:

a. $0 + 0 = 0$ $\quad$ $1 + 0 = 1$
$\quad$ $0 + 1 = 1$ $\quad$ $1 + 1 = 0$

b. There is no "carrying" from column to column.

The only surprise in the addition process is $1 + 1 = 0$. There are two reasons for this. The first is that adding five-digit strings will always give a five-digit result. The second is due to electronic properties. In a room with light switches at both ends of the room, the lights are off if both switches are off (0 position) OR if both switches are on (1 position). Hence, the addition works easily in electronic form.

Using this definition, we illustrate addition:

10010	01101
<u>11011</u>	<u>01001</u>
01001	00100

In the first example, 10010 (R) now looks as though it is 01001 (I), and a person intercepting the message will have a difficult time in determining the true message.

By adding a five-digit *key,* messages can be encrypted and interceptors cannot decode the transmitted message. For example, using the key 11101 for each letter, the word *MATH* becomes:

Key	01101000011010001000	
Add	<u>11101111011110111101</u>	
	10000111000100110101	(Send this)

An interceptor would translate this to P?IU and be confused. A person receiving the message and knowing the key merely reverses the process, as follows:

	10000111000100110101
Key	<u>11101111011110111101</u>
	01101000011010001000

Note that subtraction is identical with addition. $\begin{bmatrix} 0-0=0, 1-1=0, \\ 1-0=1, 0-1=1 \end{bmatrix}$

The message is retrieved!

You can easily create messages in code. For interest, work with two friends and try the following:

1. Choose a key known to two but not the third.

2. Encrypt a message and give it to your friends.

3. The person knowing the key will be able to decrypt the message. It will be a real challenge for the remaining person to do so.

Discussion

You can increase the complexity of the deciphering, by using vectors of length 6, 7, or 8. Discuss the effect of doing so.

In the next chapter you will see how seven-digit vectors constructed using algebraic vector properties can be used to build codes that can be corrected when errors occur.

Chapter 6

LINEAR COMBINATIONS

We have seen that if a vector $\vec{v}$ is multiplied by a scalar k, the result is a new vector $\overrightarrow{kv}$, parallel to the first but with a different magnitude. This property is maintained in two- or three-dimensional space, but suppose we move to four-dimensional space or beyond. Does the concept *parallel* mean anything? In two- or three-dimensional space we can add two non-parallel vectors $\vec{a}$ and $\vec{b}$ and form a new vector $\vec{c} = \vec{a} + \vec{b}$. This new vector $\vec{c}$ lies in the same plane as $\vec{a}$ and $\vec{b}$ but, again, what does "lie in the same plane" mean in four-dimensional space? In this chapter, we will answer these questions as well as look at the use of vectors as another tool in the world of proofs.

CHAPTER EXPECTATIONS In this chapter, you will

- perform mathematical operations on geometric vectors, **Section 6.1**
- determine the components and projection of a geometric vector, **Section 6.2, 6.3**
- represent Cartesian vectors, **Section 6.1, 6.3**
- perform mathematical operations on Cartesian vectors, **Section 6.2, 6.3**
- determine equations of lines in two- and three-dimensional space, **Section 6.4**
- prove some properties of plane figures using vector methods, **Section 6.4**

Review of Prerequisite Skills

One important use of mathematics is to describe a relation between two quantities. We say *the area of a circle depends on its radius*, or *the distance travelled at a constant speed depends on the time*, and we write formulas such as $A = \pi r^2$ or $d = vt$.

In making statements such as these, we are taking one quantity, such as the area or the distance, to be a derived quantity. In some way it is not as fundamental as the other quantity. The derived quantity is referred to as the dependent variable, because its value *depends* on the value of the other variable.

It is possible to turn things around and write $r = \sqrt{\frac{A}{\pi}}$ or $t = \frac{d}{v}$, making the radius or the time the dependent quantity. The way a formula is written is a matter of preference or circumstance, but the underlying idea of dependence is still present.

How does this basic idea of dependence apply to vectors? Under what circumstances does one vector depend on another? If vectors are independent, what are the implications? This chapter begins by examining the dependence and independence of vectors and ends with applications of vectors to geometrical proofs.

Rich Learning Link

investigate

CHAPTER 6: VECTOR SPACES

Vectors can be added to each other and they can be multiplied by real numbers. Polynomials also have these two properties, as do a number of other mathematical objects. Instead of studying all these various objects separately, mathematicians have learned much about them by studying what they all have in common. In 1888, the Italian mathematician Guiseppe Peano presented the first definition of what are now called **vector spaces.** Vector spaces provide an abstract system for studying the common properties of many different mathematical objects, including vectors.

Investigate

A vector space is a set of objects, V, called vectors, for which two operations are defined: addition and scalar multiplication. These operations must have the following ten properties:

1. Any two vectors can be added, and the result will always be in V.
2. Addition is commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. Addition is associative: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
4. There is a zero element $\vec{0}$, which has the property that $\vec{0} + \vec{v} = \vec{v}$.
5. Every vector $\vec{v}$ has an opposite $-\vec{v}$, such that $\vec{v} + (-\vec{v}) = \vec{0}$.
6. If a vector is multiplied by a real number, the result is in V.
7. Every vector is unchanged when multiplied by 1.
8. Scalar multiplication is associative: $a(b\vec{v}) = (ab)\vec{v}$.
9. Scalar multiplication distributes over vector addition: $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$.
10. Scalar multiplication distributes over scalar addition: $(a + b)\vec{v} = a\vec{v} + b\vec{v}$.

Verify that the following sets are all vector spaces: (i) the set of all vectors drawn in a plane; (ii) the set of all ordered pairs of real numbers (x, y); (iii) the set of all ordered triples of real numbers (x, y, z); (iv) the set of all polynomials with degree less than or equal to two; (v) the set of all polynomials.

DISCUSSION QUESTIONS

1. Calculate $2^{(3^4)}$ and $(2^3)^4$. Is exponentiation an associative operation on the set of real numbers?
2. The set of all real numbers is itself a vector space. Is the set of all complex numbers a vector space? What about the set of all integers?
3. How can addition and scalar multiplication be defined so that the set of all functions is a vector space?

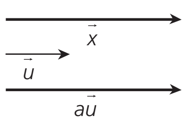
Section 6.1 — Linear Combinations of Vectors

The expression $a\vec{u} + b\vec{v}$, where a and b are scalar quantities, is a **linear combination** of the vectors $\vec{u}$ and $\vec{v}$. A linear combination of three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is written $a\vec{u} + b\vec{v} + c\vec{w}$. These expressions are *linear* because they consist only of the sum of scalar multiples of vectors and nothing else.

In the preceding chapters, it was by means of such expressions that sets of vectors were combined to form new vectors. Recall, that one way to write an algebraic vector was as a linear combination of the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$: $a\hat{i} + b\hat{j} + c\hat{k}$.

This section is concerned with the reverse problem. Under what circumstances can a given vector be expressed as a linear combination of other vectors? That is, when can a vector $\vec{x}$ be expressed in the form $\vec{x} = a\vec{u} + b\vec{v} + \ldots$? How are the coefficients of such an expression to be found? What does it mean if forming such a linear combination proves to be impossible?

We begin the investigation of these questions with the simplest case. The vector $a\vec{u}$ is a linear combination of the single vector $\vec{u}$. Under what circumstances is it possible to express a given vector $\vec{x}$ in the form $a\vec{u}$? The answer is that we write $\vec{x} = a\vec{u}$, and determine a numerical value for the scalar a only when $\vec{x}$ and $\vec{u}$ are collinear. The equation $\vec{x} = a\vec{u}$ implies that the two vectors $\vec{x}$ and $\vec{u}$ are parallel. If $\vec{x}$ and $\vec{u}$ are not collinear, $\vec{x}$ cannot be written as a scalar multiple of $\vec{u}$.



Two vectors $\vec{x}$ and $\vec{u}$ are collinear if and only if it is possible to find a non-zero scalar a such that $\vec{x} = a\vec{u}$.

EXAMPLE 1 If possible, express $\vec{x}$ as a scalar multiple of $\vec{u}$.

a. $\vec{x} = 4\hat{i} - 8\hat{j}$
$\vec{u} = 6\hat{i} - 12\hat{j}$

b. $\vec{x} = (10, -8, 3)$
$\vec{u} = (5, -4, 6)$

Solution

a. Let a be a scalar.

If $\vec{x} = a\vec{u}$

Then $(4\hat{i} - 8\hat{j}) = a(6\hat{i} - 12\hat{j})$

$4\hat{i} - 8\hat{j} = 6a\hat{i} - 12a\hat{j}$

If the vector on the left is equal to the vector on the right, then the coefficients of $\hat{i}$ and $\hat{j}$ on the left must be the same as the corresponding coefficients on the right.

Then $4 = 6a$ and $-8 = -12a$

$a = \frac{2}{3}$ $\qquad$ $a = \frac{2}{3}$

The value for a in the two cases is consistent.
Thus $\vec{x} = \frac{2}{3}\vec{u}$ and $\vec{x}$ and $\vec{u}$ are collinear.

b. Proceed as in **a.**

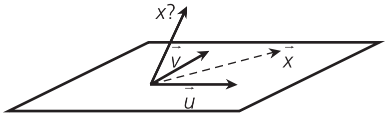
If $\vec{x} = a\vec{u}$,
then $(10, -8, 3) = a(5, -4, 6)$
$(10, -8, 3) = (5a, -4a, 6a)$.

Equating corresponding components,

$10 = 5a$ and $-8 = -4a$ and $3 = 6a$

$a = 2$ $\qquad$ $a = 2$ $\qquad$ $a = \frac{1}{2}$

Since the scalar a must be the same for all components of the vectors, there is no scalar multiple of $\vec{u}$ which equals $\vec{x}$. These two vectors are non-collinear.



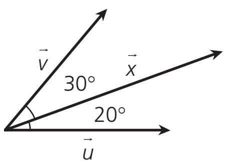
Consider now a more complicated situation. Under what circumstances is it possible to express a given vector $\vec{x}$ in terms of *two* vectors $\vec{u}$ and $\vec{v}$? The answer is that you can write $\vec{x} = a\vec{u} + b\vec{v}$ and determine numerical values for the scalars a and b only when the three vectors $\vec{x}$, $\vec{u}$, and $\vec{v}$ are *coplanar*. The equation $\vec{x} = a\vec{u} + b\vec{v}$ implies that the three vectors $\vec{x}$, $a\vec{u}$, and $b\vec{v}$ form a triangle. If $\vec{x}$ does not lie in the plane of $\vec{u}$ and $\vec{v}$, this triangle cannot exist.

Three vectors $\vec{x}$, $\vec{u}$, and $\vec{v}$ are coplanar if and only if it is possible to find non-zero scalars a and b such that $\vec{x} = a\vec{u} + b\vec{v}$.

EXAMPLE 2

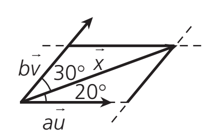
Three vectors $\vec{u}$, $\vec{v}$, and $\vec{x}$, have magnitudes $|\vec{u}| = 10$, $|\vec{v}| = 15$, and $|\vec{x}| = 24$. If $\vec{x}$ lies between $\vec{u}$ and $\vec{v}$ in the same plane, making an angle of 20º with $\vec{u}$ and 30º with $\vec{v}$, express $\vec{x}$ as a linear combination of $\vec{u}$ and $\vec{v}$.

Solution

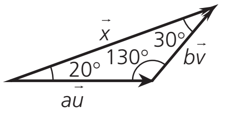


You are to find scalar multiples of $\vec{u}$ and $\vec{v}$, the sum of which equals $\vec{x}$. Let $\vec{x} = a\vec{u} + b\vec{v}$, where a and b are coefficients to be determined.

Make a parallelogram by drawing lines from the tip of $\vec{x}$ in the vector diagram, parallel to $\vec{u}$ and $\vec{v}$. The sides of this parallelogram are the required vectors $a\vec{u}$ and $b\vec{v}$, which add to $\vec{x}$. In this particular case, $a\vec{u}$ is longer than $\vec{u}$, so you should expect a to be greater than one. On the other hand, $b\vec{v}$ is shorter than $\vec{v}$, so b will be less than one.



Now draw the triangle for the addition of the vectors.



From

$$\frac{a|\vec{u}|}{\sin 30^\circ} = \frac{|\vec{x}|}{\sin 130^\circ}$$
$$a = \frac{24 \sin 30^\circ}{10 \sin 130^\circ}$$
$$\cong 1.566$$

and from

$$\frac{b|\vec{v}|}{\sin 20^\circ} = \frac{|\vec{x}|}{\sin 130^\circ}$$
$$b = \frac{24 \sin 20^\circ}{15 \sin 130^\circ}$$
$$\cong 0.714$$

Therefore, $\vec{x} = 1.566\vec{u} + 0.714\vec{v}$, and $\vec{x}$ is expressed as a linear combination of $\vec{u}$ and $\vec{v}$. Note that a is greater than one and b is less than one, as expected.

EXAMPLE 3

Determine whether or not the three vectors $\vec{u} = (3, -1, 4)$, $\vec{v} = (6, -4, -8)$, and $\vec{x} = (7, -3, 4)$ are coplanar. If they are, express $\vec{x}$ as a linear combination of $\vec{u}$ and $\vec{v}$.

Solution

We could calculate the triple scalar product. If it is zero, the vectors are coplanar, but we would be left with the problem of expressing $\vec{x}$ as a linear combination of $\vec{u}$ and $\vec{v}$. Instead, we proceed as follows.

Write $\vec{x} = a\vec{u} + b\vec{v}$. If we can find values of a and b, the vectors are coplanar. If we cannot find values for a and b, the vectors are not coplanar. We have

$$\begin{aligned}(7, -3, 4) &= a(3, -1, 4) + b(6, -4, -8)\\ &= (3a + 6b, -a - 4b, 4a - 8b)\end{aligned}$$

Equating components, we get the system of equations

$$\begin{aligned}3a + 6b &= 7\\ -a - 4b &= -3\\ 4a - 8b &= 4\end{aligned}$$

We solve any two of these equations, say the first and second.

$$\begin{aligned} 3a + 6b &= 7 \\ -3a - 12b &= -9 \\ -6b &= -2 \end{aligned}$$

so $\qquad b = \frac{1}{3}$

Substituting into either of the equations, we find $a = \frac{5}{3}$.

We now see if these values satisfy the third equation

$$\begin{aligned} 4\left(\frac{5}{3}\right) - 8\left(\frac{1}{3}\right) &= \frac{20}{3} - \frac{8}{3} \\ &= 4, \text{ as required} \end{aligned}$$

Therefore the vectors are coplanar, and $\vec{x} = \frac{5}{3}\vec{u} + \frac{1}{3}\vec{v}$.

Exercise 6.1

Part A

Communication

1. Explain why it is impossible to express $\vec{x}$ as a linear combination of $\vec{u}$ and $\vec{v}$, when $\vec{x}$ does not lie in the plane of $\vec{u}$ and $\vec{v}$.

Communication

2. The vectors $\vec{u}$, $\vec{v}$, and $\vec{x}$ are any non-zero vectors in the xy-plane. Is it always possible to express $\vec{x}$ as a linear combination of $\vec{u}$ and $\vec{v}$? Explain.

3. What information does the cross product of $\vec{u}$ and $\vec{x}$ give about the collinearity of $\vec{u}$ and $\vec{x}$?

Knowledge/Understanding

4. Write each of the following vectors as a linear combination of $\hat{i}$ and $\hat{j}$.
   a. the vector $\vec{p} = (-4, 5)$
   b. the position vector of the point $A(8, -3)$
   c. a vector directed at an angle of 45º with a magnitude of $\sqrt{2}$
   d. a vector directed at an angle of 150º with a magnitude of 6

5. a. Can every vector in the xy-plane be written as a linear combination of $\vec{u} = (1, 4)$ and $\vec{v} = (-2, 5)$? Justify your answer.
   b. Write the vector $(-567, -669)$ in terms of $\vec{u}$ and $\vec{v}$.

6. Can every vector in the xy-plane be written as a linear combination of $\vec{u} = (-4, -6)$ and $\vec{v} = (10, 15)$? Justify your answer.

Part B

Knowledge/Understanding

7. Are the following sets of vectors coplanar?

a. $(1, -1, 1), (0, 1, 1), (1, 0, 2)$

b. $(1, 0, 1), (1, 1, 1), (1, 0, -1)$

Application

8. If $\vec{u} = (2, 1, 1)$ and $\vec{v} = (-1, 1, 3)$

a. which of the following vectors can be written in the form $s\vec{u} + t\vec{v}$

(i) $(4, 2, 2)$ (ii) $(1, 2, 4)$ (iii) $(1, 5, 11)$ (iv) $(4, 5, 8)$

b. Find another vector that can be expressed in the form $s\vec{u} + t\vec{v}$.

c. Find another vector which *cannot* be expressed in the form $s\vec{u} + t\vec{v}$, and explain why it cannot.

Application

9. Given that

$$\vec{u} = x\vec{a} + 2y\vec{b}$$
$$\vec{v} = -2y\vec{a} + 3y\vec{b}$$
$$\vec{w} = 4\vec{a} - 2\vec{b}$$

where $\vec{a}$ and $\vec{b}$ are not collinear, find the values of x and y for which $2\vec{u} - \vec{v} = \vec{w}$.

Part C

10. Find values of a, b, and c which satisfy each of the following equations.

a. $a(2, 1, 0) + b(-3, 4, 5) + c(2, 0, 3) = (-4, 10, 7)$

b. $a(3, -1, 2) + b(-1, 1, 3) + c(2, 1, 5) = (2, 5, 16)$

Application

11. a. Demonstrate that the three vectors $\vec{u} = (1, 3, 2)$, $\vec{v} = (1, -1, 1)$, and $\vec{w} = (5, 1, -4)$ are mutually perpendicular.

b. Express each of the vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ as a linear combination of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.

12. If $\vec{u} = (5, -5, 2)$, $\vec{v} = (1, 8, -4)$, and $\vec{w} = (-2, -1, 2)$, express $\vec{x} = (-3, 6, 8)$

a. in terms of $\vec{u}$, $\vec{v}$ and $\vec{w}$

b. in terms of the unit vectors $\hat{u}$, $\hat{v}$, and $\hat{w}$

Thinking/Inquiry/Problem Solving

13. Vectors $\vec{u}$, $\vec{v}$, and $\vec{x}$ in the xy-plane make angles of 20º, 50º, and 130º, respectively, with the x-axis. If $|\vec{u}| = 2$, $|\vec{v}| = 10$, and $|\vec{x}| = 4$

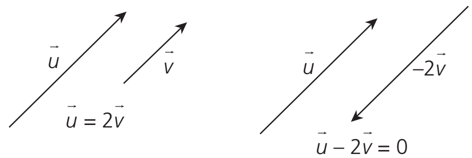
a. express $\vec{x}$ as a linear combination of $\vec{u}$ and $\vec{v}$

b. express $\vec{x}$ as a linear combination of the unit vectors $\hat{u}$ and $\hat{v}$

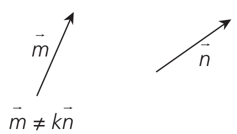
Section 6.2 — Linear Dependence and Independence

The concepts of linear dependence and independence are fundamental in vector algebra. Their importance is both theoretical and practical. Two vectors are linearly dependent if they are collinear or parallel. Three vectors are linearly dependent if they are coplanar.

Let us first consider just two vectors. For example, suppose $\vec{u}$ and $\vec{v}$ are two vectors such that $\vec{u} = 2\vec{v}$. This statement can be rewritten as $\vec{u} - 2\vec{v} = \vec{0}$. Geometrically, this means that multiplying $\vec{v}$ by the scalar -2 and adding it to $\vec{u}$ brings us back to the zero vector $\vec{0}$.



Parallel vectors



Non-parallel vectors

We can see that whenever two vectors $\vec{u}$ and $\vec{v}$ are parallel, there is a relationship $a\vec{u} + b\vec{v} = \vec{0}$, where a and b are not both zero. We then say that $\vec{u}$ and $\vec{v}$ are linearly dependent vectors.

On the other hand, consider two vectors $\vec{m}$ and $\vec{n}$ that are not parallel. There is no possible way to combine multiples of these vectors so that $a\vec{m} + b\vec{n} = \vec{0}$, unless $a = b = 0$. These vectors are called linearly independent vectors.

Linear Dependence of Two Vectors

Two vectors $\vec{u}$ and $\vec{v}$ are called linearly dependent if and only if there are scalars a and b, not both zero, such that

$$a\vec{u} + b\vec{v} = \vec{0}.$$

Two vectors are linearly independent if they are not linearly dependent.

Linear Dependence of Three Vectors

Three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent if and only if there are scalars a , b, and c, not all zero, such that

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}.$$

Three vectors are linearly independent if they are not linearly dependent.

EXAMPLE 1

a. Demonstrate that the three vectors $\vec{u} = (2, 1)$, $\vec{v} = (-1, 7)$, and $\vec{w} = (-4, 3)$ are linearly dependent by showing that it is possible to find coefficients a, b, and c, not all zero, such that the linear combination $a\vec{u} + b\vec{v} + c\vec{w}$ is equal to the zero vector.

b. Express each vector as a linear combination of the other two.

Solution

a. If the vectors are linearly dependent, then there are values a, b, c such that

$$\begin{aligned} a\vec{u} + b\vec{v} + c\vec{w} &= \vec{0} \\ a(2, 1) + b(-1, 7) + c(-4, 3) &= \vec{0} \\ (2a - b - 4c, a + 7b + 3c) &= \vec{0} \end{aligned}$$

The components of the vector on the left must be zero.

$$\begin{aligned} 2a - b - 4c &= 0 \\ a + 7b + 3c &= 0 \end{aligned}$$

Since there are three variables but only two equations, it is not possible to find unique values for a, b, and c.

eliminating a

$$\begin{aligned} 2a - b - 4c &= 0 \\ -2a - 14b - 6c &= 0 \\ -15b - 10c &= 0 \\ \therefore b &= -\tfrac{2}{3}c \end{aligned}$$

and

eliminating b

$$\begin{aligned} 14a - 7b - 28c &= 0 \\ a + 7b + 3c &= 0 \\ 15a - 25c &= 0 \\ \therefore a &= \tfrac{5}{3}c \end{aligned}$$

To avoid fractions, we choose $c = 3$, whereupon $b = -2$ and $a = 5$. Consequently,

$$5(2, 1) - 2(-1, 7) + 3(-4, 3) = \vec{0}$$

or

$$5\vec{u} - 2\vec{v} + 3\vec{w} = \vec{0}$$

Thus there are values of a, b, and c, not all zero, which make the linear combination $a\vec{u} + b\vec{v} + c\vec{w}$ equal to the zero vector. Then the three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent.

b. If $5\vec{u} - 2\vec{v} + 3\vec{w} = \vec{0}$, then

$$\vec{u} = \tfrac{2}{5}\vec{v} - \tfrac{3}{5}\vec{w}, \vec{v} = \tfrac{5}{2}\vec{u} + \tfrac{3}{2}\vec{w}, \qquad \text{and} \qquad \vec{w} = -\tfrac{5}{3}\vec{u} + \tfrac{2}{3}\vec{v}$$

EXAMPLE 2

Prove that three non-collinear vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent if and only if they are coplanar.

Solution

First, suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent, so that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ for some scalars a, b, c, not all zero. Assume that c is one of the non-zero scalars. It is then possible to divide by c and solve for $\vec{w}$.

$$\vec{w} = -\frac{a}{c}\vec{u} - \frac{b}{c}\vec{v}$$

Thus, $\vec{w}$ is expressed as a linear combination of $\vec{u}$ and $\vec{v}$, so $\vec{w}$ must lie in the plane of $\vec{u}$ and $\vec{v}$, making the three vectors coplanar.

Conversely, suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are coplanar. Then $\vec{w}$, for example, can be written as a linear combination of $\vec{u}$, and $\vec{v}$ as $\vec{w} = a\vec{u} + b\vec{v}$. Hence, $a\vec{u} + b\vec{v} - \vec{w} = \vec{0}$. Because the coefficients a, b, and -1 are not all zero, the three vectors must be linearly dependent.

We can always form a linear combination of vectors. For instance, we can always write an equation of the form $a\vec{u} + b\vec{v} = \vec{0}$ for any pair of vectors $\vec{u}$ and $\vec{v}$. If the two vectors are non-collinear, however, the *only* values of a and b that can make the equation true are $a = 0$ and $b = 0$, for then, $0\vec{u} + 0\vec{v} = \vec{0}$.

We can make a similar observation about a linear combination of three vectors. We can always write an equation of the form $a\vec{x} + b\vec{u} + c\vec{v} = \vec{0}$. But if the three vectors are not coplanar, and no two are collinear, then it is impossible to express one of the vectors in terms of the other two. Under these circumstances, the *only* values of a, b, and c that can make the equation true are $a = 0$, $b = 0$, and $c = 0$.

A linear combination of vectors will, of course, equal the zero vector if we set all the coefficients a, b, c, … equal to zero. However, if making coefficients equal zero is the *only* way the linear combination can equal the zero vector, then the vectors *cannot* be dependent. Such vectors are said to be **linearly independent.**

A set of vectors $\vec{u}, \vec{v}, \vec{w}, \vec{x}, \ldots$ is linearly independent if the *only* linear combination $a\vec{u} + b\vec{v} + c\vec{w} + d\vec{x} + \ldots$ that produces the zero vector is the one in which the coefficients $a, b, c, d, \ldots$ are all zero.

EXAMPLE 3 Prove that $\hat{i}$ and $\hat{j}$, the basis vectors for a plane, are linearly independent.

Solution

Start by forming a linear combination of $\hat{i}$ and $\hat{j}$ and setting it equal to zero.

$$a\hat{i} + b\hat{j} = \vec{0}$$

Either $a = 0$ or $a \neq 0$. Suppose that $a \neq 0$. Then $\hat{i} = -\frac{b}{a}\hat{j}$.

If $b \neq 0$, $\hat{i}$ must be a scalar multiple of $\hat{j}$.
If $b = 0$, $\hat{i}$ must equal the zero vector.
Both of these statements are false.

Consequently, the assumption $a \neq 0$ is impossible, so a must be zero. It follows that $0\hat{i} + b\hat{j} = \vec{0}$, which means that $b = 0$, since $\hat{j} \neq \vec{0}$. Thus, the values $a = 0$ and $b = 0$ are the only values of a and b for which the linear combination $a\hat{i} + b\hat{j} = \vec{0}$. Therefore, the vectors $\hat{i}$ and $\hat{j}$ must be linearly independent.

EXAMPLE 4 Show that $\vec{u} = 4\hat{i} + 8\hat{j}$ and $\vec{v} = 6\hat{i} - 3\hat{j}$ are linearly independent.

Solution

For scalars a and b, let

$$\begin{aligned} a\vec{u} + b\vec{v} &= \vec{0} \\ a(4\hat{i} + 8\hat{j}) + b(6\hat{i} - 3\hat{j}) &= \vec{0} \\ 4a\hat{i} + 8a\hat{j} + 6b\hat{i} - 3b\hat{j} &= \vec{0} \\ (4a + 6b)\hat{i} + (8a - 3b)\hat{j} &= \vec{0} \end{aligned}$$

Since $\hat{i}$ and $\hat{j}$ are linearly independent vectors, both coefficients are equal to zero.

$$4a + 6b = 0$$
$$8a - 3b = 0$$

Thus, $a = b = 0$

Then $\vec{u}$ and $\vec{v}$ are linearly independent.

EXAMPLE 5 Given that the vectors $\vec{u}$ and $\vec{v}$ are linearly independent and $(3 - s)\vec{u} + t\vec{v} = 5\vec{u} - 4s\vec{v}$, determine the values of s and t.

Solution

$$\begin{aligned} (3 - s)\vec{u} + t\vec{v} &= 5\vec{u} - 4s\vec{v} \\ (3 - s)\vec{u} - 5\vec{u} + t\vec{v} + 4s\vec{v} &= \vec{0} \\ (-2 - s)\vec{u} + (t + 4s)\vec{v} &= \vec{0} \end{aligned}$$

Since $\vec{u}$ and $\vec{v}$ are given to be linearly independent, the coefficients of $\vec{u}$ and $\vec{v}$ must be zero. Therefore,

$$-2 - s = 0 \qquad \text{and} \qquad t + 4s = 0$$
$$s = -2 \qquad\qquad\qquad t = 8$$

Two vectors are linearly dependent if they are collinear. Any two non-collinear vectors define a plane. Any pair of linearly independent vectors can be designated as basis vectors for a plane. Every vector in the plane can be expressed as a linear combination of these basis vectors. Thus, the vectors $\hat{i}$ and $\hat{j}$ are not the only vectors that could be used as basis vectors for a plane. They are just a particularly simple and convenient choice.

Three vectors are linearly dependent if they are coplanar. Three vectors are coplanar if $(\vec{u} \times \vec{v}) \bullet \vec{w} = 0$, since $(\vec{u} \times \vec{v})$ is perpendicular to the plane of $\vec{u}$ and $\vec{v}$, and if $\vec{w}$ is perpendicular to this cross product, it must lie in the plane of $\vec{u}$ and $\vec{v}$. On the other hand, if the triple scalar product of three vectors is not zero, then the three vectors are non-coplanar. Therefore, they are linearly independent. As a consequence, they can be designated as basis vectors for space, in terms of which every three-dimensional vector can be expressed.

Exercise 6.2

Part A

Communication

1. Given that $\vec{w} = a\vec{u} + b\vec{v}$, what can be said about $\vec{w}$

a. if $\vec{u}$ and $\vec{v}$ are linearly independent?

b. if $\vec{u}$ and $\vec{v}$ are linearly dependent?

Communication

2. a. Are three vectors lying in a plane always linearly dependent? Explain.

b. Given three vectors in a plane, under what circumstances is it impossible to express one of them as a linear combination of the other two?

c. Can any set of three non-collinear vectors be used as a basis for space? Explain.

Knowledge/Understanding

3. If $\vec{u}$ and $\vec{v}$ are linearly independent vectors, find the values of s and t for each of the following equations.

a. $s\vec{u} + 2t\vec{v} = \vec{0}$

b. $(s + 5)\vec{u} + (t - 3)\vec{v} = \vec{0}$

c. $(s - 2)\vec{u} = (s - t - 3)\vec{v}$

d. $s\vec{u} + 7\vec{v} = 5\vec{u} - t\vec{v}$

4. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent vectors, find the values of r, s, and t for each of the following equations.

a. $r\vec{u} + (2s - 1)\vec{v} + (r + s + t)\vec{w} = \vec{0}$

b. $(r - s - 5)\vec{u} + (r + s + 1)\vec{v} + (r + st)\vec{w} = \vec{0}$

Application

5. Given that the vectors $\vec{u}$ and $\vec{v}$ are linearly independent, determine the value of k, if possible, for each of the following equations.

a. $(k + 2)\vec{u} + (k - 2)\vec{v} = \vec{0}$

b. $(6k - 4)\vec{u} + (8 - 12k)\vec{v} = \vec{0}$

c. $(k^2 - 4)\vec{u} + (k + 2)\vec{v} = \vec{0}$

d. $k\vec{u} + 3\vec{v} = \vec{0}$

Part B

6. Given that $\vec{u}$ and $\vec{v}$ are linearly independent vectors and a and b are non-zero scalars, prove that $a\vec{u}$ and $b\vec{v}$ are linearly independent vectors.

7. Show that the representation of a three-dimensional vector in terms of $\hat{i}$, $\hat{j}$, and $\hat{k}$ is unique.

8. Show that the vectors $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$ can be used as a basis for vectors in space.

Thinking/Inquiry/Problem Solving

9. Show that the vectors $\vec{a}$, $\vec{a} \times \vec{b}$, and $\vec{a} \times (\vec{a} \times \vec{b})$ can be used as a basis for vectors in space, where $\vec{a}$ and $\vec{b}$ are any non-collinear vectors.

10. Determine whether the following sets of vectors form bases for two-dimensional space. If a set forms a basis, determine the coordinates of $\vec{v} = (8, 7)$ relative to this base.

a. $\vec{v}_1 = (1, 2)$, $\vec{v}_2 = (3, 5)$

b. $\vec{v}_1 = (3, 5)$, $\vec{v}_2 = (6, 10)$

11. Determine whether the following sets of vectors form bases for three-dimensional space. If a set forms a basis, determine the coordinates of $\vec{v} = (1, 2, 3)$ relative to the basis.

a. $\vec{v}_1 = (-1, 0, 1)$, $\vec{v}_2 = (2, 1, 1)$, $\vec{v}_3 = (3, 1, 1)$

b. $\vec{v}_1 = (1, 3, -1)$, $\vec{v}_2 = (2, 1, 1)$, $\vec{v}_3 = (-4, 3, -5)$

c. $\vec{v}_1 = (1, 0, 0)$, $\vec{v}_2 = (1, 1, 0)$, $\vec{v}_3 = (1, 1, 1)$

Part C

12. If $(3\vec{u} + 4\vec{v})$ and $(6\vec{u} - 2\vec{v})$ are linearly independent vectors, show that $\vec{u}$ and $\vec{v}$ must be linearly independent also.

Thinking/Inquiry/ Problem Solving

13. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are basis vectors for space. $\vec{d}$ is any three-dimensional vector, and $\vec{d} = k\vec{a} + l\vec{b} + m\vec{c}$. Show that this representation of $\vec{d}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is unique.

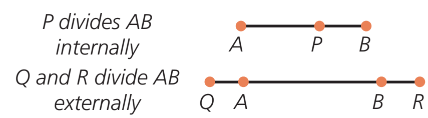
14. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are mutually perpendicular, linearly independent vectors, are the vectors $\vec{u} + \vec{v}$, $\vec{v} + \vec{w}$, and $\vec{w} + \vec{u}$ linearly dependent or linearly independent?

15. The vectors $\vec{u}$ and $\vec{v}$ are linearly independent. Find s, if the vectors $(1 - s)\vec{u} - \frac{2}{3}\vec{v}$ and $3\vec{u} + s\vec{v}$ are parallel.

Section 6.3 — Division of a Line Segment

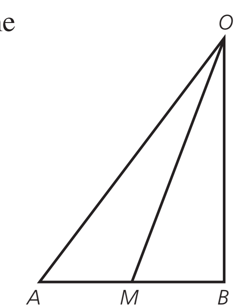
Two points determine a straight line. If a third point lies on this line, the three points are said to be collinear. When three points, A, B, and P are collinear, two of the points, say A and B, are taken as the end points of a line segment. The third point P is called a division point of the segment.

When P lies between A and B on the line segment, then P is said to divide the segment *internally*. When points, such as Q or R in the given diagram, lie on an extension of the segment outside the interval AB, then they are said to divide the segment *externally*.



The midpoint of a line segment is an example of an internal division point. The midpoint M of a segment AB lies between A and B and divides the segment exactly in half. This means that the vectors $\overrightarrow{AM}$ and $\overrightarrow{MB}$ are equal. From this equality, a formula for the position vector of the midpoint can be found.

The position vectors of the points A, B, and M relative to some origin O are, respectively, $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OM}$.



Since $\qquad \overrightarrow{AM} = \overrightarrow{MB}$

Then $\qquad \overrightarrow{OM} - \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{OM}$

$$2\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OB}$$

Therefore, $\qquad \overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$

The position vector of the midpoint is a linear combination of the position vectors of the endpoints of the line segment. The derivation of this midpoint formula is valid for vectors in both two and three dimensions.

EXAMPLE 1 Find the midpoint of the line segment from $A(10, -7, -4)$ to $B(8, 1, -6)$.

Solution

The position vectors of points A and B are $\overrightarrow{OA} = (10, -7, -4)$ and $\overrightarrow{OB} = (8, 1, -6)$. Therefore,

$$\overrightarrow{OM} = \frac{1}{2}[(10, -7, -4) + (8, 1, -6)]$$
$$= (9, -3, -5)$$

The midpoint M is $(9, -3, -5)$.

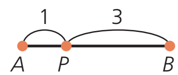
There are several equivalent ways to describe the division point of a line segment. The midpoint, for instance, can be spoken of as a ratio or written as a fraction.

M divides AB in the ratio 1:1 or $\frac{AM}{MB} = \frac{1}{1}$

In a similar manner,

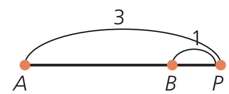
if P divides AB in the ratio 1:3 or $\frac{AP}{PB} = \frac{1}{3}$

then the points must be arranged on the line as in this division-point diagram:



In reading or making a division-point diagram, treat segments going in the same direction as the given segment as positive, and in the opposite direction as negative. Here, the segments AP and PB are both positive.

In the case of an external division of a line segment, such as that shown in this division-point diagram,



P divides AB in the ratio 3:−1 or $\frac{AP}{PB} = \frac{3}{-1}$

You can distinguish an external from an internal division by a negative sign in the ratio or fraction. The negative sign is more conveniently placed on the smaller term of the ratio. In the diagram above, P divides BA in the ratio −1:3 since BP has direction opposite to BA.

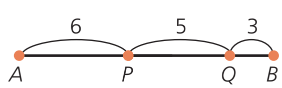
EXAMPLE 2

Points P and Q lie on line segment AB, such that $AP = 6$ units, $PQ = 5$ units, and $QB = 3$ units. In what ratio does

a. P divide AB? b. A divide BP? c. B divide PQ? d. A divide QB?

Solution

Draw a division-point diagram containing the given information:



a. $\frac{AP}{PB} = \frac{6}{8}$, so P divides AB in the ratio 6:8 or 3:4

b. $\frac{BA}{AP} = \frac{14}{-6}$, so A divides BP in the ratio 7:−3

c. $\frac{PB}{BQ} = \frac{8}{-3}$, so B divides PQ in the ratio 8:−3

d. $\frac{QA}{AB} = \frac{-11}{14}$ so A divides QB in the ratio −11:14

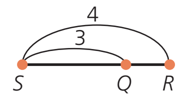
EXAMPLE 3

The points P, Q, R, and S are collinear. S divides QR in the ratio $-3{:}4$. R divides SP in the ratio $5{:}1$. In what ratio does Q divide SP?

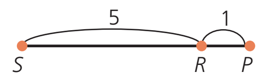
Solution

Write the fractions and draw the division-point diagrams for the two given ratios.

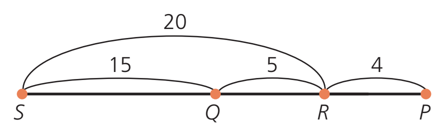
$$\frac{QS}{SR} = \frac{-3}{4}$$



$$\frac{SR}{RP} = \frac{5}{1}$$



The numbers in the ratios are not the actual lengths of the segments. Think of them as the number of equal parts into which the segment has been divided. The points S and R are common to both segments. To compare the ratios, SR must be divided into the same number of parts. Therefore, multiply the first ratio by 5 and the second by 4, and then arrange all the points on a single line.



This makes it clear that QR must contain 5 parts, and therefore that

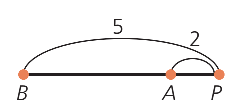
$$\frac{SQ}{QP} = \frac{15}{9}$$

so Q divides SP in the ratio 15:9 or 5:3.

EXAMPLE 4

The point P divides the line segment AB in the ratio $-2{:}5$. Express $\overrightarrow{OP}$ as a linear combination of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

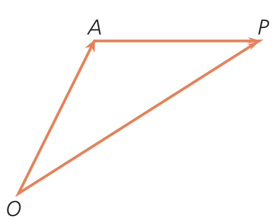
Solution

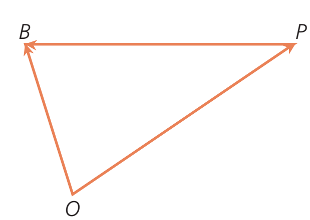


P divides AB in the ratio $-2{:}5$, so $\frac{AP}{PB} = \frac{-2}{5}$.

Therefore,

$$\begin{aligned} \overrightarrow{AP} &= -\tfrac{2}{5}\overrightarrow{PB} \\ 5(\overrightarrow{OP} - \overrightarrow{OA}) &= -2(\overrightarrow{OB} - \overrightarrow{OP}) \\ 5\overrightarrow{OP} - 5\overrightarrow{OA} &= -2\overrightarrow{OB} + 2\overrightarrow{OP} \\ 3\overrightarrow{OP} &= 5\overrightarrow{OA} - 2\overrightarrow{OB} \\ \overrightarrow{OP} &= \tfrac{5}{3}\overrightarrow{OA} - \tfrac{2}{3}\overrightarrow{OB} \end{aligned}$$

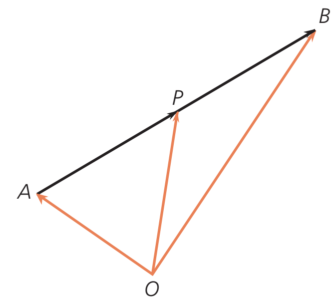




The formula for the position vector of the division point found in Example 4 can be generalized. If P divides AB in some ratio $a{:}b$, then $\frac{AP}{PB} = \frac{a}{b}$.

Therefore,

$$\begin{aligned} \overrightarrow{AP} &= \tfrac{a}{b}\overrightarrow{PB} \\ b(\overrightarrow{OP} - \overrightarrow{OA}) &= a(\overrightarrow{OB} - \overrightarrow{OP}) \\ b\overrightarrow{OP} - b\overrightarrow{OA} &= a\overrightarrow{OB} - a\overrightarrow{OP} \\ (a + b)\overrightarrow{OP} &= b\overrightarrow{OA} + a\overrightarrow{OB} \\ \overrightarrow{OP} &= \frac{b}{a + b}\overrightarrow{OA} + \frac{a}{a + b}\overrightarrow{OB} \end{aligned}$$



Division-Point Theorem

Points A, B, and P are collinear if and only if $\overrightarrow{OP} = \frac{b}{a + b}\overrightarrow{OA} + \frac{a}{a + b}\overrightarrow{OB}$.

This theorem shows that when three points are collinear, their position vectors are linearly dependent and, hence, coplanar.

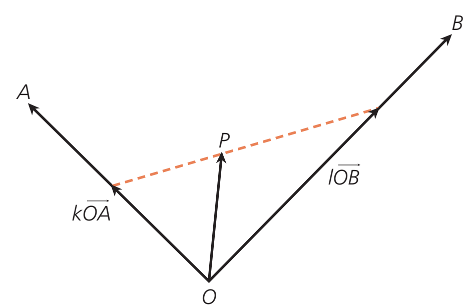
EXAMPLE 5 Prove that if $\overrightarrow{OP} = k\overrightarrow{OA} + l\overrightarrow{OB}$, and $k + l = 1$, this is sufficient to guarantee that the points A, B, and P are collinear.

Solution

Since $k + l = 1, k = 1 - l$

Then
$$\begin{aligned} \overrightarrow{OP} &= (1 - l)\overrightarrow{OA} + l\overrightarrow{OB} \\ &= \overrightarrow{OA} - l\overrightarrow{OA} + l\overrightarrow{OB} \end{aligned}$$

So
$$\begin{aligned} \overrightarrow{OP} - \overrightarrow{OA} &= l(\overrightarrow{OB} - \overrightarrow{OA}) \\ \overrightarrow{AP} &= l\overrightarrow{AB} \end{aligned}$$



Therefore, the vectors $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are parallel, and so A, B, and P are collinear. This proves the *if* part of the division-point theorem. The derivation of the division-point formula shows it is necessary for collinearity, and constitutes a proof of the *only if* part of the theorem.

Exercise 6.3

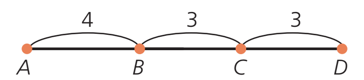
Part A

Communication

1. Can a line segment be divided in the ratio $1:-1$? Explain.

Knowledge/Understanding

2. Points A, B, C, and D are located on a line as shown in the given diagram. Determine



a. the ratio in which C divides AD

b. the ratio in which B divides AD

c. the ratio in which A divides BD

d. the ratio in which D divides AB

e. the ratio in which B divides CD

Knowledge/Understanding

3. Draw a division-point diagram for each of the following statements.

a. point A divides BC in the ratio 2:1

b. point U divides ST in the ratio $3:-1$

c. point Q divides PR in the ratio $-1:2$

d. point K divides MN in the ratio 5:8

e. point D divides EF in the ratio $-2:3$

Part B

4. If the point P divides AB in the ratio 1:2 and the point Q divides AB in the ratio $-1:2$,

a. in what ratio does A divide QB?

b. in what ratio does B divide QP?

c. in what ratio does Q divide AP?

d. in what ratio does P divide QA?

e. in what ratio does B divide PA?

5. If T divides AB in the ratio $2:-1$, prove from first principles that $\overrightarrow{OT} = 2\overrightarrow{OB} - \overrightarrow{OA}$.

6. If $\overrightarrow{OB} = \frac{2}{3}\overrightarrow{OC} + \frac{1}{3}\overrightarrow{OD}$, prove from first principles that B, C, and D are collinear points.

7. Which statements indicate that A, B, and C are collinear points?

a. $\overrightarrow{OA} = \frac{3}{4}\overrightarrow{OB} + \frac{1}{4}\overrightarrow{OC}$

b. $\overrightarrow{OC} = \frac{3}{5}\overrightarrow{OA} + \frac{3}{5}\overrightarrow{OB}$

c. $\overrightarrow{OA} = 5\overrightarrow{OB} - 4\overrightarrow{OC}$

d. $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \vec{0}$

8. In what ratio does P divide AB if

a. $\overrightarrow{OP} = \frac{2}{9}\overrightarrow{OA} + \frac{7}{9}\overrightarrow{OB}$

b. $\overrightarrow{OP} = \frac{-4}{9}\overrightarrow{OA} + \frac{13}{9}\overrightarrow{OB}$

c. $\overrightarrow{OP} = 5\overrightarrow{OA} - 4\overrightarrow{OB}$

d. $\overrightarrow{OP} = \frac{9}{7}\overrightarrow{OA} - \frac{2}{7}\overrightarrow{OB}$

9. Express $\overrightarrow{OA}$ as a linear combination of $\overrightarrow{OB}$ and $\overrightarrow{OC}$, when A divides BC in the given ratios.

a. 3:2 b. −2:3 c. 3:−2

10. Find the midpoint of the line segment joining $A(3, 4, 6)$ to $B(7, 8, -3)$.

Application

11. Find the points that trisect the line segment from $A(3, 6, 8)$ to $B(6, 0, -1)$.

Application

12. $A(2, 10)$ and $B(1, -5)$ are the endpoints of AB. Find the point that divides AB in each of the given ratios.

a. 1:5 b. 2:−1 c. −4:7 d. 3:12

Part C

13. If $\overrightarrow{OE} = \frac{-2}{5}\overrightarrow{OD} + \frac{7}{5}\overrightarrow{OF}$ and $\overrightarrow{OG} = \frac{1}{5}\overrightarrow{OD} + \frac{4}{5}\overrightarrow{OF}$,

a. in what ratio does D divide GE?

b. in what ratio does F divide GE?

Thinking/Inquiry/Problem Solving

14. If P divides AB in the ratio $a:b$, find the ratio of the areas of the triangles OAP and OPB.

Thinking/Inquiry/Problem Solving

15. Prove that if $\overrightarrow{OD} = r\overrightarrow{OA} + s\overrightarrow{OB} + t\overrightarrow{OC}$ and $r + s + t = 1$, the four points A, B, C, and D are coplanar.

Section 6.4 — Vector Proofs in Geometry

We need to be able to solve a problem in several different ways. Sometimes one solution may be more direct and easier than another. If we try different methods of solution, we gain insight into the mathematical principles involved and increase our confidence in the results.

Euclidean proofs are the usual way to establish the properties of geometrical figures. The use of vectors is an alternate way to accomplish the same result.

There are two distinct approaches that can be taken when we use vectors to do proofs. One approach is to use point-to-point vectors. The other is to use position vectors. Point-to-point vectors usually lie in the plane of a figure and join one point of the figure to another. **Position vectors**, on the other hand, point from some outside origin, which is not usually part of the figure, to points in the figure.

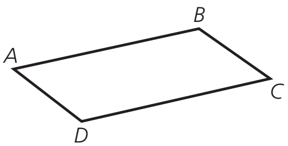
The two methods are illustrated in Examples 1 and 2 below. Remember that there are several things to do before you can actually start a proof. For instance, the proposition to be proved is usually expressed in words, so your first job is to express what is given and what is to be proved in the form of vector formulas or equations. To do this, you will need a suitably labelled diagram.

EXAMPLE 1

Two of the opposite sides of a quadrilateral are parallel and equal in length. Using point-to-point vectors, prove that the other two opposite sides are also parallel and equal in length.

Solution

Let $ABCD$ be a quadrilateral in which $AB = CD$ and $AB \parallel CD$. Using vectors, we write $\overrightarrow{AB} = \overrightarrow{DC}$. Likewise, what is to be proved can be written $\overrightarrow{AD} = \overrightarrow{BC}$.



Then
$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BD}\\ &= \overrightarrow{DC} + \overrightarrow{BD}\\ &= \overrightarrow{BD} + \overrightarrow{DC}\\ &= \overrightarrow{BC}\end{aligned}$$

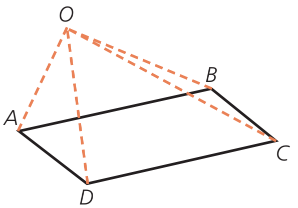
Therefore, if two of the opposite sides of a quadrilateral are parallel and equal, so are the other two opposite sides.

EXAMPLE 2

Two of the opposite sides of a quadrilateral are parallel and equal in length. Using position vectors, prove that the other two opposite sides are also parallel and equal in length.

Solution

Let $ABCD$ be a quadrilateral having $AB = CD$ and $AB \parallel CD$. Let O be an origin that is not in the plane of the quadrilateral.



As in Example 1, $\overrightarrow{AB} = \overrightarrow{DC}$ is given, and $\overrightarrow{AD} = \overrightarrow{BC}$ is to be proved.

Since
$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{DC} \\ \overrightarrow{OB} - \overrightarrow{OA} &= \overrightarrow{OC} - \overrightarrow{OD} \\ \overrightarrow{OD} - \overrightarrow{OA} &= \overrightarrow{OC} - \overrightarrow{OB} \\ \overrightarrow{AD} &= \overrightarrow{BC} \end{aligned}$$

The conclusion is the same as that of Example 1.

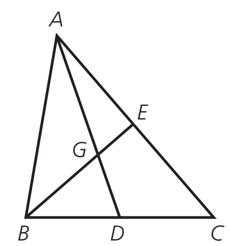
Sometimes a proof using position vectors requires the use of the division-point formula and the concept of linear independence. An example of that kind of proof is shown next.

EXAMPLE 3 Prove that the medians of a triangle intersect at a point that divides each median in the ratio 2:1.

Solution

In ΔABC, D and E are the midpoints of BC and AC, respectively. If O is a point not in the plane of the triangle, then

$$\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}, \quad \text{and} \quad \overrightarrow{OE} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OC}$$



Let $\overrightarrow{OG} = k\overrightarrow{OA} + l\overrightarrow{OD}, \ (k + l = 1)$

Then
$$\begin{aligned} &= k\overrightarrow{OA} + l\left(\frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}\right) \\ &= k\overrightarrow{OA} + \frac{1}{2}l\overrightarrow{OB} + \frac{1}{2}l\overrightarrow{OC} \end{aligned}$$

Similarly
$$\begin{aligned} \overrightarrow{OG} &= m\overrightarrow{OB} + n\overrightarrow{OE}, \ (m + n = 1) \\ &= m\overrightarrow{OB} + n\left(\frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OC}\right) \\ &= \frac{n}{2}\overrightarrow{OA} + m\overrightarrow{OB} + \frac{n}{2}\overrightarrow{OC} \end{aligned}$$

These two expressions for $\overrightarrow{OG}$ must be equal. Therefore,

$$k\overrightarrow{OA} + \frac{l}{2}\overrightarrow{OB} + \frac{l}{2}\overrightarrow{OC} = \frac{n}{2}\overrightarrow{OA} + m\overrightarrow{OB} + \frac{n}{2}\overrightarrow{OC}$$

or
$$\left(k - \frac{n}{2}\right)\overrightarrow{OA} + \left(\frac{l}{2} - m\right)\overrightarrow{OB} + \left(\frac{l}{2} - \frac{n}{2}\right)\overrightarrow{OC} = \vec{0}$$

Since the vertices of the triangle A, B, and C are not collinear, the position vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are not coplanar. Therefore, they are linearly independent vectors. This linear combination can only equal $\vec{0}$ if each of the coefficients separately equals zero:

$$k - \frac{n}{2} = 0, \frac{l}{2} - m = 0, \frac{l}{2} - \frac{n}{2} = 0$$

Since $k - \frac{n}{2} = 0, n = 2k$

Since $\frac{l}{2} - \frac{n}{2} = 0, l = n = 2k$

Now $k + l = 1$, so $k + 2k = 1$, or $k = \frac{1}{3}$

Then $l = \frac{2}{3}$

Then $\overrightarrow{OG} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OD}$, and G divides AD in the ratio 2:1.

Similarly, $\frac{l}{2} - m = 0, l = 2m$

and $\frac{l}{2} - \frac{n}{2} = 0, l = n = 2m$

Since $m + n = 1$, then $m = \frac{1}{3}, n = \frac{2}{3}$

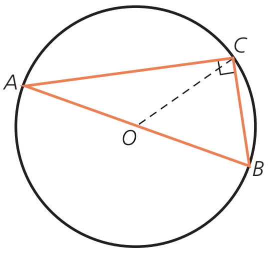
Then $\overrightarrow{OG} = \frac{1}{3}\overrightarrow{OB} + \frac{2}{3}\overrightarrow{OE}$, and G divides BE in the ratio 2:1.

If you repeat this work using $\overrightarrow{AD}$, for instance, and the third median $\overrightarrow{CF}$, the result is the same. So the point of intersection G divides each of the medians in the ratio 2:1. G is called the **centroid** of the triangle.

Another type of problem asks for a proof that two line segments are perpendicular. Such problems are handled by showing that the dot product of the corresponding vectors is zero.

EXAMPLE 4 Prove that an angle inscribed in a semicircle is a right angle.

Solution



Let O be the centre of a circle with diameter AB. Draw angle $\angle ACB$ in the semicircle. This angle is the angle between the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. If the dot product of the two vectors is zero, then $\angle C$ is a right angle.

$$\overrightarrow{CA} \bullet \overrightarrow{CB} = (\overrightarrow{OA} - \overrightarrow{OC}) \bullet (\overrightarrow{OB} - \overrightarrow{OC})$$

OA and OB are both radii and $\overrightarrow{OB} = -\overrightarrow{OA}$.

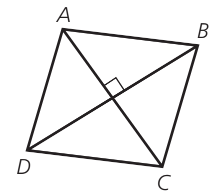
Then
$$\begin{aligned}\overrightarrow{CA} \bullet \overrightarrow{CB} &= (\overrightarrow{OA} - \overrightarrow{OC}) \bullet (-\overrightarrow{OA} - \overrightarrow{OC}) \\ &= -\overrightarrow{OA} \bullet \overrightarrow{OA} - \overrightarrow{OA} \bullet \overrightarrow{OC} + \overrightarrow{OC} \bullet \overrightarrow{OA} + \overrightarrow{OC} \bullet \overrightarrow{OC} \\ &= -\left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{OC}\right|^2 \\ &= 0, \text{ since } \left|\overrightarrow{OA}\right| \text{ and } \left|\overrightarrow{OC}\right| \text{ are radii.}\end{aligned}$$

Therefore, $\angle ACB$ is a right angle.

EXAMPLE 5 If the diagonals of a parallelogram are perpendicular, prove that the parallelogram is a rhombus.

Solution

Draw and label a diagram.



Let $ABCD$ be a parallelogram. Then opposite sides are equal vectors; for instance, $\overrightarrow{AB} = \overrightarrow{DC}$. The diagonals are perpendicular, so

$$\overrightarrow{AC} \bullet \overrightarrow{BD} = 0$$
$$(\overrightarrow{AB} + \overrightarrow{BC}) \bullet (\overrightarrow{BC} + \overrightarrow{CD}) = 0$$
$$(-\overrightarrow{CD} + \overrightarrow{BC}) \bullet (\overrightarrow{BC} + \overrightarrow{CD}) = 0$$
$$-\overrightarrow{CD} \bullet \overrightarrow{BC} - \overrightarrow{CD} \bullet \overrightarrow{CD} + \overrightarrow{BC} \bullet \overrightarrow{BC} + \overrightarrow{BC} \bullet \overrightarrow{CD} = 0$$
$$-|\overrightarrow{CD}|^2 + |\overrightarrow{BC}|^2 = 0$$

Therefore, $|\overrightarrow{BC}| = |\overrightarrow{CD}|$, so adjacent sides are equal and the figure must be a rhombus.

Exercise 6.4

Part A

Communication

1. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length
   a. using point-to-point vectors b. using position vectors

Knowledge/Understanding

2. If side BC of ΔABC is trisected by points P and Q, show that $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AP} + \overrightarrow{AQ}$
   a. using point-to-point vectors b. using position vectors

3. If D, E, and F are the midpoints of the sides of the triangle ABC, prove that $\overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$.

Part B

4. Prove that if the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

5. If G is the centroid of ΔABC and AD is one of its medians,
   a. in what ratio does D divide BC?
   b. in what ratio does G divide AD?
   c. Prove that $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$.
6. If G is the centroid of ΔABC, prove that $\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = \vec{0}$.

Knowledge/Understanding

7. Prove that the diagonals of a parallelogram bisect each other. Use the type of proof shown in Example 3.
8. If a line through the centre of a circle is perpendicular to a chord, prove that it intersects the chord at its midpoint.

Application

9. Show that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the vertices.
10. Prove that the sum of the squares of the diagonals of any parallelogram is equal to the sum of the squares of the four sides.
11. In the trapezoid $ABCD$, $\overrightarrow{AB} = n\overrightarrow{DC}$. If the diagonals BD and AC meet at K, show that
$$\overrightarrow{AK} = \frac{n}{n+1}\overrightarrow{AD} + \frac{1}{n+1}\overrightarrow{AB}$$

Application

12. ΔABC is inscribed in a circle with centre X. Define a point P by its position vector $\overrightarrow{XP} = \overrightarrow{XA} + \overrightarrow{XB} + \overrightarrow{XC}$.
   a. Show that $\overrightarrow{CP} = XA + XB$.
   b. Show that $\overrightarrow{CP} \perp \overrightarrow{AB}$, $\overrightarrow{BP} \perp \overrightarrow{AC}$, and $\overrightarrow{AP} \perp \overrightarrow{BC}$.
   c. Explain why the results of part **b** prove that the three altitudes of a triangle intersect at a common point. (P is known as the **orthocentre** of the triangle.)
13. Let $ABCD$ be a rectangle. Prove that
   a. $\overrightarrow{OA} \cdot \overrightarrow{OC} = \overrightarrow{OB} \cdot \overrightarrow{OD}$
   b. $|\overrightarrow{OA}|^2 + |\overrightarrow{OC}|^2 = |\overrightarrow{OB}|^2 + |\overrightarrow{OD}|^2$

Part C

14. A regular hexagon $ABCDEF$ has two of its diagonals, AC and BE, meeting at the point K. Determine the ratios in which K divides AC and BE.

Thinking/Inquiry/ Problem Solving

15. In a triangle ABC, the point E is selected on BC so that $BE:EC = 1:2$. The point F divides AC in the ratio 2:3. The two line segments BF and AE intersect at D.

a. Find the ratios in which D divides AE and BF.

b. Determine the ratio of the area of the quadrilateral $CEDF$ to the area of the triangle ABC.

16. In the parallelogram $ABCD$, DC is extended to E so that $DE:EC = 3:-2$. The line AE meets BC at F. Determine the ratios in which F divides BC and F divides AE.

17. In the quadrilateral $APBQ$, $|\overrightarrow{AP}| = |\overrightarrow{AQ}|$ and $|\overrightarrow{BP}| = |\overrightarrow{BQ}|$.

a. Prove that AB bisects PQ.

b. Prove that AB is perpendicular to PQ.

Thinking/Inquiry/ Problem Solving

18. Given the tetrahedron $MNPQ$ with $MN \perp PQ$ and $MP \perp NQ$, prove $MQ \perp NP$.

Key Concepts Review

The fundamental concept in this chapter is that of linear independence. If the linear combination $a\vec{u} + b\vec{v} + c\vec{w} + \ldots$ is equal to $\vec{0}$ only if the coefficients a, b, c, … are each zero, then the set of vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, … is linearly independent. You should understand what the implications are when a set of vectors is found to be linearly independent or not.

The question of linear independence is usually approached indirectly by asking if a set of vectors is linearly dependent. That consists of trying to express one of the vectors in the set in terms of the others. If the vectors are not linearly dependent, they must be linearly independent.

Two non-zero vectors $\vec{u}$ and $\vec{v}$ are linearly dependent
- if they are collinear
- if $\vec{u} = k\vec{v}$ with $k \neq 0$, or
- if the cross product $\vec{u} \times \vec{v} = \vec{0}$.

Three non-collinear vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent
- if they are coplanar
- if $\vec{u} = a\vec{v} + b\vec{w}$ where a and b are not both zero, or
- if the triple scalar product $\vec{u} \times \vec{v} \bullet \vec{w} = 0$.

Three vectors in a two-dimensional plane and four vectors in three-dimensional space are always linearly dependent.

The division of a line segment is also connected to the linear independence of vectors. In the linear combination $\overrightarrow{OP} = m\overrightarrow{OA} + n\overrightarrow{OB}$, the three points A, B, and P are collinear and their corresponding position vectors are coplanar only if the coefficients $m + n = 1$.

You should be able to express the division of a line segment AB by a point P in three equivalent ways and to convert readily from one to the other.

P divides AB in the ratio a:b, $\frac{AP}{PB} = \frac{a}{b}$, and $\overrightarrow{OP} = \frac{b}{a+b}\overrightarrow{OA} + \frac{a}{a+b}\overrightarrow{OB}$.

This is not difficult, if you pay attention to the form of the equations and the positions of the letters representing the individual points.

The concept of linear independence and the properties of division points both play a role in vector proofs of geometrical propositions. Two approaches have been illustrated:

1. using point-to-point vectors that lie in the plane of the figure
2. using position vectors from some origin to points in the figure

It should be possible to prove a proposition using either approach. However, one approach may be more difficult than the other, and unfortunately, there is no way to predict this. If you can make no progress using one method, try another.

To learn to carry out proofs successfully, there is no substitute for doing many problems. Start with the simpler proofs. Follow the examples. Expect to work through and write out the logic of a proof several times until you get it right. Persevere.

Review Exercise

1. a. Show that the vectors (2, 3) and (−4, 3) may be used as basis vectors for a plane.

   b. Express (3, −1) as a linear combination of (2, 3) and (−4, 3).

2. Classify the following sets of vectors as being linearly dependent or linearly independent. Give reasons for your answers.

   a. (3, 5, 6), (6, 10, 12), (−3, −5, 6)

   b. (5, 1, −1), (6, −5, −2), (3, 8, −2), (−40, 39, −29)

   c. (7, 8, 9), (0, 0, 0), (3, 8, 6)

   d. (7, −8), (14, 19)

   e. (0, 1, 0), (0, 0, −7), (7, 0, 0)

3. The vectors $\vec{a}$ and $\vec{b}$ are linearly independent. For what values of t are $\vec{c} = t^2\vec{a} + \vec{b}$ and $\vec{d} = (2t - 3)(\vec{a} - \vec{b})$ linearly dependent?

4. If the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent, show that $\vec{a} - 2\vec{b} - \vec{c}$, $2\vec{a} + \vec{b}$, and $\vec{a} + \vec{b} + \vec{c}$ are also linearly independent.

5. For each triangle ABC, determine the midpoints of the sides and the coordinates of the centroid.

   a. $A(0, 0)$, $B(5, -6)$, $C(2, 0)$

   b. $A(4, 7, 2)$, $B(6, 1, -1)$, $C(0, -1, 4)$

6. If $\overrightarrow{OM} = \frac{3}{5}\overrightarrow{ON} + \frac{2}{5}\overrightarrow{OP}$ and $\overrightarrow{OM} = \frac{4}{5}\overrightarrow{ON} + \frac{1}{5}\overrightarrow{OQ}$,

   a. in what ratio does P divide NQ?

   b. in what ratio does Q divide NM?

7. If M divides AB in the ratio 1:7, show from first principles that $\overrightarrow{OM} = \frac{7}{8}\overrightarrow{OA} + \frac{1}{8}\overrightarrow{OB}$.

8. a. Prove from first principles that the points M, N, and Q are collinear if $\overrightarrow{ON} = -\frac{2}{9}\overrightarrow{OM} + \frac{11}{9}\overrightarrow{OQ}$.

   b. Express $\overrightarrow{OM}$ as a linear combination of $\overrightarrow{ON}$ and $\overrightarrow{OQ}$.

9. The point P divides the sides AC of the triangle ABC in the ratio 3:4 and Q divides AB in the ratio 1:6. Let R be the point of intersection of CQ and BP. Determine the ratios in which R divides CQ and BP.

10. In the parallelogram $ABCD$, E divides AB in the ratio 1:4 and F divides BC in the ratio 3:1. The line segments DE and AF meet at K. In what ratio does K divide DE, and in what ratio does K divide AF?

11. Prove that the median to the base of an isosceles triangle is perpendicular to the base.

12. Prove that a line that passes through the centre of a circle and the midpoint of a chord is perpendicular to the chord.

13. Prove that the medians to the equal sides of an isosceles triangle are equal.

14. Prove that the sum of the squares of the diagonals of a quadrilateral is equal to twice the sum of the squares of the line segments joining the midpoints of the opposite sides.

15. In ΔABC, the points D, E, and F are the midpoints of sides BC, CA, and AB, respectively. The perpendicular at E to AC meets the perpendicular at F to AB at the point Q.

    a. Prove that $\overrightarrow{AB} \bullet (\overrightarrow{QD} - \frac{1}{2}\overrightarrow{AC}) = 0$.

    b. Prove that $\overrightarrow{AC} \bullet (\overrightarrow{QD} - \frac{1}{2}\overrightarrow{AB}) = 0$.

    c. Use parts **a** and **b** to prove that $\overrightarrow{CB} \bullet \overrightarrow{QD} = 0$.

    d. Explain why these results prove that the perpendicular bisectors of the sides of a triangle meet at a common point. (Q is called the **circumcentre** of the triangle.)

Rich Learning Link wrap-up

investigate and apply

CHAPTER 6: VECTOR SPACES

When you try to make exactly 49 cents from the coins in your pocket, there are two ways that you might be unsuccessful. You might not have enough coins (you will need at least seven), or you might have the wrong coins (seven dimes will not work). When you try to express a vector as a linear combination of a set of vectors, the same problems can occur; your set may not have enough vectors or it may have unsuitable vectors. In a particular vector space V, a set of vectors B that has just the right number of vectors (no more, no less) of the right type (linearly independent) so that every other vector in V can be written as a linear combination of the vectors in B, is called a **basis** of V.

INVESTIGATE AND APPLY

1. What is the smallest set of coins you would need in order to be able to make any amount of change less than one dollar?
2. a) Show that every ordered pair of real numbers can be written as a linear combination of (1, 0) and (0, 1). So {(1, 0), (0, 1)} is a basis of the set of all ordered pairs. It is called the standard basis of this vector space.

   b) Show that {(2, 3), (1, −1)} is also a basis of the set of all ordered pairs.
3. Give two examples of bases of the set of all ordered triples of real numbers.
4. How many different bases could a particular vector space have? What do they all have in common?
5. a) Find a basis of the vector space of all polynomials with degree less than or equal to two.

   b) Let n be a fixed whole number. Find a basis of the vector space of all polynomials of degree less than or equal to n.

INDEPENDENT STUDY

Investigate: How is the dimension of a vector space defined? What are the dimensions of the vector spaces we have encountered so far? Can vector spaces be infinitely dimensional?

Investigate: Is the set of all matrices of a fixed size a vector space? If it is, what is an example of a basis?

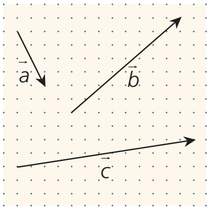
Chapter 6 Test

Achievement Category	Questions
Knowledge/Understanding	3, 4
Thinking/Inquiry/Problem Solving	7, 8
Communication	1, 2
Application	5, 6

1. Three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent. Explain this concept using
   a. an algebraic example
   b. a geometric example

2. P divides QR in the ratio $10:-3$.
   a. Express $\overrightarrow{OP}$ as a linear combination of $\overrightarrow{OQ}$ and $\overrightarrow{OR}$.
   b. Express $\overrightarrow{OR}$ as a linear combination of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

3. a. Copy the three vectors shown in the given diagram onto graph paper and draw $\vec{c}$ accurately as a linear combination of $\vec{a}$ and $\vec{b}$.
   b. Determine values of r and s where $\vec{c} = r\vec{a} + s\vec{b}$.



4. The vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are coplanar and have magnitudes 5, 12, and 18 respectively. $\vec{u}$ lies between $\vec{v}$ and $\vec{w}$, making an angle of 35º with $\vec{v}$ and 20º with $\vec{w}$. Express $\vec{u}$ as a linear combination of $\vec{v}$ and $\vec{w}$.

5. F divides AP in the ratio $13:-8$ and F divides PG in the ratio $4:-3$.
   a. Draw a division-point diagram showing the relative positions of the four points.
   b. In what ratio does P divide AG?

6. a. Form two point-to-point vectors out of the three points $A(-4, 2, -8)$, $B(-1, -4, -2)$ and $P(1, -8, 2)$. Demonstrate that the two vectors are collinear.
   b. Express $\overrightarrow{OP}$ as a linear combination of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

7. $ABCD$ is a quadrilateral. P, Q, R, and S are the midpoints of its sides. Prove using vectors that $PQRS$ is a parallelogram.

8. In ΔABC, D lies on AB and E lies on AC such that $\overrightarrow{DE} = k\overrightarrow{BC}$. Prove that $\overrightarrow{AD} = k\overrightarrow{AB}$ and $\overrightarrow{AE} = k\overrightarrow{AC}$, using the fact that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly independent.

Extending and Investigating

ERROR–CORRECTING CODES

In the last chapter, you saw that 0 - 1 strings can be used to create vectors for communication. Suppose that a satellite sends electronic messages using this system from outer space to a receiver on earth. The message passes through electronic interference on its way and may be distorted, so a vector 0001000 is received as 0001001. Since we have no access to the original message we need to be able to determine whether or not the message received is unchanged from the one that was sent. Further, we wish to correct it if it is changed.

While this seems like an impossible task, it can be done if we are clever in constructing the vectors we use. The following system is the basis for the system originally used by NASA and provides a one-error correcting code. It does not correct if there are two or more errors. It uses the same addition as the earlier example but also employs the dot product of vectors. Hence for vectors, we have

$$\begin{aligned}(1001101) \bullet (1101011) &= 1 + 0 + 0 + 1 + 0 + 0 + 1 \\ &= 1. \qquad \text{(Because } 1 + 0 = 1 \text{ and } 1 + 1 = 0\text{)}\end{aligned}$$

To create the vectors, we use the matrix

$$\begin{matrix}100\\010\\001\end{matrix}$$

together with all possible linear combinations of the vertical vectors making up the matrix.

For example,

$$\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

We obtain

$$B\begin{cases}1001101 & b_1\\0101011 & b_2\\0010111 & b_3\end{cases}$$

This is a set B of three vectors, b_1, b_2, b_3, that allows us to determine the validity of a vector received and to correct it if exactly one error occurs.

The vectors used for transmitting are formed as follows. By transposing the last four columns in B (that is, changing rows to columns) and then appending the 4×4 matrix with 1 in the diagonal and 0 elsewhere, we have the following vectors:

v_1:1101000

v_2:1010100

v_3:0110010

v_4:1110001

These vectors have the property that their dot product with each of the vectors b_1, b_2, b_3 is 0. You should verify that this is so.

By adding $v_1 + v_2$, we create $v_5 = 0111100$, and by taking all possible combinations of them (two, three, or four together) we can create a total of 15 vectors, all with the same property, that their dot product with each of b_1, b_2, b_3 is 0.

With this system we have only 15 vectors, but we have shown how to create a system using the properties of matrices, vectors, combinations of vectors, and dot product. Now suppose that $v_6 = v_1 + v_3 = (1011010)$ is sent from space but in passing through a field of lightning is changed to (1010010). How can we tell whether it was originally sent in this form and, if it wasn't, how it has been changed? We simply take the dot product of the vector received with each of b_1, b_2, b_3 , as follows:

$$(1010010) \times (1001101) = 1$$

$$(1010010) \times (0101011) = 1$$

$$(1010010) \times (0010111) = 0$$

Because we do not obtain 0 in each dot product there is an error. Where is the error? The vertical vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ matches the vector in the array B in column 4. This is the location of the error. By changing the 0 in the fourth position to 1, the vector received is corrected to that which was sent.

In this example, you see that the clever use of number properties together with the algebra of vectors allows for the creation of impressive technological applications. In more advanced applications, the approach described here is used in systems requiring the correction of errors when it is impossible to obtain original messages for comparison.

Chapter 7

LINES IN A PLANE

When we solve geometric problems in two-dimensional space, Euclid's methods are usually sufficient for problems involving polygons and circles. For solving problems involving curves such as parabolas, ellipses, and hyperbolas, however, the analytic geometry of Descartes, using the language of algebra, is a superior tool. Both Euclidean and analytic methods are used for solving problems in three-dimensional space as well, but vector methods are more powerful than either the Euclidean or the analytic method. There are well-established formulas for finding the slope or direction of a line in two-dimensional space, but how do you express direction in three-dimensional space? There are also formulas for lines in two-dimensional space, but are there corresponding formulas for lines in three-dimensional space? In this chapter, we will use vectors to develop these formulas and to solve problems involving points and lines in two and three dimensions.

CHAPTER EXPECTATIONS In this chapter, you will

- determine equations of lines in two- and three-dimensional space, **Section 7.1, 7.2, 7.3, 7.4**
- solve problems involving intersections of lines and planes, **Section 7.4**

Review of Prerequisite Skills

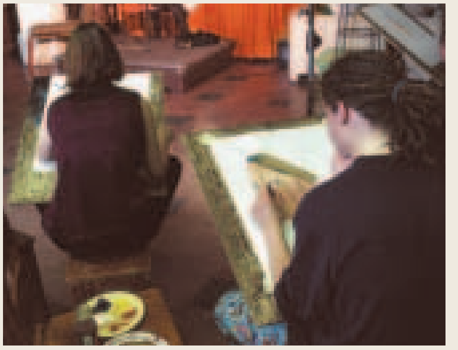
In this chapter and the next, vectors are used to investigate the geometry of straight lines and Euclidean planes in two and three dimensions. Lines are not vectors, but vectors are used to describe lines. Their similarities and differences are presented in the following table.

Lines	Vectors
Lines are bi-directional. A line defines a direction, but there is nothing to distinguish forward from backward.	Vectors are unidirectional. A vector defines a direction with a clear distinction between forward and backward.
A line is infinite in extent in both directions. A line segment has a finite length.	Vectors have a finite magnitude.
Lines and line segments have a definite location. The opposite sides of a parallelogram are two different line segments.	A vector has no fixed location. The opposite sides of a parallelogram are described by the same vector.
Two lines are the same when they have the same direction and same location. Such lines are said to be coincident.	Two vectors are the same when they have the same direction and the same magnitude. Such vectors are said to be equal.

The equation of a straight line in a plane in the form $y = mx + b$ is familiar from earlier mathematics courses. This equation is not suitable for describing the equation of a line in space. In this chapter, a new form of the equation of a line based on vectors is developed, one that can be extended from two to three dimensions. We will also develop the principal concepts needed to solve problems about the intersections of and distances between straight lines in both two and three dimensions.

Rich Learning Link investigate

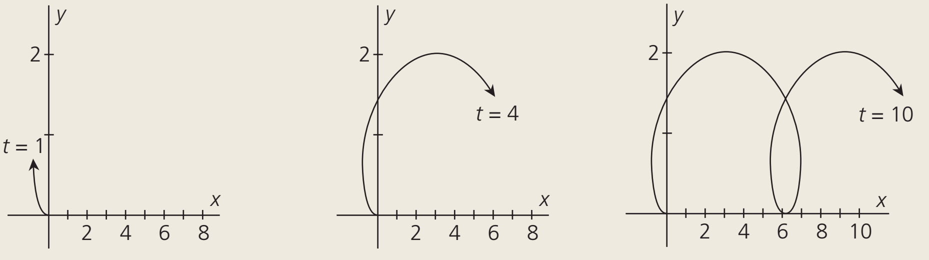
CHAPTER 7: EQUATIONS OF LINES



Some forms of mathematics use creativity and imagination in a way that is similar to artistic creation. It is not uncommon to hear mathematicians refer to theorems as *elegant* or even *beautiful*. Sometimes mathematicians produce interesting, even beautiful, images.

Investigate and Inquire

One way to create interesting images is through the use of parametric equations. These will be defined more precisely in Chapter 8, but one example of a set of parametric equations is $x = t - 2\sin(t)$, $y = 1 - \cos(t)$, $t \geq 0$. Here, each value of t gives a point (x, y). For example, if $t = \pi$ radians, then $(x, y) = (\pi, 2)$. As t increases, the point moves through the plane. The result is shown below.



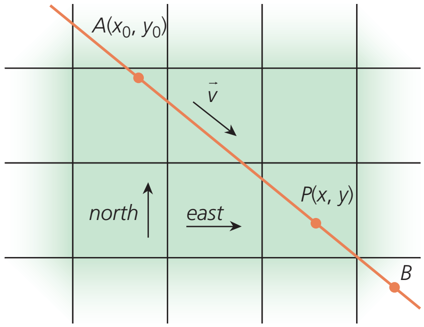
This figure is called a trochoid. Try drawing this on a graphing calculator. (*Note:* when working with parametric curves, we will evaluate trigonometric functions using radians.)

Many parametric equations can be interpreted as the position of a particle, at time t, as it moves through the plane.

DISCUSSION QUESTIONS

1. How can parametric equations be used to describe a curve in three dimensions? What about four or more dimensions?
2. When has mathematics required you to be creative, to use your imagination? Why does math sometimes seem unlike art?
3. How has mathematics been used in artistic practices?

Section 7.1 — Parametric and Vector Equations of a Line in a Plane



Imagine you are travelling at a constant speed along a perfectly straight highway that runs south and east from point A toward point B. As you travel, your position, P, changes from moment to moment, depending on how much time, t, has passed since leaving point A. The x- and y-coordinates of your position depend on t, but how? What are the equations which relate x and y to t?

Your velocity $\vec{v}$ is a vector (v_x, v_y). In this vector, v_x is the eastward (x) component, which will be positive in this example. v_y is the northward (y) component, which will be negative in this example, since you are travelling to the south.

Consider first your motion toward the east. The distance that you have travelled east is the difference $x - x_0$ between your present position at point P and your starting position at point A. This distance is equal to $v_x t$, where v_x is the eastward component of your velocity, and t is the length of time you have been travelling: $x - x_0 = v_x t$.

Therefore $\qquad x = x_0 + v_x t$

In like manner $\qquad y = y_0 + v_y t$

This pair of equations gives your position $P(x, y)$ on the highway at any time t, after starting from $A(x_0, y_0)$.

The highway from A to B is a straight line. It is important to realize that the equations derived above represent a new and different way to describe this straight line. Unlike the familiar formula $y = mx + b$, which expresses y as a function of x, here each of the coordinates x and y is expressed separately in terms of a third variable, t.

In mathematics, when you describe a relation between two variables in an indirect manner using a third variable, the third variable is called a **parameter.** Equations that show how the two variables depend on that parameter are called **parametric equations.**

The parametric equations of a straight line in a plane have the form

$$x = x_0 + at$$
$$y = y_0 + bt$$

where (x, y) is the position vector of any point on the line
(x_0, y_0) is the position vector of some particular point on the line
(a, b) is a *direction vector* for the line
and $t \in R$ is the parameter.

The parameter t, which represents the travel time above, is a real number that can take on any value. Just as each point in time corresponds to a position on the highway, each value of t corresponds to a particular point on the line, and each point on the line is characterized by a unique value of t.

EXAMPLE 1

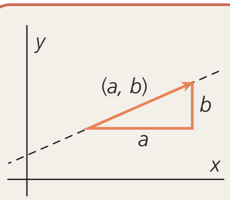
Highway 33 from Regina to Stoughton, Saskatchewan, is an almost straight line. Suppose you travel on this highway with a constant velocity (expressed in component form, where east and north are positive) $\vec{v} = (85, -65)$ km/h. How far south of Regina are you when you are at a position 102 km east of Regina?

Solution

The parametric equations are $x = x_0 + 85t$, $y = y_0 - 65t$, with $(x_0, y_0) = (0, 0)$. Using the x-component of the velocity, $102 = 0 + 85t$, then $t = \frac{6}{5}$. Therefore, it takes $\frac{6}{5}$ of an hour, or 72 minutes, for you to reach a point on the highway that is 102 km east of Regina.

Then $y = 0 + (-65)\left(\frac{6}{5}\right)$ or $y = -78$. Consequently, at this point in time, you are at a position on the highway that is 78 km south of Regina.

The velocity vector, which in Example 1 was parallel to the highway, is an example of a direction vector. In general, any vector $\vec{d} = (a, b)$ parallel to a line may be used as a direction vector for the line. By choosing the vector $(85, -65)$ in the example, we indicate that the units are in kilometres and hours. This could be, for example, a vector from one point to another on the line. The diagram below shows how the direction vector for a line is related to its slope.



Any vector that is parallel to a line may be used as a direction vector for the line.
A line with direction vector (a, b) has slope $\frac{b}{a}$, provided $a \neq 0$.

EXAMPLE 2

State a direction vector for

a. the line that passes through the points $C(3, 4)$ and $D(7, 2)$

b. a line that has slope $-\frac{5}{3}$

c. a vertical line passing through the point $(-6, 5)$

Solution

a. The vector $\overrightarrow{CD}$ has components $(7 - 3, 2 - 4) = (4, -2)$. This vector or any scalar multiple of it such as $(2, -1)$ would be a suitable direction vector.

b. A line with a slope of $-\frac{5}{3}$ has a rise of -5 and a run of 3. The vector $(3, -5)$ is parallel to this line and would be a suitable direction vector.

c. A vector parallel to a vertical line has a horizontal component of zero. The simplest such vector is $(0, 1)$. So, even though the slope of a vertical line does not exist, a direction vector does. The point the line goes through is irrelevant.

EXAMPLE 3

A line passes through the point $(5, -2)$ with direction vector $(2, 6)$.

a. State the parametric equations of this line.

b. What point on the line corresponds to the parameter value $t = 3$?

c. Does the point $(1, -8)$ lie on this line?

d. Find the y-intercept and the slope of the line. Then, write the equation of the line in the form $y = mx + b$.

Solution

a. It is given that $(x_0, y_0) = (5, -2)$ and $(a, b) = (2, 6)$. The parametric equations of the line are

$$x = 5 + 2t$$
$$y = -2 + 6t, t \in R$$

b. When $t = 3$,

$$x = 5 + 2(3) = 11$$
$$y = -2 + 6(3) = 16$$

Therefore, the point $(11, 16)$ on the line corresponds to the parameter value $t = 3$.

c. To determine if $(1, -8)$ lies on the line, try to find its parameter value. Substitute $(1, -8)$ for (x, y) and solve for t.

$$1 = 5 + 2t \qquad -8 = -2 + 6t$$
$$t = -2 \qquad\qquad t = -1$$

There is no single parameter value that satisfies both equations. Therefore, the point $(1, -8)$ does not lie on the line.

d. To find the y-intercept, set $x = 0$ and find the values of t and then y.

$$0 = 5 + 2t \qquad \text{so} \quad t = -\frac{5}{2}$$
$$y = -2 + 6\left(-\frac{5}{2}\right) \quad \text{so} \quad y = -17$$

Since the direction vector is $(2, 6)$, the slope is $\frac{6}{2}$ or 3. Using the y-intercept -17, the equation of the line is $y = 3x - 17$.

Let us now look at the parametric equations of a line from a vector viewpoint. Recall that the ordered pair (x, y) can be reinterpreted as the position vector of the point $P(x, y)$. Therefore, the parametric equations of a line are equations about the x- and y-components of vectors. Consequently, we can combine the two parametric equations into one vector equation.

$$x = x_0 + at,\ y = y_0 + bt \ \text{ becomes } \ (x, y) = (x_0 + at, y_0 + bt)$$
$$\text{or } \ (x, y) = (x_0, y_0) + t(a, b)$$

The vector equation of a straight line in a plane has the form

$$\vec{r} = (x_0, y_0) + t(a, b)$$

where $\vec{r} = (x, y)$ is the position vector of any point on the line,
(x_0, y_0) is the position vector of some particular point on the line,
(a, b) is a direction vector for the line,
and $t \in R$.

EXAMPLE 4 State a vector equation of the line passing through the points $P(4, 1)$ and $Q(7, -5)$.

Solution

The vector $\overrightarrow{PQ}$ from one point to the other on the line may be used as a direction vector, $\vec{d}$, for the line.

$$\vec{d} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= (7, -5) - (4, 1)$$
$$= (3, -6)$$

Then, a vector equation of the line is $\vec{r} = (4, 1) + t(3, -6)$, $t \in R$. You could also have used a shorter direction vector and the other point, so another vector equation of this line is $\vec{r} = (7, -5) + s(1, -2)$, $s \in R$.

As Example 4 shows, the vector equation of a line has an unusual feature. Since any vector parallel to the line will do as a direction vector, and any point on the line can serve as the particular point required in the equation, two vector equations may look entirely different, yet still represent the same line. It is important, then, to determine whether or not two different vector equations in fact represent two different lines.

EXAMPLE 5

Are the lines represented by the following vector equations coincident? That is, do these equations represent the same straight line?

a. $\vec{r} = (3, 4) + s(2, -1)$ b. $\vec{r} = (-9, 10) + t(-6, 3)$

Solution

Check the direction vectors first.

a. $\vec{d}_1 = (2, -1)$ b. $\vec{d}_2 = (-6, 3)$

Since $\vec{d}_2 = -3\vec{d}_1$, the direction vectors of the two lines are parallel.

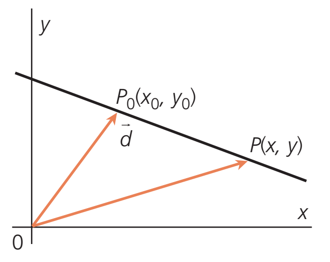
To decide if the lines are coincident, we check to see whether a point on one of the lines satisfies the vector equation of the other. The point (3, 4) is on the first line. If it is also on the second line, then $(3, 4) = (-9, 10) + t(-6, 3)$.

Then $3 = -9 - 6t$ and $4 = 10 + 3t$

$t = -2$ $\quad$ $t = -2$

Since the same parameter value is obtained from each equation, the point (3, 4) from the first line does lie on the second line, and the lines are coincident. (You may check in the same way that (−9, 10) from the second line lies on the first line with $s = -6$.)

To summarize using vector language, the vector equation of a line is a formula that gives the position vector $\overrightarrow{OP}$ of any point on the line. The diagram shows that $\overrightarrow{OP}$ is the sum of the vectors $\overrightarrow{OP_0}$ to the line and $\overrightarrow{P_0P}$ *along* the line: $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$.



$\overrightarrow{OP_0}$ is the position vector (x_0, y_0) of a particular point on the line. $\overrightarrow{P_0P}$ is a scalar multiple of some direction vector (a, b) for the line. Consequently,

$$\overrightarrow{OP} = \overrightarrow{OP_0} + t\vec{d}$$

or $$(x, y) = (x_0, y_0) + t(a, b)$$

or $$\vec{r} = \vec{r_0} + t\vec{d}$$

Exercise 7.1

Part A

Communication

1. What is a direction vector? What is a parameter? What role do these quantities play in the equation of a line?

Knowledge/ Understanding

2. State a direction vector for each of the following lines.

a. a line parallel to $x = 9 - 3t$, $y = -4 + t$

b. the line through $(6, 4)$ and $(-2, -6)$

c. the line $y = 3x + 6$

d. a line parallel to $\vec{r} = (1, 7) + t(4, 3)$

e. a horizontal line

f. a vertical line

3. State the coordinates of two points on each of the following lines.

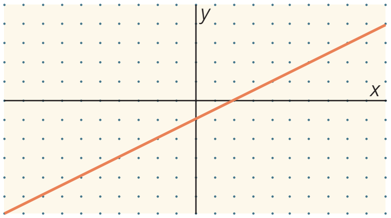
a. $x = 3 - 8t$, $y = 4t$

b. $\vec{r} = (4, 0) + t(0, 5)$

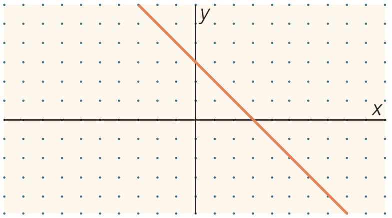
Knowledge/ Understanding

4. State a parametric equation and a vector equation for each of the following lines.

a.



b.



5. Graph the following lines.

a. $x = -1 - 5t$
$y = 6 + 2t$

b. $\vec{r} = (-3, -4) + t(4, 3)$

6. For each of the following, find the parametric equations of the line that passes through the point P with direction vector $\vec{d}$. In each case, find two points on the line different from P.

a. $P(1, 1)$, $\vec{d} = (4, -4)$

b. $P(5, 0)$, $\vec{d} = (1, 3)$

7. State parametric equations

a. for the x-axis

b. for a line parallel to but not coincident with the x-axis

Part B

8. For each of the following lines, find the vector equation that passes through the point P with direction vector $\vec{d}$.

   a. $P(-2, 7), \vec{d} = (3, -4)$
   b. $P\left(2, \frac{3}{4}\right), \vec{d} = \left(\frac{2}{3}, 6\right)$
   c. $P(1, -1), \vec{d} = (-\sqrt{3}, 3)$
   d. $P(0, 0), \vec{d} = (-2, 3)$

Thinking/Inquiry/ Problem Solving

9. For each of the following lines, state a direction vector with integer components. If possible, name a point on the line with integer coordinates.

   a. $x = \frac{1}{3} + 2t, y = 3 - \frac{2}{3}t$
   b. $\vec{r} = \left(\frac{1}{3}, \frac{1}{2}\right) + t\left(\frac{1}{3}, \frac{1}{4}\right)$
   c. $\vec{r} = \left(\frac{1}{2}, 3\right) + t\left(-\frac{1}{2}, 5\right)$

Thinking/Inquiry/ Problem Solving

10. For each of the following, determine which pairs of lines are parallel and which are perpendicular.

   a. $x = 1 - 3t, y = 7 + 4t$ and $x = 2 - 4s, y = -3s$
   b. $\vec{r} = (1, 7) + t(-3, 4)$ and $\vec{r} = (2, 0) + s(3, -4)$
   c. $\vec{r} = (1, 7) + t(-3, 4)$ and $\vec{r} = (2, 0) + s(4, -3)$

11. Find a vector equation of the line that passes through the point (4, 5) and is perpendicular to the line $\vec{r} = (1, 8) + t(3, 7)$.

12. Find the points where each of the following lines intersects the x- and y-axes. Graph the line.

   a. $x = 6, y = 1 + 7t$
   b. $\vec{r} = (-5, 10) + t(1, 5)$
   c. $\vec{r} = (2, 3) + t(3, -1)$

Application

13. Show that both lines $\vec{r} = (3, 9) + t(2, 5)$ and $\vec{r} = (-5, 6) + u(3, -1)$ contain the point (1, 4). Find the acute angle of intersection of these lines to the nearest degree.

14. The angle α, $0° \leq \alpha \leq 180°$, that a line makes with the positive x-axis is called the **angle of inclination** of the line.

   a. Find the angle of inclination of each of the following lines.
      (i) $\vec{r} = (2, -6) + t(3, -4)$
      (ii) $\vec{r} = (6, 1) + t(5, 1)$
   b. Prove that the tangent of the angle of inclination is equal to the slope of the line.

Application

15. You are driving from point $A(24, 96)$ on a map grid toward point B with a velocity defined by $\vec{d}$ (85, −65) km/h.

a. State the parametric equations of the highway line.

b. How long have you been travelling when you reach a point P 102 km east of where you started at point A?

c. What are the coordinates of your position P at that time?

Thinking/Inquiry/ Problem Solving

16 **a.** By eliminating the parameter t from the parametric equations of a line, show that the equation of a line can be written in the form $\frac{x - x_0}{a} = \frac{y - y_0}{b}$ (provided neither a nor b is zero). This is known as the **symmetric equation** of a line.

b. Find a symmetric equation for each of the following lines.

(i) $x = 5 - 8t, y = -3 + 5t$ (ii) $\vec{r} = (0, -4) + t(4, 1)$

c. Find a symmetric equation for the line through the points $A(7, -2)$ and $B(-5, -4)$.

Part C

17. a. Show that $P(5, 8)$ and $Q(17, -22)$ are points on the line that passes through $A(7, 3)$ with direction vector $(2, -5)$.

b. Describe the line segment from $P(5, 8)$ to $Q(17, -22)$ using parametric equations with suitable restrictions on the parameter.

Thinking/Inquiry/ Problem Solving

18. **a.** Suppose $\vec{p}$ and $\vec{q}$ are the position vectors of points P and Q in the plane. Show that the line that passes through P and Q has the vector equation $\vec{r} = (1 - t)\vec{p} + t\vec{q}$.

b. For what values of t does the point R with position vector $\vec{r}$ lie between points P and Q on the line?

c. When $t = 2$, draw a vector diagram that shows where point R with position vector $\vec{r}$ lies on the line relative to points P and Q.

d. For what values of t does the point R with position vector $\vec{r}$ lie closer to Q than P?

19. a. Find the vector equations of the two lines that bisect the angles between the lines

$\vec{r}_1 = (5, 2) + t(-3, 6)$

$\vec{r}_2 = (5, 2) + u(11, 2)$

b. Sketch all four lines.

c. Are the two lines that bisect the angles made by the intersecting lines always perpendicular? Explain.

Section 7.2 — The Scalar Equation of a Line in a Plane

Another way to form the equation of a line is to use a vector that is perpendicular to the line rather than one that is parallel to the line. Any vector that is perpendicular to a line is called a **normal vector** or simply a **normal** to the line.

EXAMPLE 1

Find a normal to the line

a. $y = -2x + 5$

b. $(x, y) = (2, -3) + t(2, 5), t \in R$

Solution

a. The slope of the given line is -2. The slope of a line perpendicular to the given line is $\frac{1}{2}$. A vector normal to the line is, therefore, (2, 1).

b. The direction vector is (2, 5). The dot product of (2, 5) and any normal vector (n_1, n_2) must be zero.

$$(2, 5) \bullet (n_1, n_2) = 0$$
$$2n_1 + 5n_2 = 0$$

One of the many ways this equation can be satisfied is by choosing $n_1 = 5$ and $n_2 = -2$. Then, $(5, -2)$ is a normal to the line with direction vector (2, 5).

The dot product of a normal vector and a direction vector is always zero because they are perpendicular. This is the key to the use of normal vectors in two dimensions.

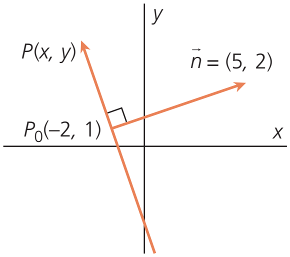
EXAMPLE 2

Find the equation of the straight line with normal (5, 2), which passes through the point $(-2, 1)$. Write the equation of the line in the form $Ax + By + C = 0$.

Solution

For a point $P(x, y)$ on the line, a direction vector is defined by

$$\overrightarrow{P_0P} = (x + 2, y - 1).$$



This vector is perpendicular to the normal. The dot product of these vectors must be zero.

$$(5, 2) \bullet (x + 2, y - 1) = 0$$
$$5(x + 2) + 2(y - 1) = 0$$
$$5x + 10 + 2y - 2 = 0$$
$$5x + 2y + 8 = 0$$

This is the equation of the line through $(-2, 1)$ with normal (5, 2).

In the equation found in Example 2, we can see that the components of the normal end up as the coefficients of the x- and y-terms. The following derivation demonstrates that this will always be the case.

The vector $\overrightarrow{P_0P}$ along a line from a fixed point $P_0(x_0, y_0)$ to any other point $P(x, y)$ must be perpendicular to the normal $\vec{n} = (A, B)$.

Then
$$\vec{n} \bullet \overrightarrow{P_0P} = 0$$
$$(A, B) \bullet (x - x_0, y - y_0) = 0$$
$$A(x - x_0) + B(y - y_0) = 0$$
$$Ax + By + (-Ax_0 - By_0) = 0$$
$$Ax + By + C = 0, \text{ where } C = -Ax_0 - By_0.$$

The scalar or Cartesian equation of a straight line in a plane has the form

$$Ax + By + C = 0$$

where the vector (A, B) is a normal to the line.

EXAMPLE 3

Find the scalar equation of the straight line with normal $(-6, 4)$ that passes through the point $(-3, -7)$.

Solution

Since $(-6, 4) = -2(3, -2)$ we can use $(3, -2)$ as a normal to the line. The equation must be of the form

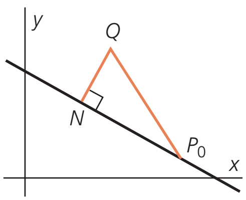
$$3x - 2y + C = 0$$

Since the point $(-3, -7)$ lies on the line, its coordinates must satisfy the following equation.

$$3(-3) - 2(-7) + C = 0$$
$$C = -5$$

The equation of the line is $3x - 2y - 5 = 0$.

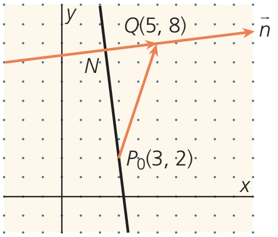
When the equation of a line is expressed in scalar form, it is a relatively straightforward task to find the distance from a point to the line. The shortest distance from the point Q to the line l is QN, measured along the normal through Q. This distance is shorter than the distance from Q to any other point P_0 on the line. (Why?)



EXAMPLE 4 Find the distance from the point $Q(5, 8)$ to the line $7x + y - 23 = 0$.

Solution

In the diagram, the required distance is QN, where N is the point where the normal through Q meets the line. Then QN is the magnitude of the projection of $\overrightarrow{P_0Q}$ onto the normal to the line, where P_0 is any point on the line. Choosing P_0 to be $(3, 2)$ gives $\overrightarrow{P_0Q} = (2, 6)$. Also, $\vec{n} = (7, 1)$, so



$$\begin{aligned} QN &= \left|\text{proj}(\overrightarrow{P_0Q} \text{ onto } \vec{n})\right| \\ &= \left|\frac{(2, 6) \cdot (7, 1)}{\sqrt{7^2 + 1^2}}\right| \\ &= \left|\frac{14 + 6}{\sqrt{50}}\right| \\ &= 2\sqrt{2} \end{aligned}$$

The distance from the point $Q(5, 8)$ to the line $7x + y - 23 = 0$ is $2\sqrt{2}$ units.

By working through the steps of the solution to Example 4 in general terms, we can find a simple formula for the distance from a point $Q(x_1, y_1)$ to a line with scalar equation $Ax + By + C = 0$. Letting $P_0(x_0, y_0)$ be a point on the line, the distance, d, is

$$\begin{aligned} d &= \left|\text{proj}(\overrightarrow{P_0Q} \text{ onto } \vec{n})\right| \\ &= \left|\frac{\overrightarrow{P_0Q} \cdot \vec{n}}{|\vec{n}|}\right| \\ &= \frac{|(x_1 - x_0, y_1 - y_0) \cdot (A, B)|}{\sqrt{A^2 + B^2}} \\ &= \frac{|Ax_1 + By_1 - Ax_0 - By_0|}{\sqrt{A^2 + B^2}} \end{aligned}$$

Since $P_0(x_0, y_0)$ is on the line, it satisfies the equation of the line, so

$$Ax_0 + By_0 + C = 0$$

or $\quad C = -Ax_0 - By_0$

Then the distance is $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

The distance from the point (x_1, y_1) to the line $Ax + By + C = 0$ is given by the formula

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Exercise 7.2

Part A

Communication

1. Explain why there is one and only one scalar equation of a given line, whereas there are many different parametric and vector equations for the line.

Communication

2. State a normal of the line that is

a. perpendicular to $2x - 4y + 5 = 0$

b. parallel to $2x - 4y + 5 = 0$

c. perpendicular to $\vec{r} = (2, -5) + t(4, -2)$

d. parallel to $\vec{r} = (2, 5) + t(4, -2)$

Knowledge/Understanding

3. For each of the following, find the scalar equation of the line that passes through the point P_0 and has normal vector $\vec{n}$.

a. $P_0(4, -2), \vec{n} = (2, 7)$

b. $P_0\left(\frac{1}{2}, 2\right), \vec{n} = (-4, 0)$

c. $P_0(3, 3), \vec{n} = (1, 1)$

d. $P_0\left(\frac{1}{3}, \frac{1}{3}\right), \vec{n} = (-1, 1)$

4. For each of the following, find a normal vector, a direction vector, and a point on each line.

a. $4x + 3y - 12 = 0$

b. $3x - 6y = 14$

c. $x = 5$

d. $y = 3x - 10$

5. Prove that both $(-b, a)$ and $(b, -a)$ are perpendicular to (a, b) for all a and b.

6. Find the Cartesian equation of each of the following lines.

a. $(x, y) = (4, -6) + t(8, 2)$

b. $x = 3 + 18t, y = 4 + 9t$

c. $\vec{r} = (2, 7) + t(2, 7)$

d. $x = 2t, y = -2$

7. Find the scalar equation of the line that passes through $(2, -6)$ and

a. is parallel to $2x - 3y + 8 = 0$

b. is perpendicular to $3x - 2y + 12 = 0$

c. has a direction vector $(2, -3)$

d. has a normal vector $(3, -2)$

8. Find the scalar equation of the line through $(8, -2)$ that is parallel to the line $x = -4 - 5t, y = 11 + 3t$ by first finding the symmetric equation of this line, and then simplifying it.

Part B

9. Find vector, parametric, and symmetric equations of the following lines.

   a. $5x - 3y + 15 = 0$ b. $-4x + 6y + 9 = 0$

Thinking/Inquiry/Problem Solving

10. Prove that the shortest distance from a point to a line is the distance measured along the perpendicular from the point to the line.

Knowledge/Understanding

11. For each of the following, find the distance from $Q(3, -2)$ to each line.

   a. $3x - 2y - 6 = 0$ b. $\frac{x-3}{2} = \frac{y-4}{7}$

   c. $\vec{r} = (-3, -7) + t\left(\frac{1}{5}, \frac{1}{6}\right)$ d. $x = -5$

12. Find the distance from each of the following points to the line $6x + 3y - 10 = 0$.

   a. $(4, 7)$ b. $(4, -8)$ c. $(0, 5)$ d. $\left(5, -\frac{20}{3}\right)$

Part C

13. a. Prove that two lines in a plane are parallel if and only if their normals are parallel.

    b. Prove that two lines in a plane are perpendicular if and only if their normals are perpendicular.

Application

14. a. Show that the equation of a line that has an angle of inclination α can be expressed in the form $x \sin \alpha - y \cos \alpha + C = 0$. (See Exercise 7.1, Question 14.)

    b. Find the angle of inclination of $2x + 4y + 9 = 0$.

    c. Find the scalar equation of the line through the point $(6, -4)$ with an angle of inclination of 120º.

Thinking/Inquiry/Problem Solving

15. Draw any line through point $A(2, 2)$. Through point $B(8, 10)$, draw a normal to the line through A, meeting it at the point $N(x, y)$.

    a. Show that N is a point on the circle defined by $\overrightarrow{AN} \cdot \overrightarrow{BN} = 0$

    b. Describe the relationship between this circle and the points A and B.

16. $\vec{n}$ is a normal to a line and $\overrightarrow{OP}$ is the position vector of a point $P(x, y)$ on the line.

    a. Using diagrams, show that the line goes through the origin when $\vec{n} \cdot \overrightarrow{OP} = 0$.

    b. Prove that the line goes through the origin if and only if $\vec{n} \cdot \overrightarrow{OP} = 0$.

Section 7.3 — Equations of a Line in 3-Space

In generalizing the equations for a line from a two-dimensional plane to a three-dimensional space, we must introduce a *z*-coordinate for points and a *z*-component for vectors. The equations are otherwise very similar, except that there is no scalar equation of a line in space because a line in space does not have a unique normal.

The vector equation of a straight line in space has the form

$$\overrightarrow{OP} = \overrightarrow{OP_0} + t\vec{d}$$

or $\vec{r} = \vec{r_0} + t\vec{d}$

or $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$

where $\vec{r}$ **is the position vector of any point on the line**

$\vec{r_0}$ **is the position vector of some particular point on the line**

$\vec{d}$ **is a direction vector for the line**

and $t \in R$.

The numbers *a*, *b*, and *c*, which are the components of the direction vector, are known as **direction numbers** of the line.

EXAMPLE 1

Determine a direction vector for

a. the line that passes through the points $P(6, -4, 1)$ and $Q(2, -8, -5)$
b. a line perpendicular to the *xz*-plane

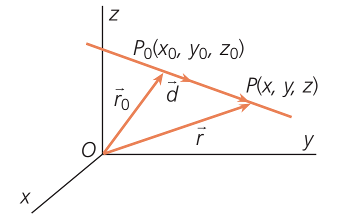
Solution

a. $$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP}\\ &= (2, -8, -5) - (6, -4, 1)\\ &= (-4, -4, -6)\end{aligned}$$

This vector or, better, $(-2, -2, -3)$ or, better still, $(2, 2, 3)$ could be used as a direction vector for this line.

b. A vector perpendicular to the *xz*-plane is parallel to the *y*-axis. A suitable direction vector is, therefore, $(0, 1, 0)$.

The position vector $\overrightarrow{OP}$ to a general point $P(x, y, z)$ on the line can be expressed as $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$. $\overrightarrow{OP_0}$ is the position vector of a particular point $P_0(x_0, y_0, z_0)$ on the line. $\overrightarrow{P_0P}$ is some scalar multiple of a direction vector.



EXAMPLE 2

a. Find a vector equation of the line that passes through the point $P(1, 0, -1)$ and has direction numbers $(1, 2, 3)$.

b. Does the point $Q(-3, -8, -13)$ lie on this line?

Solution

a. A vector equation of the line is $\vec{r} = (1, 0, -1) + t(1, 2, 3)$.

b. The point $Q(-3, -8, -13)$ lies on the line only if there is a value of the parameter t such that

$$(-3, -8, -13) = (1, 0, -1) + t(1, 2, 3)$$

Then $-3 = 1 + t$ and $-8 = 2t$ and $-13 = -1 + 3t$
or $t = -4$ $\quad t = -4$ $\quad t = -4$

This vector equation is satisfied by $t = -4$, so the point Q does lie on the line.

When each component of the vector equation is written out separately, the resulting equations are the parametric equations of a straight line in space.

The parametric equations of a straight line in space have the form

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

where (x_0, y_0, z_0) are the coordinates of some particular point on the line, and a, b, and c are direction numbers for the line,
and $t \in R$.

Solving each of the parametric equations for the parameter t gives

$$t = \frac{x - x_0}{a},\ t = \frac{y - y_0}{b},\ \text{and } t = \frac{z - z_0}{c}$$

provided that none of a, b, or c is zero. These expressions give an alternate form for equations of a straight line in space.

The symmetric equations of a straight line in space have the form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where (x_0, y_0, z_0) are the coordinates of some particular point on the line, and a, b, and c are direction numbers for the line with a, b, and $c \neq 0$.

EXAMPLE 3

Find vector, parametric, and symmetric equations of the line that passes through the points $A(2, -2, -8)$ and $B(5, -2, -14)$.

Solution

Since $\overrightarrow{AB} = (3, 0, -6)$, then $(1, 0, -2)$ is a direction vector for the line. Using A as the fixed point, a vector equation of the line is $\vec{r} = (2, -2, -8) + t(1, 0, -2)$.

Then the corresponding parametric equations are

$$x = 2 + t$$
$$y = -2$$
$$z = -8 - 2t$$

and the corresponding symmetric equations are

$$\frac{x - 2}{1} = \frac{z + 8}{-2}, \ y = -2$$

There is no symmetric expression for y because the corresponding direction number $b = 0$. In cases like this, when y does not change with t, you must still state its value. (If two direction numbers are 0, there is no symmetric equation.)

EXAMPLE 4

Write a vector equation for the line $-x = y + 2 = z$.

Solution

Rewriting the equations,

$$\frac{x - (0)}{-1} = \frac{y - (-2)}{1} = \frac{z - (0)}{1}$$

Then by inspection, a vector equation of the line is

$$\vec{r} = (0, -2, 0) + t(-1, 1, 1)$$

EXAMPLE 5

Do the equations $\frac{x - 5}{2} = \frac{y + 4}{-5} = \frac{z + 1}{3}$ and $\frac{x + 1}{-4} = \frac{y - 11}{10} = \frac{z + 4}{-6}$ represent the same line?

Solution

The direction vector of the second line $(-4, 10, -6)$ is -2 times the direction vector of the first line $(2, -5, 3)$, so the lines are parallel. They are coincident if the point $(-1, 11, -4)$ on the second line satisfies the equation of the first line.

$$\frac{(-1) - 5}{2} = -3, \frac{(11) + 4}{-5} = -3, \frac{(-4) + 1}{3} = -1$$

The fractions are not equal, therefore the lines are parallel and distinct.

EXAMPLE 6 Find vector, parametric, and symmetric equations of the y-axis, if possible.

Solution

The y-axis goes through the origin and has direction $\hat{j} = (0, 1, 0)$. A vector equation for the y-axis is $\vec{r} = (0, 1, 0)$ or simply $\vec{r} = t\hat{j}$. Parametric equations are $x = 0$, $y = t$, $z = 0$. It has no symmetric equation because two of the direction numbers are zero.

Exercise 7.3

Part A

Communication

1. Why does a line in space have a vector equation and a parametric equation, but no scalar equation?

Knowledge/Understanding

2. Find a direction vector for a line

a. parallel to $\vec{r} = (7, -9, 3) + t(-4, 2, -5)$

b. through $(0, 6, 3)$ and $(7, 4, 6)$

c. parallel to $-x = \frac{y - 3}{2} = \frac{z}{4}$

3. Give the coordinates of two points on each of the following lines.

a. $\vec{r} = (1, 1, 2) + t(3, -1, -1)$

b. $x = 4 - 2t$, $y = -2 + 5t$, $z = 5 + 4t$

c. $\frac{x - 4}{3} = \frac{y + 5}{4} = \frac{z + 1}{-1}$

Knowledge/Understanding

4. For each of the following, find vector, parametric, and, if possible, symmetric equations of the line that passes through P_0 and has direction vector $\vec{d}$.

a. $P_0(2, 4, 6)$, $\vec{d} = (-1, -3, 2)$

b. $P_0(0, 0, -5)$, $\vec{d} = (-1, 4, 1)$

c. $P_0(1, 0, 0)$, $\vec{d} = (0, 0, -1)$

Application

5. List the points on the line $\vec{r} = (-2, 4, 3) + t(3, -1, 5)$ for even integer values of t from -6 to $+6$.

Application

6. a. Which of the following points lies on the line $x = 2t$, $y = 3 + t$, $z = 1 + t$?

$P(2, 4, 2)$ $\quad Q(-2, 2, 1)$ $\quad R(4, 5, 2)$ $\quad S(6, 6, 2)$

b. If the point $(a, b, -3)$ lies on the line, find the values of a and b.

Part B

7. Find parametric equations for the line that passes through the point $(0, -1, 1)$ and the midpoint of the line segment from $(2, 3, -2)$ to $(4, -1, 5)$.

Thinking/Inquiry/ Problem Solving

8. Find symmetric equations for the line through the origin that is parallel to the line through the points $(4, 3, 1)$ and $(-2, -4, 3)$.

9. For each of the following pairs of equations, determine whether they represent the same line, parallel lines, or neither of these.

a. $\vec{r} = (1, 0, 3) + s(3, -6, 3)$ and $\vec{r} = (2, -2, 5) + t(2, -4, 2)$

b. $\vec{r} = (2, -1, 4) + s(3, 0, 6)$ and $\vec{r} = (-3, 0, 1) + t(2, 0, 2)$

c. $\vec{r} = (1, -1, 1) + s(6, 2, 0)$ and $\vec{r} = (-5, -3, 1) + t(-9, -3, 0)$

10. Describe in words the lines having the following parametric equations. Sketch the lines.

a. $x = t, y = 2, z = -1$

b. $x = 0, y = 1 + t, z = 1 - t$

c. $x = -5, y = 2 + t, z = 2 + t$

Thinking/Inquiry/ Problem Solving

11. a. Describe the set of lines in space that have one direction number equal to zero.

b. Describe the set of lines in space that have two direction numbers equal to zero.

Part C

12. Find the symmetric equations of the line that passes through the point $(-6, 4, 2)$ and is perpendicular to both of the lines

$$\frac{x}{-4} = \frac{y+10}{-6} = \frac{z+2}{3} \text{ and } \frac{x-5}{3} = \frac{y-5}{2} = \frac{z+5}{4}.$$

13. a. Show that the points $A(-9, -3, -16)$ and $B(6, 2, 14)$ lie on the line that passes through $(0, 0, 2)$ and has direction numbers $(3, 1, 6)$.

b. Describe the line segment from A to B using parametric equations with suitable restrictions on the parameter.

Thinking/Inquiry/ Problem Solving

14. Find an equation of the line through the point $(4, 5, 5)$ that meets the line $\frac{x-11}{3} = \frac{y+8}{-1} = \frac{z-4}{1}$ at right angles.

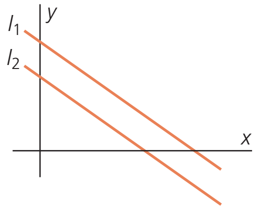
15. a. Prove that the distance from a point Q in space to a line through a point P with direction vector $\vec{d}$ is equal to $\frac{|\overrightarrow{PQ} \times \vec{d}|}{|\vec{d}|}$.

b. Find the distance from the point $Q(1, -2, -3)$ to the line $\vec{r} = (3, 1, 0) + t(1, 1, 2)$.

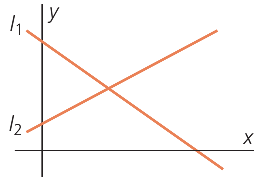
c. Find the distance between the parallel lines $\vec{r} = (-2, 2, 1) + t(7, 3, -4)$ and $r = (2, -1, -2) + u(7, 3, -4)$.

Section 7.4 — The Intersection of Two Lines

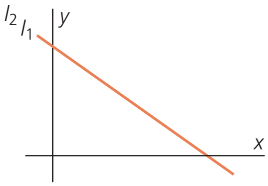
What are the possible ways that two lines in a plane can intersect? They can be parallel (and distinct), intersecting at no points; they can cross, intersecting at a single point; or they can be coincident, thereby having an infinite number of common points.



parallel



intersecting



coincident

When the equations of two lines are expressed in scalar form, you can find their point of intersection by the familiar method of elimination.

EXAMPLE 1

Find the intersection of the lines

$$2x + 3y - 30 = 0$$

and $$x - 2y + 13 = 0$$

Solution

Solving,

$$\begin{array}{r} 2x + 3y - 30 = 0 \\ \underline{-2x + 4y - 26 = 0} \\ 7y - 56 = 0 \\ y = 8 \end{array}$$

Substitute in the second equation,

$$\begin{array}{r} x - 2(8) + 13 = 0 \\ x = 3 \end{array}$$

Therefore, the point of intersection is (3, 8).

EXAMPLE 2

Find the intersection of the lines

$$\vec{r} = (18, -2) + t(3, -2)$$
$$\vec{r} = (-5, 4) + s(2, 1)$$

Solution

First write the parametric equations of the lines.

line 1 $x = 18 + 3t$, $y = -2 - 2t$ **line 2** $x = -5 + 2s$, $y = 4 + s$

Equating the expressions for x and y,

$$\begin{aligned} 18 + 3t &= -5 + 2s \\ -2 - 2t &= 4 + s \end{aligned} \quad \text{or} \quad \begin{aligned} 3t - 2s + 23 &= 0 \\ 2t + s + 6 &= 0 \end{aligned}$$

Solving, $s = 4$ and $t = -5$.

Substituting these into line 1 or into line 2, the coordinates of the intersection point are (3, 8).

Like lines in a plane, lines in space can be parallel, intersecting at a point, or coincident. But there is also a new possibility: they can be **skew.** Skew lines are not parallel. Nevertheless, they do not intersect, because they lie in different planes. They just pass by each other like the vapour trails left by two aircraft flying at different altitudes.

EXAMPLE 3 Find the intersection of

line 1 $\begin{cases} x = -1 + 3t \\ y = 1 + 4t \\ z = -2t \end{cases}$ and **line 2** $\begin{cases} x = -1 + 2s \\ y = 3s \\ z = -7 + s \end{cases}$

Solution

Equating the expressions for x, y, and z gives

$$\begin{aligned} -1 + 3t &= -1 + 2s \\ 1 + 4t &= 3s \\ -2t &= -7 + s \end{aligned} \quad \text{or} \quad \begin{aligned} 3t - 2s &= 0 \\ 4t - 3s + 1 &= 0 \\ 2t + s - 7 &= 0 \end{aligned}$$

Solve for s and t using the second and third equations.

Equation 2 $4t - 3s + 1 = 0$

(–2) × Equation 3 $\underline{-4t - 2s + 14 = 0}$

$$-5s + 15 = 0$$
$$s = 3$$

Substituting,

$$4t - 3(3) + 1 = 0$$
$$t = 2$$

Verify that $t = 2$ and $s = 3$ satisfy the first equation.

$$\begin{aligned} 3t - 2s &= 3(2) - 2(3) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

Therefore, the two lines intersect at a unique point, which is the point determined by $t = 2$ on line 1, and $s = 3$ on line 2.
The point of intersection is $(5, 9, -4)$.

EXAMPLE 4 Find the intersection of

line 1 $\vec{r} = (2, 1, 0) + t(1, -1, 1)$
line 2 $\vec{r} = (3, 0, -1) + s(2, 3, -1)$

Solution

The direction vectors are not parallel, so the lines either intersect or are skew.
The parametric equations are

line 1 $x = 2 + t$, $y = 1 - t$, $z = t$ **line 2** $x = 3 + 2s$, $y = 3s$, $z = -1 - s$

Equating the expressions for x, y, and z gives

$$\begin{aligned} 2 + t &= 3 + 2s \\ 1 - t &= 3s \\ t &= -1 - s \end{aligned} \qquad \text{or} \qquad \begin{aligned} t - 2s - 1 &= 0 \\ t + 3s - 1 &= 0 \\ t + s + 1 &= 0 \end{aligned}$$

Solving the first and second equations,

$$\begin{array}{rr} \textbf{Equation 1} & t - 2s - 1 = 0 \\ \textbf{(–1)} \times \textbf{Equation 2} & \underline{-t - 3s + 1 = 0} \\ & -5s = 0 \\ & s = 0, \text{ so } t = 1 \end{array}$$

Finally, check to see if these values of t and s satisfy the third equation.

$$\begin{aligned} t + s + 1 &= (1) + (0) + 1 \\ &= 2 \\ &\neq 0 \end{aligned}$$

The values of t and s do not satisfy the third equation. Therefore, the lines have no point of intersection. They are skew lines.

It is now time to place the subject of this section, intersections of lines, into a more general context. The scalar equation of a line is an example of a **linear equation.**

A linear equation is an equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + ... = k$$

where the x_i, ... are variables
and the a_i, ... and k are constants.

The intersection problems considered in this section are elementary examples of **linear systems.** A linear system is a set of two or more linear equations and may involve thousands of equations. Problems requiring the solution of a linear system arise in disciplines such as engineering, economics, physics, and biology. In this section, the focus has been on the geometrical interpretation of a linear system and its solutions.

A system of linear equations may have
- **i) no solution**
- **ii) a unique solution**
- **iii) an infinite number of solutions**

A linear system is said to be **consistent** if it has at least one solution. Otherwise the system is **inconsistent.**

Solving a linear system when the number of equations is large is an extremely challenging problem, particularly if all coefficients are non-zero. Try to imagine the amount of work required to solve ten equations with ten variables. Fortunately, in real life many of the coefficients are zero.

You can get a reasonable picture of a real situation from the following examples. Suppose that a grocery store that stocks 50 different items has 50 customers. The first buys six different items worth \$20, the second buys ten different items worth \$37, the third buys seven different items worth \$52, and so on. From this we can construct 50 equations in 50 variables, assuming that no two customers make identical purchases. From these equations we can determine the cost of each item.

Now picture the situation if there are 50 000 items in the store, or imagine the task of solving for 1 000 000 forces acting on the beams in a large building. It is true that many of the coefficients are 0. A system involving a large number of equations and having many coefficients equal to 0 is referred to as a **sparse** system.

Solving such a system involves computer applications and clever algorithms. The study of linear systems is a highly developed area, and people skilled in analyzing such systems are greatly in demand.

Exercise 7.4

Part A

Communication

1. Line 1 intersects both the x-axis and the y-axis. Line 2 intersects only the z-axis. Neither contains the origin. Must the two lines be parallel or skew, or can they intersect?

Knowledge/Understanding

2. Find the intersection point of each of the following pairs of lines. Graph the lines and identify the intersection point.

a. $2x + 5y + 15 = 0$

$3x - 4y + 11 = 0$

b. $\vec{r} = (-3, -6) + s(1, 1)$

$\vec{r} = (4, -8) + t(1, 2)$

3. Determine whether the following pairs of lines are coincident, parallel and distinct, or neither.

a. $\frac{x-3}{10} = \frac{y-8}{-4}$

$\frac{x-33}{-5} = \frac{y+4}{2}$

b. $x = 6 - 18s,\ y = 12 + 3s$

$x = 8 - 6t,\ y = 4 + 9t$

c. $x = 8 + 12s,\ y = 4 - 4s,\ z = 3 - 6s$

$x = 2 - 4t,\ y = 2 + t,\ z = 6 + 2t$

d. $\frac{x+4}{3} = \frac{y-12}{4} = \frac{z-3}{6}$

$\frac{x}{\frac{1}{2}} = \frac{y-10}{\frac{2}{3}} = z + 5$

Part B

Knowledge/Understanding

4. Find the intersection of each pair of lines. If they do not meet, determine whether they are parallel and distinct or skew.

a. $\vec{r} = (-2, 0, -3) + t(5, 1, 3)$

$\vec{r} = (5, 8, -6) + u(-1, 2, -3)$

b. $x = 1 + t,\ y = 1 + 2t,\ z = 1 - 3t$

$x = 3 - 2u,\ y = 5 - 4u,\ z = -5 + 6u$

c. $\vec{r} = (2, -1, 0) + t(1, 2, -3)$

$\vec{r} = (-1, 1, 2) + u(-2, 1, 1)$

d. $(x, y, z) = (1 + t, 2 + t, -t)$

$(x, y, z) = (3 - 2u, 4 - 2u, -1 + 2u)$

e. $\frac{x-3}{4} = y - 2 = z - 2$

$\frac{x-2}{-3} = \frac{y+1}{2} = \frac{z-2}{-1}$

5. Consider the lines $\vec{r} = (1, -1, 1) + t(3, 2, 1)$ and $\vec{r} = (-2, -3, 0) + u(1, 2, 3)$.
   a. Find their point of intersection.
   b. Find a vector equation for the line perpendicular to both of the given lines that passes through their point of intersection.

6. Show that the lines $\vec{r} = (4, 7, -1) + t(4, 8, -4)$ and $\vec{r} = (1, 5, 4) + u(-1, 2, 3)$ intersect at right angles and find the point of intersection.

7. If they exist, find the x-, y-, and z-intercepts of the line $x = 24 + 7t$, $y = 4 + t$, $z = -20 - 5t$.

Application

8. Find the point at which the normal through the point $(3, -4)$ to the line $10x + 4y - 101 = 0$ intersects the line.

Part C

Communication

9. What are the possible ways that three lines in a plane can intersect? Describe them all with diagrams.

Communication

10. What are the possible ways that three lines in space can intersect? Describe them all with diagrams.

11. Find the equation of the line through the point $(-5, -4, 2)$ that intersects the line at $\vec{r} = (7, -13, 8) + t(1, 2, -2)$ at 90º. Determine the point of intersection.

Application

12. Find the points of intersection of the line $\vec{r} = (0, 5, 3) + t(1, -3, -2)$ with the sphere $x^2 + y^2 + z^2 = 6$. Is the segment of the line between the intersection points a diameter of the sphere?

13. Find a vector equation for the line through the origin that intersects both of the lines $\vec{r} = (2, -16, 19) + t(1, 1, -4)$ and $\vec{r} = (14, 19, -2) + u(-2, 1, 2)$.

Thinking/Inquiry/
Problem Solving

14. a. Determine the point N at which the normal through the origin intersects the line $Ax + By + C = 0$ in the xy-plane.
    b. Find the magnitude of the position vector $\overrightarrow{ON}$ of point N.

15. The common perpendicular of two skew lines with direction vectors $\overrightarrow{d_1}$ and $\overrightarrow{d_2}$ is the line that intersects both the skew lines and has direction vector $\vec{n} = \overrightarrow{d_1} \times \overrightarrow{d_2}$. Find the points of intersection of the common perpendicular with each of the lines $(x, y, z) = (0, -1, 0) + s(1, 2, 1)$ and $(x, y, z) = (-2, 2, 0) + t(2, -1, 2)$.

Thinking/Inquiry/ Problem Solving

16. The distance between the skew lines $\vec{r} = \overrightarrow{OP} + t\vec{d_1}$ and $\vec{r} = \overrightarrow{OQ} + s\vec{d_2}$ is $\left|\text{Proj}(\overrightarrow{PQ} \text{ onto } \vec{n})\right|$ or $\frac{|\overrightarrow{PQ} \bullet \vec{n}|}{|\vec{n}|}$ where $\vec{n} = \vec{d_1} \times \vec{d_2}$.

Find the distance between the lines

a. $\vec{r} = (0, -2, 6) + t(2, 1, -1)$ and $\vec{r} = (0, -5, 0) + s(-1, 1, 2)$

b. $x = 6, y = -4 - t, z = t$ and $x = -2s, y = 5, z = 3 + s$

Key Concepts Review

This chapter has illustrated how the algebraic description of straight lines can be formulated in terms of vectors. The form of the vector equation of a line, $\vec{r} = \vec{r_0} + t\vec{d}$, is the same whether the line lies in a plane or in a three-dimensional space. This equation also describes a line in more abstract, higher dimensional spaces, where the vectors have more than three components.

To master this material, learn the various forms of the equation of a line.

the **vector equation**	$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$ or $\vec{r} = \vec{r_0} + t\vec{d}$
the **parametric equations**	$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$
the **symmetric equations**	$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$
the **scalar equation**	$Ax + By + C = 0$ (in two dimensions only)

Notice the position in each equation of the components (a, b, c) of the direction vector and the coordinates (x_0, y_0, z_0) of a point on the line. Work at converting from one form to another by inspection.

Try visualizing lines in three-dimensional space, perhaps using the lines along the corners of a room as coordinate axes. Practice sketching graphs of lines in two and three dimensions, remembering to move parallel to the axes when you plot coordinates of points or components of direction vectors.

Review Exercise

1. Consider any line in space that does not pass through the origin.
   a. Is it possible for this line to intersect just one coordinate axis? exactly two? all three? none at all?
   b. Is it possible for this line to intersect just one coordinate plane? exactly two? all three? none at all?

2. Find a vector equation of the line
   a. that passes through the points $(3, 9)$ and $(-4, 2)$
   b. that passes through the point $(-5, -3)$ and is parallel to the line $\vec{r} = (4, 0) + t(0, 5)$
   c. that is perpendicular to the line $2x - 5y - 6 = 0$ and passes through the point $(0, -3)$

3. Find parametric equations of the line
   a. that passes through $(-9, 8)$ with slope $-\frac{2}{3}$
   b. that passes through $(3, -2)$ and is perpendicular to the line $\vec{r} = (4, -1) + t(3, 2)$
   c. through the points $(4, 0)$ and $(0, -2)$

4. Find a vector equation of the line
   a. that passes through the points $(2, 0, -3)$ and $(-3, 2, -2)$
   b. that has an x-intercept of -7 and a z-intercept of 4
   c. that is parallel to $\frac{x - 5}{4} = \frac{y + 2}{-2} = \frac{z + 6}{5}$ and passes through the point $(0, 6, 0)$

5. Find parametric equations of the line
   a. that is parallel to the line $\frac{x + 1}{-3} = \frac{y + 2}{-2} = z + 3$ and passes through the origin
   b. that passes through the point $(6, -4, 5)$ and is parallel to the y-axis
   c. that has a z-intercept of -3 and direction vector $(1, -3, 6)$

6. Find the Cartesian equation of the line
   a. that passes through the point $(-1, -2)$ and is parallel to the line $3x - 4y + 5 = 0$
   b. that passes through the point $(-7, 3)$ and is perpendicular to the line $x = 2 + t$, $y = -3 + 2t$
   c. that passes through the origin and is perpendicular to the line $x + 4y + 1 = 0$

7. a. Find the parametric equations of the line l that passes through the point $A(6, 4, 0)$ and is parallel to the line passing through $B(-2, 0, 4)$ and $C(3, -2, 1)$.
   b. If $(-4, m, n)$ is a point on l, find m and n.

8. Determine if the following pairs of lines are parallel and distinct, coincident, perpendicular, or none of these.
   a. $\vec{r} = (2, 3) + t(-3, 1)$ and $\vec{r} = (-1, 4) + u(6, -2)$
   b. $x = 1 + 2t$, $y = -3 - t$ and $x = u$, $y = \frac{1}{3} + 2u$
   c. $\frac{x - 1}{2} = \frac{y + 4}{1}$, $z = 1$ and $x = 4t$, $y = 1 + 2t$, $z = 6$
   d. $(x, y, z) = (1, 7, 2) + t(-1, -1, 1)$ and $(x, y, z) = (-3, 0, 1) + u(2, -2, -2)$

9. At what points does the line $\frac{x + 4}{2} = \frac{y - 6}{-1} = \frac{z + 2}{4}$ meet the coordinate planes?

10. In the xy-plane,
    a. find the Cartesian equation of the line $\vec{r} = (2, 3) + t(-1, 5)$
    b. find a vector equation of the line $5x - 2y + 10 = 0$
    c. find a vector equation of the line $y = \frac{3}{4}x + \frac{1}{2}$

11. Given the line $\vec{r} = (12, -8, -4) + t(-3, 4, 2)$,
    a. find the intersections with the coordinate planes, if any
    b. find the intercepts with the coordinate axes, if any
    c. graph the line in an x-, y-, z-coordinate system

12. Find the direction cosines and the direction angles (to the nearest degree) of the direction vectors of the following lines.
    a. $\frac{x - 3}{5} = \frac{y + 6}{2} = \frac{z - 1}{-1}$
    b. $x = 1 + 8t$, $y = 2 - t$, $z = 4 - 4t$
    c. $\vec{r} = (-7, 0, 0) + t(4, 1, 0)$

13. Find the intersection, if any, of

a. the line $\vec{r} = (0, 0, 2) + t(4, 3, 4)$ and the line $\vec{r} = (-4, 1, 0) + u(-4, 1, -2)$

b. the line $x = t$, $y = 1 + 2t$, $z = 3 - t$ and the line $x = -3$, $y = -6 + 2u$, $z = 3 - 6u$

14. Find the shortest distance between

a. the points $(2, 1, 3)$ and $(0, -4, 7)$

b. the point $(3, 7)$ and the line $2x - 3y = 7$

c. the point $(4, 0, 1)$ and the line $\vec{r} = (2, -2, 1) + t(1, 2, -1)$

d. the point $(1, 3, 2)$ and the line $\frac{x-1}{-1} = \frac{y-3}{1} = \frac{z-7}{2}$

15. Find the coordinates of the foot of the perpendicular from $Q(3, 2, 4)$ to the line $\vec{r} = (-6, -7, -3) + t(5, 3, 4)$.

Rich Learning Link wrap-up

investigate and apply

CHAPTER 7: EQUATIONS OF LINES

Beauty is a cultural concept. What is beautiful to one person may not appeal to another. Nevertheless, when people consider the aesthetic qualities of an object, they usually consider some relationship between the complexity of the object and its orderliness. Repetition and symmetry are two ways in which an object may contain order.

Investigate and Apply

The exercises below require you to draw graphs. A graphing utility such as a graphing calculator will be helpful. Most graphing calculators have a mode setting that allows you to draw parametric curves. Remember to use radian mode.

1. Graph $x = 6 \cos t$, $y = 6 \sin t$, $0 \leq t \leq 2\pi$.
2. Find and verify parametric equations for an ellipse.
3. Graph $x = \frac{1}{2}t \cos t$, $y = \frac{1}{2}t \sin t$, $0 \leq t \leq 4\pi$.
4. Graph $x = \frac{4}{\cos t}$, $y = 3 \tan t$, $0 \leq \text{t} \leq 2\pi$.
5. Graph $x = 2t - 2 \sin t$, $y = 2 - 2 \cos t$, $t \geq 0$. The resulting shape is called a cycloid. It is a type of trochoid. It represents the path of a point on the edge of a circle as the circle rolls along the x-axis.
6. a) Graph an epicycloid: $x = 5 \cos t - 2 \cos\left(\frac{5t}{2}\right)$, $y = 5 \sin t - 2 \sin\left(\frac{5t}{2}\right)$, $0 \leq t \leq 4\pi$.

   b) Graph an epitrochoid: $x = 5 \cos t - 4 \cos\left(\frac{5t}{2}\right)$, $y = 5 \sin t - 4\, sin\left(\frac{5t}{2}\right)$, $0 \leq t \leq 4\pi$.
7. Graph a tricuspoid: $x = 6 \cos t + 3 \cos(2t)$, $y = 6 \sin t - 3 \sin(2t)$, $0 \leq t \leq 2\pi$.
8. Graph a Lissajous curve: $x = 8 \sin(3t + 1)$, $y = 8 \sin t$, $0 \leq t \leq 2\pi$.
9. What does $x = 6 \cos t$, $y = 6 \sin t$, $z = t$, $t \geq 0$ describe?

INDEPENDENT STUDY

Investigate: Find parametric equations for an Astroid, a hypocycloid, a Nephroid, a Plateau curve, and the Folium of Descartes.

Experiment: Create a parametric curve different from the ones you have seen here. Try to create one with features that make your curve unique.

Investigate: What are polar coordinates? What curves can be described using polar coordinates? ●

Chapter 7 Test

Achievement Category	Questions
Knowledge/Understanding	1, 4
Thinking/Inquiry/Problem Solving	5, 7
Communication	3
Application	2, 6

1. A line goes through the points (9, 2) and (3, 4). Determine
   a. its vector equation
   b. its parametric equations
   c. its symmetric equation
   d. its scalar equation

2. Find the scalar equation of the line which is perpendicular to the line $2x - 3y + 18 = 0$ and has the same y-intercept as the line $(x, y) = (0, 1) + t(-3, 4)$.

3. Find any two of the three intersections of the line $\frac{x-2}{6} = \frac{y-4}{3} = \frac{z+2}{-3}$ with the coordinate planes, and graph the line.

4. Find the distance from the point $(1, -2, -3)$ to the line $x = y = z - 2$.

5. A line through the origin has direction angles $\beta = 120°$ and $\gamma = 45°$. Find a vector equation for the line.

6. Determine the point of intersection of the two lines $(x, y, z) = (-2, 0, -3) + t(5, 1, 3)$ and $\frac{x-5}{-1} = \frac{y-8}{2} = \frac{z+6}{-3}$

7. Let $l_1{:}x = -8 + t, y = -3 - 2t, z = 8 + 3t$ and $l_2{:}\frac{x-1}{2} = \frac{y+1}{1} = \frac{z}{3}$ be two lines in three-dimensional space.
   a. Show that l_1 and l_2 are skew lines (that is, neither parallel nor intersecting).
   b. State the coordinates of P_1, the point on l_1 determined by $t = -2$.
   c. Determine the coordinates of P_2, the point on l_2 such that $P_1 P_2$ is perpendicular to l_2.

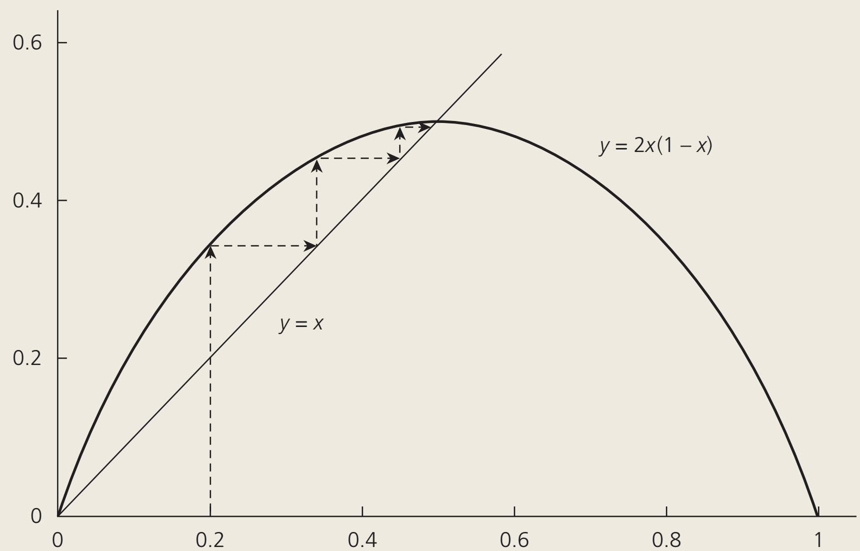
Extending and Investigating

CHAOS

The simple quadratic function $f(x) = rx(1 - x)$ where r is a specified constant can be used to demonstrate some of the most interesting ideas in modern mathematics. You can easily check that the graph of this function is symmetric about $x = \frac{1}{2}$ with a maximum value of $\frac{r}{4}$. We are interested in the function when $0 < x < 1$ and $0 < r < 4$ so that $0 < f(x) < 1$.

We can define a sequence by specifying x_0 and then each subsequent term by $x_n = f(x_{n-1})$, $n = 1, 2, \ldots$
If we start with $r = 2.0$ and $x_0 = 0.2$, we get
$x_1 = 2 \times 0.2 \times 0.8 = 0.32$, $x_2 = 0.4352$, $x_3 = 0.491602$, ... All terms are between 0 and 1.
It is easy to calculate the terms of this sequence on a spreadsheet. Note that the sequence can be written as $x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots$

We can trace the development of the sequence on a plot of the function shown below. The line $y = x$ is also shown on the plot.



We start at the point (0.2, 0). The next point is $(0.2, f(0.2))$. We then move horizontally to the line $y = x$ to get the point $(f(0.2), f(0.2))$. The next point is $(f(0.2), f(f(0.2))$and so on. The points on the curve have coordinates (x_n, x_{n+1}), $n = 0, 1, \ldots$.

The key question is to determine what happens as n gets large. In this case, we see that the terms of the sequence will approach 0.5. If we start with any value $0 < x_0 < 1$, then the sequence will converge to the same value — this is easy to see from the graph and to check with a spreadsheet calculation.

This does not seem very interesting. However, let's see what happens if we change the value of the multiplier to and carry out the same calculations with $x_0 = 0.2$. The first few terms of the sequence are

n	0	1	2	3	4	5	6
x_n	0.2	0.512	0.800	0.513	0.799	0.513	0.799

If you construct the corresponding plot, you will see that there are now two points of convergence at 0.513 and 0.799. When n is large, the sequence oscillates between these two values. Most starting values produce the same limiting behaviour, but some, such as $x_0 = \frac{5}{16}$, produce a single point of convergence. Can you figure out why?

If we further increase r to 3.5, we find that there are four points of convergence for most starting points. In fact by slowly increasing r, we can get 8, 16, 32 , … points of convergence. The big surprise occurs about $r = 3.57$. Suddenly, there is no apparent pattern in the sequence for many starting points and there are no points of convergence. Different starting values lead to different sequences. The sequence is called *chaotic*. Try generating this sequence on a spreadsheet with $x_0 = 0.2$.

Even stranger, if we look at larger values of r, there are some values for which the sequence is chaotic and some for which there are regular oscillations. Write a spreadsheet program to try some values of x_0 and r. Can you produce the plot of the sequence as described above?

In the chaotic case, a very small change in the value of r can lead to a complete change of behaviour of the sequence. For mathematicians used to continuous behaviour, this abrupt change is fascinating. The study of this sequence and its generalizations is called *chaos theory*, a very active branch of modern mathematics.

Chapter 8

EQUATIONS OF PLANES

The concepts of point and line are as fundamental to geometry in three-dimensional space as they are in two-dimensional space. In three-dimensional space, however, we have another fundamental issue to consider, that of the plane. Is there such a thing as the equation or equations of a plane? Does a plane have a direction in space? In this chapter, we will look at the relationships between a point and a plane, a line and a plane, two planes, and even three planes.

CHAPTER EXPECTATIONS In this chapter, you will

- determine the vector, parametric, and scalar equations of planes, **Section 8.1, 8.2**
- determine the intersection of a line and a plane in three-dimensional space, **Section 8.3**
- determine the intersection of two or three planes, **Section 8.4**
- solve systems of linear equations involving up to three unknowns using row reduction of matrices, with and without the aid of technology, **Section 8.4**
- interpret row reduction of matrices as the creation of a new linear system equivalent to the original, **Section 8.4**
- interpret linear equations in two and three unknowns, **Section 8.5**

Review of Prerequisite Skills

Two points or one vector (a direction vector) and one point determine a line, a one-dimensional object. A third point not on that line opens the door to a second dimension. Thus, three non-collinear points or two non-collinear vectors and a point determine a plane, a two-dimensional object. Similarly, four non-coplanar points or three non-coplanar vectors determine a three-dimensional space. These concepts can be generalized to higher dimensional spaces, which despite their abstract nature have a surprising number of applications in the physical sciences, engineering, and economics.

The equation of a two-dimensional plane in a three-dimensional space has several forms. These are developed in the first part of this chapter in much the same way as those of a straight line.

Recall in particular

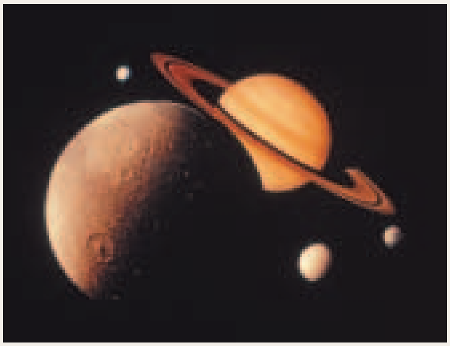
the **vector equation**	$\vec{r} = \vec{r}_0 + t\vec{d}$
the **parametric equations**	$x = x_0 + at$ $y = y_0 + bt$ $z = z_0 + ct$
the **scalar equation of a line** in **two-dimensional space**	$Ax + By + C = 0$

Equations of lines and planes are essential parts of computer systems used by engineers and architects for computer-assisted design.

The remainder of this chapter is concerned with how lines can intersect with planes and planes can intersect with other planes. Intersection problems are geometrical models of linear systems. Therefore, this chapter includes an introduction to systematic methods for solving linear systems using matrices.

Rich Learning Link investigate

CHAPTER 8: SUN ELEVATION



Astronomers have always sought methods for determining and predicting the positions of objects in the sky. One reason for doing this has been to test and improve upon our models of the solar system. In this way, astronomers have already learned, among other things, that the earth revolves around the sun once every 365.25 solar days in an elliptical orbit at a distance varying between 147.1 and 152.1 million kilometres. The earth rotates on its axis once every 23 hours, 56 minutes, and 4 seconds. This is known as a sidereal day. The axis of rotation is tilted 23.45° from perpendicular to the plane of the earth's orbit. We shall investigate, here and in the wrap-up at the end of the chapter, how to determine the angle of elevation of the sun at any given time of any given day at any given place on the surface of the earth.

Investigate and Inquire

The angle of elevation of the sun is the angle between the vector from the earth to the sun and the plane tangent to the surface of the earth. Planes will be studied in this chapter. Here we shall set out some groundwork for our calculations.

We will make the following assumptions to simplify our calculations: the earth is a perfect sphere that orbits the sun every 365 days in a perfect circle whose radius is 150 million kilometres. We shall let d be the number of solar days past December 21, the date of the winter solstice in the northern hemisphere. We will assume that at noon on December 21, the north pole is pointed away from the sun as much as possible. We let h be the number of hours (positive or negative) from noon. Noon here means the time when the sun is highest, regardless of the standardized time-zone time. It is the time halfway between sunrise and sunset. Let θ be the latitude of the observer.

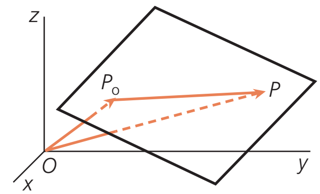
If we place the sun at the origin of a three-dimensional Cartesian coordinate system, we can parameterize the earth's orbit in the xy-plane as $x = -150 \sin\left(\frac{360d}{365}\right)$, $y = 150 \sin\left(\frac{360d}{365}\right)$ (See the Rich Learning Link on parametric curves in Chapter 7.) The vector $\vec{s}$, from the earth to the sun, will be $\vec{s} = (-x, -y, 0) = \left(150 \sin\left(\frac{360d}{365}\right), -150 \cos\left(\frac{360d}{365}\right), 0\right)$. Why is it $(-x, -y, 0)$ and not $(x, y, 0)$?

DISCUSSION QUESTIONS

1. What is the angle of elevation of the sun at sunrise and sunset? How might we interpret a negative angle of elevation?

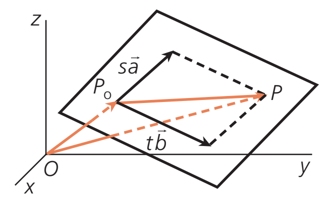
2. Why is there a difference between the length of the solar day (the 24 hours between successive noons) and the sidereal day?

Section 8.1 — The Vector Equation of a Plane in Space



The vector equation of a plane gives the position vector $\overrightarrow{OP}$ of any point $P(x, y, z)$ in the plane. It is constructed in the same way as the vector equation of a line. First, write the position vector $\overrightarrow{OP}$ as the sum of two vectors: $\overrightarrow{OP_0}$, the vector from the origin to some particular point $P_0(x_0, y_0, z_0)$ in the plane, and $\overrightarrow{P_0P}$, the vector from the particular point P_0 to the general point P.

$$\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$$



Now, choose two non-collinear vectors in the plane as basis vectors for the plane. Call them $\vec{a}$ and $\vec{b}$. These two vectors are known as direction vectors for the plane. Express the point-to-point vector $\overrightarrow{P_0P}$ as a linear combination of $\vec{a}$ and $\vec{b}$. We write

$$\overrightarrow{P_0P} = s\vec{a} + t\vec{b}$$

Therefore, $\overrightarrow{OP} = \overrightarrow{OP_0} + s\vec{a} + t\vec{b}$

or, letting $\vec{r}$ and $\vec{r_0}$ stand for the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OP_0}$, respectively,

$$\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$$

The vector equation of a plane has the form

$$\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$$

where $\vec{a}$ **and** $\vec{b}$ **are direction vectors for the plane,**
$\vec{r_0}$ **is the position vector of a particular point in the plane**
and $s, t \in R$.

The coefficients s and t in the vector equation of a plane are parameters. There are two parameters because a plane is two-dimensional. The parametric equations of the plane are equations for the components of $\vec{r}$.

The parametric equations of a plane have the form

$$x = x_0 + sa_1 + tb_1$$
$$y = y_0 + sa_2 + tb_2$$
$$z = z_0 + sa_3 + tb_3$$

where (a_1, a_2, a_3) **and** (b_1, b_2, b_3) **are components of the direction vectors** $\vec{a}$ **and** $\vec{b}$ **for the plane,**
(x_0, y_0, z_0) **are components of the position vector of a specific point in the plane,**
and $s, t \in R$.

In reality, a plane is a flat surface that extends infinitely in all directions. In the diagrams on page 276, we have depicted a plane using a parallelogram. This gives a three-dimensional perspective to the diagrams and suggests that the plane may be oriented at some angle to the coordinate axes. Although not true graphs, such diagrams are adequate for analyzing most problems about lines and planes in three dimensions.

EXAMPLE 1

Find vector and parametric equations of the plane that contains the three points $A(1, 0, -3)$, $B(2, -3, 1)$, and $C(3, 5, -3)$.

Solution

The point-to-point vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ both lie in the plane. They are

$$\overrightarrow{AB} = (1, -3, 4)$$
$$\overrightarrow{AC} = (2, 5, 0)$$

Since these vectors are non-collinear, they can serve as direction vectors for the plane. Taking point A as the given point, $\vec{r}_0 = \overrightarrow{OA} = (1, 0, -3)$. Therefore, a vector equation of the plane is

$$\vec{r} = (1, 0, -3) + s(1, -3, 4) + t(2, 5, 0)$$

The parametric equations can be written down by inspection.

$$x = 1 + s + 2t$$
$$y = -3s + 5t$$
$$z = -3 + 4s$$

It should be clear that the vector and parametric equations of a plane are not unique. In Example 1, if $\overrightarrow{BA}$ and $\overrightarrow{BC}$ had been chosen as direction vectors and point B as the given point, then the vector equation would have been

$$r = (2, -3, 1) + s(-1, 3, -4) + t(1, 8, -4)$$

When two equations look entirely different, how do you decide if they represent the same plane? This question will be addressed in the next section.

EXAMPLE 2 Does the point $(4, 5, -3)$ lie in the plane $\vec{r} = (4, 1, 6) + p(3, -2, 1) + q(-6, 6, -1)$?

Solution

The parametric equations are

$$x = 4 + 3p - 6q$$
$$y = 1 - 2p + 6q$$
$$z = 6 + p - q$$

If the point lies in the plane, the coordinates of the point, $(4, 5, -3)$, must satisfy these equations. Substitution gives

$$\begin{aligned} 4 &= 4 + 3p - 6q \\ 5 &= 1 - 2p + 6q \\ -3 &= 6 + p - q \end{aligned} \qquad \text{or} \qquad \begin{aligned} 3p - 6q &= 0 \\ -2p + 6q &= 4 \\ p - q &= -9 \end{aligned}$$

Solving the first two equations gives $p = 4$, $q = 2$. But these values of p and q do not satisfy the third equation. Therefore, the point does not lie in the plane. You can also see that these values of p and q produce $z = 8$ for the z-coordinate of the point, not $z = -3$ as they should.

EXAMPLE 3 Find the vector equation of the plane that contains the two parallel lines

$$l_1: \vec{r} = (2, 4, 1) + t(3, -1, 1)$$
$$l_2: \vec{r} = (1, 4, 4) + t(-6, 2, -2)$$

Solution

We take $(2, 4, 1)$ from l_1 as the position vector $\vec{r}_0$ of a given point on the plane and $(3, -1, 1)$ as $\vec{a}$, one of the direction vectors.

For the second direction vector, use the point-to-point vector between the given points on the two lines, $(1, 4, 4) - (2, 4, 1) = (-1, 0, 3)$. A vector equation of the plane is thus

$$\vec{r} = (2, 4, 1) + t(3, -1, -1) + s(-1, 0, 3).$$

Exercise 8.1

Part A

Communication

1. Why does the vector equation of a plane have two parameters while the vector equation of a line has only one?

Communication

2. **a.** State two direction vectors for the xz-coordinate plane.

b. What do all direction vectors for the xz-coordinate plane have in common?

Knowledge/Understanding

3. State two direction vectors for each of the following planes.

a. $\vec{r} = (9, 5, 2) + s(-3, 5, 2) + t(-6, 1, 2)$

b. a plane parallel to the plane $x = 3 + 5s + t$
$y = -2 - 5s + 6t$
$z = -2 + 3s - 2t$

c. the plane containing the intersecting lines $\vec{r} = (6, 5, -2) + s(4, -2, 1)$ and $\vec{r} = (-10, -3, 1) + t(-1, 5, 2)$

Knowledge/Understanding

4. State two points that lie on each of the following planes.

a. $\vec{r} = (9, 4, -3) + t(-2, 2, 1) + p(0, -2, 6)$

b. $\vec{r} = (0, 1, 0) + t(1, 0, -2) + p(0, 0, 4)$

c. $x = 3 + 5s + t$
$y = -2 - 5s + 6t$
$z = -2 + 3s - 2t$

d. the xz-plane

Knowledge/Understanding

5. Write parametric equations for each of these planes.

a. $\vec{r} = (-4, -6, 3) + s(5, 2, 3) + t(-4, -6, 3)$

b. $\vec{r} = (0, 0, 1) + s(0, 2, 0) + t(3, 0, 0)$

c. the xz-plane

Knowledge/Understanding

6. Write a vector equation for each of these planes.

a. $x = -4 + s + 3t$
$y = -1 + 3s - 4t$
$z = 3 + 4s - t$

b. $x = 7s$
$y = 4$
$z = -2t$

c. the xz-plane

Part B

7. Determine a vector equation of each of the following planes.
   a. the plane through the point $(-4, 5, 1)$ parallel to the vectors $(-3, -5, 3)$ and $(2, -1, -5)$
   b. the plane containing the two intersecting lines $\vec{r} = (4, 7, 3) + t(1, 4, 3)$ and $\vec{r} = (-1, -4, 6) + s(-1, -1, 3)$
   c. the plane containing the line $\vec{r} = (-3, 4, 6) + t(-5, -2, 3)$ and the point $(8, 3, 5)$
   d. the plane containing the two parallel lines $\vec{r} = (0, 1, 3) + t(-6, -3, 6)$ and $\vec{r} = (-4, 5, -4) + s(4, 2, -4)$
   e. the plane containing the three points $(2, 6, -5)$, $(-3, 1, -4)$, and $(6, -2, 2)$.

8. Determine parametric equations of each of the following planes.
   a. the plane through the point $(7, -5, 2)$ parallel to the vectors $(4, -1, 1)$ and $(-3, 4, 4)$
   b. the plane containing the two intersecting lines $\vec{r} = (5, 4, 2) + t(4, -2, 1)$ and $\vec{r} = (7, 4, -7) + s(-3, 1, 4)$
   c. the plane containing the line $\vec{r} = (1, 3, -1) + t(2, 2, -5)$ and the point $(8, 3, 5)$
   d. the plane containing the two parallel lines $\vec{r} = (3, 2, 2) + t(-9, 6, -6)$ and $\vec{r} = (1, 6, -6) + s(6, -4, 4)$
   e. the plane containing the three points $(2, 6, -5)$, $(-3, 1, -4)$, and $(6, -2, 2)$

9. Determine the vector equation of each of the following planes.
   a. the plane parallel to the *yz*-plane containing the point $(6, 4, 2)$
   b. the plane containing the origin and the points $(3, 3, 3)$ and $(8, -1, -1)$
   c. the plane containing the *x*-axis and the point $(-1, -4, -7)$

10. a. Explain why the three points $(2, 3, -1)$, $(8, 5, -5)$, and $(-1, 2, 1)$ *do not* determine a plane.
    b. Explain why the line $\vec{r} = (4, 9, -3) + t(1, -4, 2)$ and the point $(8, -7, 5)$ *do not* determine a plane.

Application

11. Find vector and parametric equations of the plane that contains the line $x = 7 - t$, $y = -2t$, $z = -7 + t$ and that does not intersect the z-axis.

12. Demonstrate that a plane with a vector equation of the form $\vec{r} = (a, b, c) + s(d, e, f) + t(a, b, c)$ passes through the origin.

Part C

Thinking/Inquiry/ Problem Solving

13. a. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are the position vectors of three points A, B, and C. Show that $\vec{r} = p\vec{a} + s\vec{b} + t\vec{c}$, where $p + s + t = 1$ is an equation of the plane containing these three points.

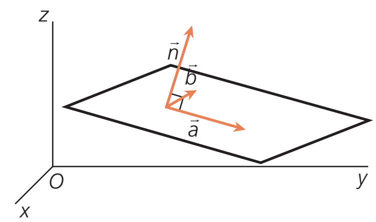
b. What region of the plane is determined by the equation, when the parameters s and t are restricted to the values $0 \leq s \leq 1$, and $0 \leq t \leq 1$? (*Hint:* replace p with $(1 - s - t)$.)

14. a. The equation $\vec{r} = \vec{r_0} + t\vec{d}$ is a vector equation of a line and $\vec{q}$ is the position vector of a point Q not on the line. Show that $\vec{r} = k\vec{r_0} + l\vec{q} + t\vec{d}$, where $k + l = 1$ is an equation of the plane containing the line and the point.

b. What region of the plane is determined by the equation, when the parameter k is restricted to $0 \leq k \leq 1$? (*Hint:* replace l by $(1 - k)$.)

Section 8.2 — The Scalar Equation of a Plane in Space

Any vector that is perpendicular to a plane is a **normal vector** or simply a **normal** to the plane. You can find the normal to a plane by finding the cross product of the two direction vectors of the plane. Since every vector in the plane can be represented as a linear combination of the direction vectors, the normal is perpendicular to every vector in the plane.



EXAMPLE 1

a. Find a normal to the plane with vector equation $\vec{r} = (3, 0, 2) + s(2, 0, -1) + t(6, 2, 0)$.

b. Show that the normal is perpendicular to every vector in the plane.

Solution

a. The two direction vectors of the plane are $(2, 0, -1)$ and $(6, 2, 0)$. The cross product of the direction vectors is $(2, 0, -1) \times (6, 2, 0) = (2, -6, 4)$. Thus, a normal to the plane is $(2, -6, 4)$ or $(1, -3, 2)$.

b. Any vector in the plane can be written as a linear combination of the two direction vectors, say $\vec{v} = p(2, 0, -1) + q(6, 2, 0)$. To show that the normal is perpendicular to $\vec{v}$, find the dot product.

$$\begin{aligned}\vec{v} \cdot \vec{n} &= [p(2, 0, -1) + q(6, 2, 0)] \cdot (1, -3, 2)\\ &= p(2, 0, -1) \cdot (1, -3, 2) + q(6, 2, 0) \cdot (1, -3, 2)\\ &= p(0) + q(0)\\ &= 0\end{aligned}$$

Since the dot product is zero, the two vectors must be perpendicular. This result is independent of the values of p and q.

You can use the fact that the normal to a plane is perpendicular to every vector in the plane to derive the scalar equation of a plane. Let $P(x, y, z)$ be any point in a plane with normal (A, B, C), and let $P_0(x_0, y_0, z_0)$ be some particular point in the plane. The vector $\overrightarrow{P_0P}$ must lie in the plane because its endpoints do. Therefore, it must be perpendicular to the normal (A, B, C), and their dot product must be zero.

$$\begin{aligned}(A, B, C) \cdot \overrightarrow{P_0P} &= 0\\ (A, B, C) \cdot (x - x_0, y - y_0, z - z_0) &= 0\\ A(x - x_0) + B(y - y_0) + C(z - z_0) &= 0\\ Ax + By + Cz + (-Ax_0 - By_0 - Cz_0) &= 0\end{aligned}$$

The quantity in brackets is a constant because the components of the normal and the coordinates of the given point have particular numerical values.
Letting $D = (-Ax_0 - By_0 - Cz_0)$ the result is

$$Ax + By + Cz + D = 0$$

The scalar or Cartesian equation of a plane in space is

$$Ax + By + Cz + D = 0$$

where (A, B, C) is a vector normal to the plane.

Unlike the vector equation, the scalar equation of a plane is unique. For instance, the equations $x + 2y + 3z + 4 = 0$ and $2x + 4y + 6z + 8 = 0$ represent the same plane, since one equation is a multiple of the other.

EXAMPLE 2

a. Find the scalar equation of the plane with vector equation $\vec{r} = (3, 0, 2) + p(2, 0, -1) + q(6, 2, 0)$.

b. Show that $\vec{r} = (-1, -2, 1) + s(5, 3, 2) + t(2, 4, 5)$ is another vector equation of the same plane.

Solution

a. In Example 1, a normal to this plane was found to be $(1, -3, 2)$. Therefore,

$$(A, B, C) = (1, -3, 2)$$

and $$x - 3y + 2z + D = 0$$

The vector $(3, 0, 2)$ is given as the position vector of a point on this plane. Then

$$(3) - 3\,(0) + 2\,(2) + D = 0$$
$$D = -7$$

Therefore, the scalar equation is $x - 3y + 2z - 7 = 0$.

b. For $\vec{r} = (-1, -2, 1) + s(5, 3, 2) + t(2, 4, 5)$, the normal is $(5, 3, 2) \times (2, 4, 5) = (7, -21, 14)$.
Therefore, $(7, -21, 14)$ or $(1, -3, 2)$ is a normal to the plane.
The scalar equation of the plane is

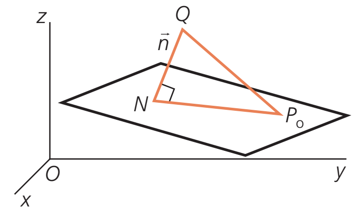
$$x - 3y + 2z + D = 0$$

Now, substitute the point $(-1, -2, 1)$ into this equation to find D.

$$(-1) - 3(-2) + 2(1) + D = 0$$
$$D = -7$$

The scalar equation of this plane is $x - 3y + 2z - 7 = 0$, so the two vector equations represent the same plane, or the planes represented by the two vector equations are coincident.

The distance from a point to a plane in three dimensions is calculated in much the same way as the distance from a point to a line in two dimensions. It is measured along the normal to the plane. If Q is some point not in the plane and P_0 is any point in the plane, then the distance $|QN|$ from Q to the plane is the projection of $\overrightarrow{P_0Q}$ onto the normal $\vec{n}$.



EXAMPLE 3 Find the distance from the point $Q(1, 3, -2)$ to the plane $4x - y - z + 6 = 0$.

Solution

The distance is the projection of $\overrightarrow{P_0Q}$ onto the normal $(4, -1, -1)$. For P_0, choose any point in the plane, say $(0, 0, 6)$. Then $\overrightarrow{P_0Q} = (1, 3, -8)$. The distance is then

$$\begin{aligned} d &= \left|\text{Proj}(\overrightarrow{P_0Q} \text{ onto } \vec{n})\right| \\ &= \frac{|\overrightarrow{P_0Q} \bullet \vec{n}|}{|\vec{n}|} \\ &= \frac{|(1, 3, -8) \bullet (4, -1, -1)|}{\sqrt{(4)^2 + (-1)^2 + (-1)^2}} \\ &= \frac{9}{\sqrt{18}} \\ &= \frac{3}{\sqrt{2}} \\ &= \frac{3\sqrt{2}}{2} \end{aligned}$$

A general formula can be derived by following the same steps. If P_0Q is the vector from some point P_0 on the plane $Ax + By + Cz + D = 0$ to a point $Q(x_1, y_1, z_1)$ off the plane, then the distance d from Q to the plane is the projection of P_0Q onto the normal (A, B, C).

$$\begin{aligned} d &= \left|\text{Proj}(\overrightarrow{P_0Q} \text{ onto } \vec{n})\right| \\ &= \frac{|\overrightarrow{P_0Q} \bullet \vec{n}|}{|\vec{n}|} \\ &= \frac{|(x_1 - x_0, y_1 - y_0, z_1 - z_0) \bullet (A, B, C)|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|Ax_1 + By_1 + Cz_1 + (-Ax_0 - By_0 - Cz_0)|}{\sqrt{A^2 + B^2 + C^2}} \end{aligned}$$

Since P_0 is a point in the plane, it satisfies the equation of the plane, so $Ax_0 + By_0 + Cz_0 + D = 0$ or $D = -Ax_0 - By_0 - Cz_0$. Substituting this into the above equation gives the following result.

The distance from the point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is given by the formula

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Note the structure. The numerator uses the equation of the plane, with the coordinates of the point off the plane substituted for x, y, and z. The denominator is the magnitude of the normal.

In the special case when the point $Q(x_1, y_1, z_1)$ is the origin, the distance to the plane $Ax + By + Cz + D = 0$ is

$$d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}$$

EXAMPLE 4

What is the distance between the planes $2x - y - 2z + 3 = 0$ and $4x - 2y - 4z - 9 = 0$?

Solution

The planes are parallel, since $\vec{n}_2 = (4, -2, -4)$ is a multiple of $\vec{n}_1 = (2, -1, -2)$. The distance between the planes is the distance from a point in the first plane to the second plane. The point $(0, 3, 0)$ is on the first plane. Then

$$\begin{aligned} d &= \frac{|4(0) - 2(3) - 4(0) - 9|}{\sqrt{(4)^2 + (-2)^2 + (-4)^2}} \\ &= \frac{|-15|}{\sqrt{36}} \\ &= \frac{5}{2} \end{aligned}$$

Exercise 8.2

Part A

Knowledge/Understanding

1. For each of the following, find the scalar equation of the plane that passes through the point P_0 and has normal $\vec{n}$.

a. $P_0(2, 1, -3)$, $\vec{n} = (7, 1, -1)$
b. $P_0(5, 1, 9)$, $\vec{n} = (1, 0, 0)$
c. $P_0(0, 6, -2)$, $\vec{n} = (2, 0, 3)$
d. $P_0(0, 0, 0)$, $\vec{n} = (2, -1, 4)$

Knowledge/ Understanding

2. Determine the scalar equation of the plane that passes through $(1, -2, 3)$ and has a normal

a. parallel to the y-axis

b. perpendicular to the xy-plane

c. parallel to the normal of the plane $x - y - 2z + 19 = 0$

Communication

3. a. Find the scalar equation of the plane that passes through the origin and has a normal $\vec{n} = (A, B, C)$.

b. How can you tell by inspection of the scalar equation of a plane whether or not the plane passes through the origin?

Communication

4. a. What is the orientation of a plane in space when two of the three variables x, y, and z are missing from its scalar equation?

b. What is the orientation of a plane in space when only one of the three variables x, y, or z is missing from its scalar equation?

5. Find the scalar equation of each of the following planes. State which of the planes, if any, are coincident.

a. $\vec{r} = (-8, -1, 8) + s(-5, 1, 4) + t(3, 2, -4)$

b. $\vec{r} = (-2, -2, 5) + s(3, 1, -1) + t(4, 1, -4)$

c. $\vec{r} = (2, 0, 0) + s(0, 4, 0) + t(0, 0, -3)$

d. $\vec{r} = (-8, 2, 0) + s(4, 0, 3) + t(0, -2, -5)$

e. $\vec{r} = (2, -11, -17) + s(0, 5, 13) + t(0, 3, 10)$

f. $\vec{r} = (13, 0, -12) + s(-1, 8, -4) + t(11, 3, -12)$

6. Find the scalar equation of each of the following planes.

a. $x = 4 + 3s - 2t$
$y = 2 + 4s + 4t$
$z = 1 - 2s - 3t$

b. $x = -2t$
$y = 2 - s - 3t$
$z = 5 + 3s$

7. For each of the following, find the scalar equation of the plane that passes through the given points.

a. $(1, 1, -1), (1, 2, 3), (3, -1, 2)$
b. $(2, -2, 4), (1, 1, -4), (3, 1, -6)$
c. $(1, 1, 1), (-1, 1, 1), (2, 1, 2)$
d. $(1, 3, 0), (0, 5, 2), (3, 4, -2)$

Part B

8. What is the scalar equation of the plane that contains the x-axis and the point $(4, -2, 1)$?

Application **9.** Find the scalar equation of the plane that contains the intersecting lines

$$\frac{x-2}{1} = \frac{y}{2} = \frac{z+3}{3} \text{ and } \frac{x-2}{-3} = \frac{y}{4} = \frac{z+3}{2}$$

Application **10.** Determine whether the following pairs of planes are coincident, parallel and distinct, or neither.

a. $x + 3y - z - 2 = 0$ and $2x + 6y - 2z - 8 = 0$

b. $2x + y + z - 3 = 0$ and $6x + 2y + 2z - 9 = 0$

c. $3x - 3y + z - 2 = 0$ and $6x - 6y + 2z - 4 = 0$

d. $2x - 4y + 2z - 6 = 0$ and $3x - 6y + 3z - 9 = 0$

11. Find a vector equation for the plane with scalar equation

a. $2x - y + 3z - 24 = 0$ b. $3x - 5z + 15 = 0$

12. Which of the following lines is parallel to the plane $4x + y - z - 10 = 0$? Do any of the lines lie in the plane?

a. $\vec{r} = (3, 0, 2) + t(1, -2, 2)$

b. $x = -3t, y = -5 + 2t, z = -10t$

c. $\frac{x-1}{4} = \frac{y+6}{-1} = \frac{z}{1}$

13. The angle between two planes is defined as the angle between their normals. Determine the angle θ $(0 \le \theta \le 90°)$, to the nearest degree, between the given planes.

a. $2x + 3y - z + 9 = 0$ and $x + 2y + 4 = 0$

b. $x - y - z - 1 = 0$ and $2x + 3y - z + 4 = 0$

Part C

14. If the positive z-axis points up, show that the line $x = 0, y = t, z = 2t$

a. is parallel to and below the plane $2x - 10y + 5z - 1 = 0$

b. is parallel to and above the plane $x + 4y - 2z - 7 = 0$

15. a. Find an equation for the set of points $P(x, y, z)$ that are equidistant from the points $A(1, 2, 3)$ and $B(4, 0, 1)$.

b. What does this equation represent geometrically?

16. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are the position vectors of three points A, B, and C, respectively.

   a. Show that the scalar equation of the plane through A, B, and C can be expressed in the form $(\vec{r} - \vec{a}) \cdot (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}) = 0$.

   b. Find the scalar equation of the plane through the points $A(8, 4, -3)$, $B(5, -6, 1)$, and $C(-4, 1, 2)$.

Thinking/Inquiry/Problem Solving

17. Show that as k varies, the plane $2x + 3y + kz = 0$ rotates about a line through the origin in the xy-coordinate plane. Find parametric equations for this line.

Thinking/Inquiry/Problem Solving

18. When the coefficients A, B, and C in the scalar equation of a plane are the components of a unit normal, what is a geometrical interpretation for the constant D?

Thinking/Inquiry/Problem Solving

19. If a, b, and c are the x-intercept, the y-intercept, and the z-intercept of a plane, respectively, and d is the distance from the origin to the plane, show that

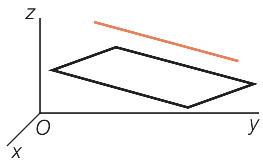
$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Thinking/Inquiry/Problem Solving

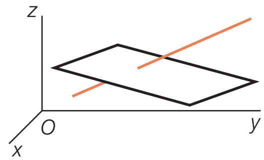
20. Find a formula for the scalar equation of a plane in terms of a, b, and c, where a, b, and c are the x-intercept, the y-intercept, and the z-intercept of a plane, respectively.

Section 8.3 — The Intersection of a Line and a Plane

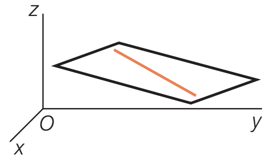
What are the possible ways that a line and a plane in three dimensions can intersect? The line can be parallel to the plane, intersecting it at no points. It can cut through the plane, intersecting it at one point. It can lie in the plane, in which case every point on the line is a point of intersection.



line is parallel to the plane



line intersects the plane



line lies in the plane

EXAMPLE 1

Find the intersection of the line with parametric equations $x = 1 + 2t$, $y = -6 + 3t$, $z = -5 + 2t$ and the plane whose scalar equation is $4x - 2y + z - 19 = 0$.

Solution

In terms of t, the coordinates of a point on the given line are $(x, y, z) = (1 + 2t, -6 + 3t, -5 + 2t)$. This point will lie on the plane if, for some particular value of t, these coordinates satisfy the equation of the plane. Substituting,

$$\begin{aligned} 4(1 + 2t) - 2(-6 + 3t) + (-5 + 2t) - 19 &= 0 \\ 4 + 8t + 12 - 6t - 5 + 2t - 19 &= 0 \\ 4t - 8 &= 0 \\ t &= 2 \end{aligned}$$

Therefore, the point on the line with parameter $t = 2$ is the point at which the line intersects the plane. Its coordinates are

$$\begin{aligned} x &= 1 + 2(2) = 5 \\ y &= -6 + 3(2) = 0 \\ z &= -5 + 2(2) = -1 \end{aligned}$$

The point of intersection of the line and the plane is $(5, 0, -1)$.

EXAMPLE 2

Find the intersection of the line $x = 2t$, $y = 1 - t$, $z = -4 + t$ and the plane $x + 4y + 2z - 4 = 0$.

Solution

We find the parameter value of the point of intersection by substituting the point $(2t, 1 - t, -4 + t)$ into the equation of the plane.

$$\begin{aligned}(2t) + 4(1 - t) + 2(-4 + t) - 4 &= 0\\ 2t + 4 - 4t - 8 + 2t - 4 &= 0\\ 0t &= 8\end{aligned}$$

There is no value of t which satisfies this equation, so there is no point at which the line intersects the plane.

This means that the line must be parallel to the plane. Its direction vector, $\vec{m} = (2, -1, 1)$, must be perpendicular to the normal to the plane, $(1, 4, 2)$. Indeed,

$$\begin{aligned}\vec{m} \bullet \vec{n} &= (2, -1, 1) \bullet (1, 4, 2)\\ &= 2 - 4 + 2\\ &= 0\end{aligned}$$

EXAMPLE 3 Find the intersection of the line $x = -4 + 3t$, $y = 0$, $z = t$ and the plane $x - 2y - 3z + 4 = 0$.

Solution

Substitute the point $(-4 + 3t, 0, t)$ into the equation of the plane to find the parameter value of the point of intersection.

$$\begin{aligned}(-4 + 3t) - 2(0) - 3(t) + 4 &= 0\\ -4 + 3t - 3t + 4 &= 0\\ 0t &= 0\end{aligned}$$

In this case, the equation is satisfied for *all* values of t. Therefore, *every* point on the line is an intersection point, and the line lies in the plane.

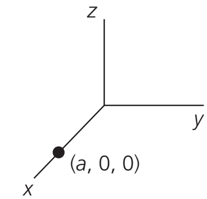
The intersection of the line and the plane is the entire line itself. You can confirm this conclusion by checking that the particular point $(-4, 0, 0)$ on the line is a point in the plane, and that the direction vector of the line, $(3, 0, 1)$, is perpendicular to the normal to the plane, $(1, -2, -3)$.

The x-, y-, and z-axes are lines in space. The intersections of a plane with these special lines are of particular importance. A plane may intersect an axis at a point, or a plane may be parallel to or contain an axis. These intersections are the key to making sketches of planes in three dimensions.

EXAMPLE 4 Determine the x-, y-, and z-intercepts of the plane $3x - 8y - 8z + 24 = 0$. Make a sketch of the plane.

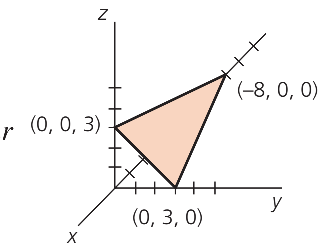
Solution

To find the x-intercept, set y and z equal to zero.



$$3x - 8(0) - 8(0) + 24 = 0$$
$$3x + 24 = 0$$
$$x = -8$$

The x-intercept of this plane is the point -8. Likewise, the y- and z-intercepts are 3 and 3, respectively.



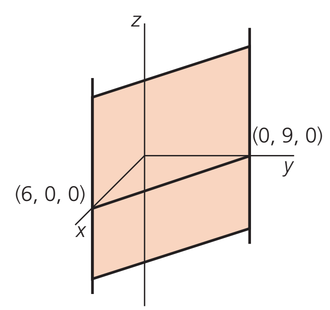
Now, plot these on the coordinate axes, join them with straight line segments, and sketch the plane as a *triangular surface*. This figure is a three-dimensional representation of the plane, which extends infinitely in the directions shown by the orientation of the triangle.

Note that the sides of the triangle formed by the line segments joining the intercepts are segments of the lines in which the plane intersects each of the three coordinate planes.

EXAMPLE 5 Find the intersections of the plane $3x + 2y - 18 = 0$ with the three coordinate axes. Make a sketch of the plane.

Solution

The normal to this plane, $(3, 2, 0)$, has no component in the z direction. Therefore, the plane must be parallel to the z-axis, and there is no z-intercept. By inspection, the x- and y-intercepts are 6 and 9.



Plot the intercepts. Then, through the intercepts, draw lines parallel to the z-axis. The flat region between the parallel lines is a representation of the plane in three dimensions.

Keep in mind that the plane extends infinitely up and down and left and right, in the directions shown by the orientation of the shaded area. The line joining the intercepts is the line in which the plane intersects the xy-plane. The vertical lines through the intercepts are the lines in which the plane intersects the xz-plane and the yz-plane.

As observed, a plane not only intersects a coordinate axis in a point, but it also intersects a coordinate plane in a line. Clearly, knowing how to find these intersection lines would help us make the sketch of a plane. Fortunately, there is a simple way to find the equations of these lines.

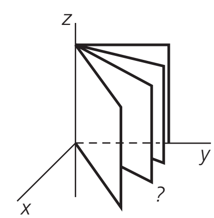
The *xy*-coordinate plane, for example, is the plane where the *z*-coordinate of every point is zero. The scalar equation of the *xy*-plane is $z = 0$. By setting *z* equal to zero in the equation of a plane, we are singling out those points in the plane that lie in the *xy*-coordinate plane. These are exactly the points on the intercept line, and by setting $z = 0$ we obtain the equation.

In Example 4, for instance, the plane intersects the *xy*-coordinate plane in the line $3x - 8y - 8(0) + 24 = 0$ or $3x - 8y + 24 = 0$. In Example 5, the plane intersects the *xy*-coordinate plane in the line $3x + 2y - 18 = 0$ (there is no variable *z* in the equation of this plane, so setting *z* equal to zero does not change the equation).

EXAMPLE 6 Sketch the plane $5x - 2y = 0$.

Solution

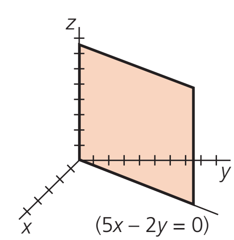
Since $D = 0$, the point $(0, 0, 0)$ satisfies the equation of the plane. So this plane contains the origin. Consequently the *x*- and *y*-intercepts are both zero. The normal to this plane is $(5, -2, 0)$, so as with Example 5, this plane is parallel to the *z*-axis. But if the plane is parallel to the *z*-axis and contains the origin, it must contain the entire *z*-axis. You can reach the same conclusion by observing that every point $(0, 0, z)$ on the *z*-axis satisfies the equation of the plane.



The set of planes with this property is illustrated in the given diagram.

Sketch the plane as a parallelogram, with the intersection line and the *z*-axis as sides. This parallelogram-shaped region represents a section of the plane $5x - 2y = 0$ in three dimensions.

From this set of planes, we choose the one which intercepts the *xy*-plane along the line with equation $5x - 2y = 0$.



Exercise 8.3

Part A

Knowledge/Understanding

1. For each of the following, find the intersection of the line and the plane.
   a. $x = 4 - t$, $y = 6 + 2t$, $z = -2 + t$ and $2x - y + 6z + 10 = 0$
   b. $x = 3 + 4t$, $y = -2 - 6t$, $z = \frac{1}{2} - 3t$ and $3x + 4y - 7z + 7 = 0$

c. $x = 5 + t, y = 4 + 2t, z = 7 + 2t$ and $2x + 3y - 4z + 7 = 0$

d. $\vec{r} = (2, 14, 1) + t(-1, -1, 1)$ and $3x - y + 2z + 6 = 0$

e. $\vec{r} = (5, 7, 3) + t(0, 1, -1)$ and $z + 5 = 0$

2. a. Does the line $\vec{r} = (-2, 6, 5) + t(3, 2, -1)$ lie in the plane $3x - 4y + z + 25 = 0$?

   b. Does the line $\vec{r} = (4, -1, 2) + t(3, 2, -1)$ lie in the plane $3x - 4y + z - 17 = 0$?

3. Where does the plane $3x - 2y - 7z - 6 = 0$ intersect

   a. the x-axis? b. the y-axis? c. the z-axis?

Part B

4. a. In what point does the plane $\vec{r} = (6, -4, 3) + s(-2, 4, 7) + t(-7, 6, -3)$ intersect

   i) the x-axis ii) the y-axis iii) the z-axis

   b. In what line does this plane intersect the

   i) the xy-plane ii) the yz-plane iii) the xz-plane

5. Where does the line $\vec{r} = (6, 10, 1) + t(3, 4, -1)$ meet

   a. the xy-plane b. the xz-plane c. the yz-plane

Communication

6. State whether it is possible for the lines and planes described below to intersect in one point, in an infinite number of points, or in no points.

   a. a line parallel to the x-axis and a plane perpendicular to the x-axis

   b. a line parallel to the y-axis and a plane parallel to the y-axis

   c. a line perpendicular to the z-axis and a plane parallel to the z-axis

Application

7. Find the point of intersection of the plane $3x - 2y + 7z - 31 = 0$ with the line that passes through the origin and is perpendicular to the plane.

Application

8. Find the point at which the normal to the plane $4x - 2y + 5z + 18 = 0$ through the point $(6, -2, -2)$ intersects the plane.

9. For each of the following planes, find the x-, y-, and z-intercepts and make a three-dimensional sketch.

   a. $12x + 3y + 4z - 12 = 0$ b. $x - 2y - z - 5 = 0$

   c. $2x - y + z + 8 = 0$ d. $4x - y + 2z - 16 = 0$

10. For each of the following planes, find the x-, y-, and z-intercepts, if they exist, and the intersections with the coordinate planes. Then make a three-dimensional sketch of the plane.

a. $x + y - 4 = 0$ b. $x - 3 = 0$ c. $2y + 1 = 0$

d. $3x + z - 6 = 0$ e. $y - 2z = 0$ f. $x + y - z = 0$

Part C

11. For what values of k will the line $\frac{x - k}{3} = \frac{y + 4}{2} = \frac{z + 6}{1}$ intersect the plane $x - 4y + 5z + 5 = 0$

a. in a single point?

b. in an infinite number of points?

c. in no points?

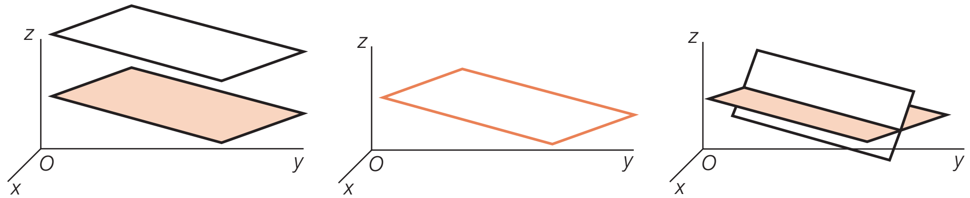
Thinking/Inquiry/Problem Solving

12. A plane has an x-intercept of a, a y-intercept of b, and a z-intercept of c, none of which is zero. Show that the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Section 8.4 — The Intersection of Two Planes

technology APPENDIX P. 532

What are the possible ways two planes can intersect? They can be parallel and distinct, hence not intersecting. They can be coincident, intersecting at every point. They can intersect in a line.



planes are parallel *planes are coincident* *planes intersect in a line*

If the normals are collinear, the planes are parallel and distinct or coincident. If the normals are not collinear, the planes must intersect in a line.

EXAMPLE 1 Find the intersection of the two planes $2x - 2y + 5z + 10 = 0$ and $2x + y - 4z + 7 = 0$.

Solution

The equations of the two planes constitute a linear system of two equations with three variables.

The normals of the two planes are $(2, -2, 5)$ and $(2, 1, -4)$. These are not collinear, so the planes intersect in a line. To find its equation, we solve the equations.

Subtracting we obtain

$$\begin{aligned} 2x - 2y + 5z + 10 &= 0 \\ 2x + y - 4z + 7 &= 0 \\ -3y + 9z + 3 &= 0 \end{aligned}$$

Then $y = 1 + 3z$

The value of y depends on the value of z. But there are no constraints on z.

Let $z = 2t, t \in R$

Then $y = 1 + 6t$

Substituting in equation two,

$$\begin{aligned} 2x + (1 + 6t) - 8t + 7 &= 0 \\ x &= -4 + t \end{aligned}$$

Parametric equations of the line of intersection of the two planes are $x = -4 + t, y = 1 + 6t, z = 2t$.

The solution of systems of linear equations is such an important topic that several different methods to handle this problem have evolved. One of them makes use of **matrices**. For our purposes, a **matrix** is a rectangular array of numbers made to facilitate the solution of a linear system.

Consider, for instance, the linear system dealt with in Example 1. From the coefficients of x, y, and z in the two equations

$$\begin{aligned} 2x - 2y + 5z &= -10 \\ 2x + y - 4z &= -7 \end{aligned} \qquad \text{you can form the matrix} \qquad \begin{bmatrix} 2 & -2 & 5 \\ 2 & 1 & -4 \end{bmatrix}$$

This is a 2×3 matrix – it has two rows and three columns. It is called the **coefficient matrix** of the system. Each coefficient is an **element** of the matrix. The row and column position of each matrix element indicates the equation and the term to which the coefficient belongs.

The constant terms of the equations (which are here written to the right of the equal signs) can be included by adding another column to the coefficient matrix.

$$\left[\begin{array}{rrr|r} 2 & -2 & 5 & -10 \\ 2 & 1 & -4 & -7 \end{array}\right]$$

This matrix is called the **augmented matrix** of the system. The vertical bar in the matrix shows where the equal signs in the system are located.

The matrix method of solving the system of Example 1 starts with the augmented matrix and proceeds by performing arithmetic operations on its rows. The first operation is to subtract the elements of the second row from those of the first, and then replace the second row with this difference.

$$R_1 - R_2 \qquad \left[\begin{array}{rrr|r} 2 & -2 & 5 & -10 \\ 0 & -3 & 9 & -3 \end{array}\right]$$

Observe how this operation on the rows of the matrix is expressed in symbolic form: R_1 and R_2 stand for the two rows. By placing $R_1 - R_2$ beside row 2, we indicate where the result is to be placed.
This step is the counterpart of subtracting the equations in Example 1. These operations on the matrix have made the element in the lower left corner equal to zero, which is equivalent to eliminating x in the corresponding equation.

The next step is to divide each of the elements of the second row by -3. We write

$$R_2 \div (-3) \qquad \left[\begin{array}{rrr|r} 2 & -2 & 5 & -10 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

This is equivalent to the removal of the factor -3 from the result of the subtraction in Example 1.

In order to make the element in row 1, column 2 equal to zero, we multiply the second row by 2, add it to the first row, and replace the first row with this sum.

We write

$$2R_2 + R_1 \qquad \left[\begin{array}{ccc|c} 2 & 0 & -1 & -8 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

This is equivalent to eliminating y in the first equation. Such operations on the rows of the matrix are legitimate, because they match similar operations that could be done on the corresponding equations. Lastly, divide the elements of the first row by 2.

$$R_1 \div 2 \qquad \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & -4 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

At this point the matrix has served its purpose. The two equations corresponding to this matrix are

$$x - \frac{1}{2}z = -4 \qquad \text{or} \qquad x = -4 + \frac{1}{2}z$$

$$y - 3z = 1 \qquad\qquad y = 1 + 3z$$

Here, x and y are both functions of z, but there are no restrictions on z. So, setting $z = 2t$, the equations of the line of intersection are $x = -4 + t$, $y = 1 + 6t$, $z = 2t$ as before.

The matrix method of solving a system of linear equations, illustrated above, is referred to as **Gauss-Jordan elimination.** *A* 2×4 matrix of the form

$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \end{array}\right]$$

can be written down directly from the original equations of the linear system to be solved and then changed into **reduced row-echelon form**

$$\left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \end{array}\right]$$

This form is only one step removed from the solution of the system. The actual operations performed on the rows will depend on what the coefficients are.

The permissible operations that can be performed on the rows of a matrix arise from the algebraic operations that can be performed on the equations of the corresponding linear system.

Row Operations

1. **Any row can be multiplied (or divided) by a non-zero constant.**
2. **Any row can be replaced by the sum (or difference) of that row and a multiple of another row.**
3. **Any two rows can be interchanged.**

technology

Using the matrix methods described above, the solution of a linear system can be systematized so that it can be programmed on a calculator or computer.

This makes it possible to find solutions to systems with many equations and variables, such as those in economics or statistics, which would be difficult, if not impossible, to work out by hand.

technology

The box on page 299 shows how to use a calculator to solve a system of linear equations. If you have a calculator that can perform matrix operations, try using it to work through the example above before continuing.

EXAMPLE 2 Find the intersection of the two planes $4x + 7y - 33z + 17 = 0$ and $-8x - 5y + 3z - 7 = 0$ using Gauss-Jordan elimination.

Solution

The equations of the two planes form the linear system

$$\begin{aligned} 4x + 7y - 33z &= -17 \\ -8x - 5y + 3z &= 7 \end{aligned}$$

The augmented matrix of this linear system is $\left[\begin{array}{rrr|r} 4 & 7 & -33 & -17 \\ -8 & -5 & 3 & 7 \end{array}\right]$

The solution is achieved by starting with the augmented matrix and carrying out the following row operations to change the matrix into reduced row-echelon form.

$$2R_1 + R_2 \quad \left[\begin{array}{rrr|r} 4 & 7 & -33 & -17 \\ 0 & 9 & -63 & -27 \end{array}\right]$$

$$R_2 \div 9 \quad \left[\begin{array}{rrr|r} 4 & 7 & -33 & -17 \\ 0 & 1 & -7 & -3 \end{array}\right]$$

$$R_1 - 7R_2 \quad \left[\begin{array}{rrr|r} 4 & 0 & 16 & 4 \\ 0 & 1 & -7 & -3 \end{array}\right]$$

$$R_1 \div 4 \quad \left[\begin{array}{rrr|r} 1 & 0 & 4 & 1 \\ 0 & 1 & -7 & -3 \end{array}\right]$$

The final matrix corresponds to the equations

$$\begin{aligned} x + 4z &= 1 \qquad \text{or} \qquad & x &= 1 - 4z \\ y - 7z &= -3 & y &= -3 + 7z \end{aligned}$$

Parametric equations of the line of intersection result when z is set equal to t. They are $x = 1 - 4t$, $y = -3 + 7t$, $z = t$.

EXAMPLE 3 Find the intersection of the two planes $x + 4y - 3z + 6 = 0$ and $2x + 8y - 6z + 11 = 0$.

Solution

The augmented matrix of this system is $\left[\begin{array}{rrr|r} 1 & 4 & -3 & -6 \\ 2 & 8 & -6 & -11 \end{array}\right]$

The first operation is to put a zero in the lower left corner of the matrix

$$2R_1 - R_2 \quad \left[\begin{array}{ccc|c} 1 & 4 & -3 & -6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

There is no need to go further. The second row of this matrix corresponds to the equation $0z = -1$, but there is no value of z for which this equation is true. Hence, there is no solution, and the planes do not intersect. They must be parallel. If an elementary row operation makes all the elements of a row zero, this indicates that one equation is a multiple of the other and the planes are coincident. We could say that the normals to the planes, $(1, 4, -3)$ and $(2, 8, -6)$, are collinear, so the planes are parallel and distinct or coincident. Since $(2, 8, -6, 11) \neq 2(1, 4, -3, 6)$, the planes are distinct.

CALCULATOR APPLICATION

Some calculators can put a matrix into reduced row-echelon form and thereby help you to find the solution to a linear system. To solve the linear system of Example 1, for instance, start with the augmented matrix

$$\left[\begin{array}{ccc|c} 4 & 7 & -33 & -17 \\ -8 & -5 & 3 & 7 \end{array}\right]$$

and follow the following steps (the instructions are for a TI-83 Plus calculator).

1. To define the matrix,
   press **2nd** **MATRIX**, select EDIT, select matrix **A**, and press **ENTER**.
   To set its dimensions,
   press 2 **ENTER** and 4 **ENTER**.
   To enter its elements,
   press 4 **ENTER**, then 7 **ENTER**, etc., for all eight elements.
   Then press **2nd** **QUIT** to return to the home screen.

2. To put the matrix in reduced row-echelon form,
   press **2nd** **MATRIX**, select MATH, then cursor down to **B:rref(**
   and press **ENTER**).
   To select which matrix to reduce,
   press **2nd** **MATRIX**, select NAMES, select matrix **A**,
   and press **ENTER**.
   To complete and execute the instruction,
   press **)** and press **ENTER**.

The result is $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -7 & -3 \end{array}\right]$

Now write the corresponding equations and complete the solution.

Exercise 8.4

Part A

Communication

1. Explain why two planes can never intersect in a single point.

2. Do the following pairs of planes intersect in a straight line?

a. $-6x + 12y - 9z + 9 = 0$ and $4x - 8y + 6z + 9 = 0$

b. $2x - y - 2z + 3 = 0$ and $6x - 3y - 6z + 9 = 0$

c. $\vec{r} = (6, 0, 1) + p(1, 1, 2) + q(4, 2, 3)$
and $\vec{r} = (1, 1, -9) + s(5, 3, 5) + t(3, 1, 1)$

d. $\vec{r} = (1, 1, 1) + p(0, 0, 1) + q(0, 1, 0)$
and $\vec{r} = (0, 0, 0) + s(0, 0, 1) + t(1, 0, 0)$

Knowledge/Understanding

3. Determine which of the following pairs of planes are parallel. For each pair that is not parallel, find the parametric equations of the line of intersection. Use algebraic elimination.

a. $x + y - 3z = 4$ and $x + 2y - z = 1$

b. $5x - 2y + 2z + 1 = 0$ and $5x - 2y + 2z - 3 = 0$

c. $x - 3y - z + 3 = 0$ and $2x + 4y - z - 5 = 0$

d. $x + y + z = 1$ and $x = 0$

e. $x + 3y - z - 4 = 0$ and $2x + 6y - 2z - 8 = 0$

Part B

4. Write the augmented matrix for each of the following linear systems.

a. $3x - 7y + z = 12$
$x + y - 2z = -3$

b. $-4x + 3y + 2z = 4$
$2y - 5z = 5$

c. $x + 4z = 16$
$y - 8z = -2$

d. $5y - 2z + 6x = 4$
$3z + 5y - 2x = -4$

5. Write the system of equations that corresponds to each of these matrices.

a. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 9 \\ 0 & 1 & -6 & 4 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 8 & -2 & 3 & -6 \\ 2 & -6 & -6 & 9 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 5 & 0 & -10 & 8 \\ 0 & 3 & -4 & 6 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 9 & 0 \end{array}\right]$

Knowledge/Understanding

6. Use Gauss-Jordan elimination to find the vector equation of the line of intersection of each pair of planes.

a. $x + 2y + 7z = 4$
$x + 3y + 3z = 1$

b. $x - 4y + 3z = 5$
$2x + y + 6z = 0$

c. $2x + 8y + 2z = 7$
$x + 4y - z = 3$

d. $4x - 8y - 3z = 6$
$-3x + 6y + z = -2$

e. $3x + 2y - 6z = 5$
$2x + 3y - 9z = -10$

f. $6x + 8y - 3z = 9$
$10x - 2y - 5z = 15$

Part C

Communication

7. What is the geometrical interpretation of the system of equations that corresponds to these matrices?

a. $\left[\begin{array}{ccc|c} 2 & 6 & -2 & 5 \\ 6 & 5 & 1 & -5 \\ 2 & -3 & 4 & 3 \end{array}\right]$

b. $\left[\begin{array}{cc|c} -1 & 8 & 5 \\ 4 & 3 & 2 \\ -3 & 2 & -2 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 6 & 3 & -4 & 10 \\ 3 & -6 & 4 & 22 \\ 8 & -7 & 1 & 15 \\ 5 & 2 & -5 & -9 \end{array}\right]$

Thinking/Inquiry/Problem Solving

8. a. Let $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ be two non-parallel planes in space. Show that for any fixed k,

$$(A_1x + B_1y + C_1z + D_1) + k(A_2x + B_2y + C_2z + D_2) = 0$$

is the equation of the plane through the intersection of the two planes. As k varies, this equation generates the family of all such planes (except the second plane itself).

b. Find the scalar equation of the plane that passes through the origin and the line of intersection of the planes $3x + 4y - 7z - 2 = 0$ and $2x + 3y - 4 = 0$.

c. Find the scalar equation of the plane that is parallel to the line $x = 2y = 3z$ and passes through the line of intersection of the planes $4x - 3y - 5z + 10 = 0$ and $4x - y - 3z + 15 = 0$.

Application

9. Find the scalar equation of the plane that is perpendicular to the plane $\vec{r} = (-2, 1, 3) + s(5, -2, -2) + t(-1, 0, 1)$ and intersects it along the line $\vec{r} = (9, -1, -5) + p(2, -2, 2)$.

Section 8.5 — The Intersection of Three Planes

technology APPENDIX P. 540

What are the possible ways three planes can intersect? Before reading further, try to discover as many as you can.

One of the ways that three planes can intersect is in a single point. The three coordinate planes, for instance, intersect in a single point, namely the origin. You can find a point of intersection using algebraic elimination or by using matrices and Gauss-Jordan elimination. Examples 1 and 2 illustrate these methods.

EXAMPLE 1 Find the point of intersection of the three planes using algebraic elimination.

① $x - 3y - 2z = -9$
② $2x - 5y + z = 3$
③ $-3x + 6y + 2z = 8$

Solution

For these equations, it appears that z is the easiest variable to eliminate.

Add ① and ③

$$\begin{aligned} x - 3y - 2z &= -9 \\ -3x + 6y + 2z &= 8 \end{aligned}$$

④ $-2x + 3y = -1$

Multiply ② by 2 and add to ①

$$\begin{aligned} x - 3y - 2z &= -9 \\ 4x - 10y + 2z &= 6 \end{aligned}$$

⑤ $5x - 13y = -3$

Multiply ④ by 5 and ⑤ by 2 and add

$$\begin{aligned} -10x + 15y &= -5 \\ 10x - 26y &= -6 \\ -11y &= -11 \\ y &= 1 \end{aligned}$$

Substitute $y = 1$ in ⑤

$$\begin{aligned} 5x - 13 &= -3 \\ x &= 2 \end{aligned}$$

Substitute $y = 1$ and $x = 2$ in ②

$$\begin{aligned} 4 - 5 + z &= 3 \\ z &= 4 \end{aligned}$$

The planes intersect at (2, 1, 4).

Gauss-Jordan elimination in a case like this consists of putting a matrix

$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right] \quad \text{into the reduced row-echelon form:} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

The equations of the three planes constitute a linear system. The augmented matrix for this system is a 3×4 matrix.

$$\left[\begin{array}{rrr|r} 1 & -3 & -2 & -9 \\ 2 & -5 & 1 & 3 \\ -3 & 6 & 2 & 8 \end{array}\right]$$

To accomplish this in an orderly manner, consider the elements one at a time in the order indicated by the numbers. For each one, carry out the row operations that will turn that element into zero.

$$\left[\begin{array}{ccc|c} * & \#4 & \#5 & * \\ \#1 & * & \#6 & * \\ \#2 & \#3 & * & * \end{array}\right]$$

EXAMPLE 2

Find the point of intersection of the three planes of Example 1 using Gauss-Jordan elimination.

① $x - 3y - 2z = -9$

② $2x - 5y + z = 3$

③ $-3x + 6y + 2z = 8$

Solution

Starting with the augmented matrix, the calculations are

$$\left[\begin{array}{rrr|r} 1 & -3 & -2 & -9 \\ 2 & -5 & 1 & 3 \\ -3 & 6 & 2 & 8 \end{array}\right]$$

$$-2R_1 + R_2 \quad \left[\begin{array}{rrr|r} 1 & -3 & -2 & -9 \\ 0 & 1 & 5 & 21 \\ -3 & 6 & 2 & 8 \end{array}\right]$$

$$3R_1 + R_3 \quad \left[\begin{array}{rrr|r} 1 & -3 & -2 & -9 \\ 0 & 1 & 5 & 21 \\ 0 & -3 & -4 & -19 \end{array}\right]$$

$$3R_2 + R_3 \quad \left[\begin{array}{rrr|r} 1 & -3 & -2 & -9 \\ 0 & 1 & 5 & 21 \\ 0 & 0 & 11 & 44 \end{array}\right]$$

$$R_3 \div 11 \quad \left[\begin{array}{rrr|r} 1 & -3 & -2 & -9 \\ 0 & 1 & 5 & 21 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$3R_2 + R_1 \quad \left[\begin{array}{rrr|r} 1 & 0 & 13 & 54 \\ 0 & 1 & 5 & 21 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$-13R_3 + R_1 \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 5 & 21 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$-5R_3 + R_2 \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

The final matrix corresponds to the equations $x = 2$, $y = 1$, $z = 4$. Therefore, the solution is (2, 1, 4), as before.

As you can see from Example 2, it can be a complicated and lengthy process to work out a problem by hand using Gauss-Jordan elimination. Using a calculator with matrix functions makes the work faster and easier (see the box on page 307). Try solving the problem in Example 2 using a calculator.

When working without a calculator, it is usually simpler to do **Gaussian elimination**. This consists of using matrix methods to get just the three zeros in the lower left corner; that is, putting the augmented matrix in row-echelon form.

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right] \quad \text{which in Example 2 is} \quad \left[\begin{array}{ccc|c} 1 & -3 & -2 & -9 \\ 0 & 1 & 5 & 21 \\ 0 & 0 & 11 & 44 \end{array}\right]$$

Then, continue by writing the corresponding equations,

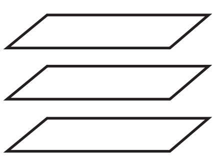
$$\begin{aligned} x - 3y - 2z &= -9 \\ y + 5z &= 21 \\ 11z &= 44 \end{aligned}$$

and finally finish the problem by doing the substitutions as in Example 1. The remaining examples in this section illustrate this method of solving a linear system.

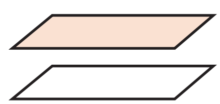
Now that you have the tools to solve systems of three linear equations, it is time to return to the question that started this section: What are the possible ways three planes can intersect?

To answer this question, consider the normals of the three planes.

When the normals of all three are parallel, the possibilities are



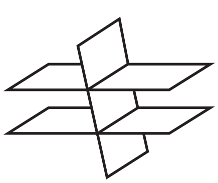
3 planes are parallel and distinct; no intersection



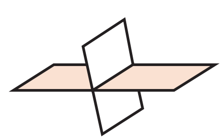
2 planes are coincident, the other parallel; no intersection

3 planes are coincident;. intersection: a plane

When only two of the normals of the planes are parallel, the possibilities are

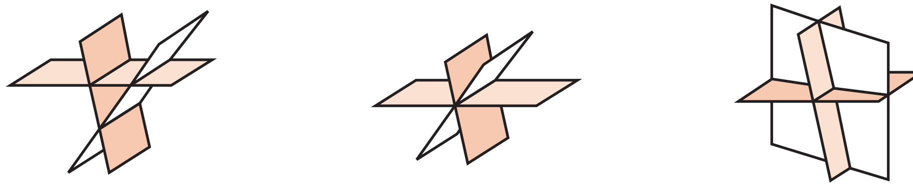


two planes are parallel and distinct, the other crossing; no common intersection



two planes are coincident; the other crossing; intersection: a line

When none of the normals are parallel, the possibilities are



normals coplanar; no intersection — *normals coplanar; intersection: a line* — *normals are not parallel and non-coplanar; intersection: a point*

EXAMPLE 3 Find the intersection of the following planes using Gaussian elimination.

$$\begin{aligned} x + y + 2z &= -2 \\ 3x - y + 14z &= 6 \\ x + 2y &= -5 \end{aligned}$$

Solution

By inspection, none of the normals are collinear. Solving,

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & -2 \\ 3 & -1 & 14 & 6 \\ 1 & 2 & 0 & -5 \end{array}\right]$$

$$\begin{array}{r} \\ 3R_1 - R_2 \\ R_1 - R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 2 & -2 \\ 0 & 4 & -8 & -12 \\ 0 & -1 & 2 & 3 \end{array}\right]$$

$$R_2 \div 4 \left[\begin{array}{rrr|r} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & 2 & 3 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R_2 + R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding equations are

$$\begin{aligned} x + y + 2z &= -2 \\ y - 2z &= -3 \\ 0z &= 0 \end{aligned}$$

Since the third equation is true for any value of z, set $z = t$, and then solve for x and y in terms of t.

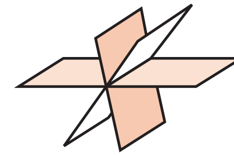
$$y = -3 + 2t$$

and $$x + (-3 + 2t) + 2t = -2$$

$$x = 1 - 4t$$

The solution is then

$x = 1 - 4t,\ y = -3 + 2t,\ z = t$



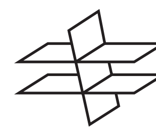
The three planes, none of which are parallel, intersect in a single line, as shown in the diagram.

EXAMPLE 4 Determine the intersection of the following planes.

$$\begin{aligned} x - 2y + 3z &= 9 \\ x + y - z &= 4 \\ 2x - 4y + 6z &= 5 \end{aligned}$$

Solution

The normal vectors of the three planes are $\vec{n}_1 = (1, -2, 3)$, $\vec{n}_2 = (1, 1, -1)$, and $\vec{n}_3 = (2, -4, 6)$. Since $\vec{n}_3 = 2\vec{n}_1$, but the third equation is not twice the first, the two corresponding planes are parallel and distinct. The third plane intersects them, as shown in the diagram. Consequently, there is no solution.



Alternatively, using Gaussian elimination, we obtain

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 1 & 1 & -1 & 4 \\ 2 & -4 & 6 & 5 \end{array}\right] \quad \text{row-echelon form} \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & -13 \end{array}\right]$$

Without proceeding further, we can see that the last row corresponds to the equation $0z = -13$, which has no solution.

EXAMPLE 5 Determine the intersection of the following planes.

$$\begin{aligned} x - y + 4z &= 5 \\ 3x + y + z &= -2 \\ 5x - y + 9z &= 1 \end{aligned}$$

Solution

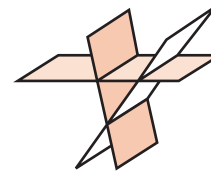
None of the normals are collinear.

$$\begin{aligned} (\vec{n}_1 \times \vec{n}_2) \bullet \vec{n}_3 &= (1, -1, 4) \times (3, 1, 1) \bullet (5, -1, 9) \\ &= (-5, 11, 4) \bullet (5, -1, 9) \\ &= 0 \end{aligned}$$

The normals are coplanar.

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 5 \\ 3 & 1 & 1 & -2 \\ 5 & -1 & 9 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ R_2 - 3R_1 \\ R_3 - 5R_1 \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 4 & 5 \\ 0 & 4 & -11 & -17 \\ 0 & 4 & -11 & -24 \end{array}\right]$$



$$\begin{array}{r} \\ \\ R_2 - R_3 \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 4 & 5 \\ 0 & 4 & -11 & -17 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

The third row corresponds to the equation $0z = 7$, which has no solution. Therefore, the three planes intersect in pairs in three parallel lines, as shown in the diagram.

To check that the lines are indeed parallel, calculate the cross products of the normals:

$$\vec{n}_1 \times \vec{n}_2 = (1, -1, 4) \times (3, 1, 1) = (-5, 11, 4)$$
$$\vec{n}_2 \times \vec{n}_3 = (3, 1, 1) \times (5, -1, 9) = (10, -22, -8)$$
$$\vec{n}_3 \times \vec{n}_1 = (5, -1, 9) \times (1, -1, 4) = (5, -11, -4)$$

The normals are all multiples of the same vector, so this confirms the nature of the intersection.

technology

CALCULATOR APPLICATION

The steps to put a 3×4 matrix into reduced row-echelon form are almost identical to those for a matrix. To solve the linear system of Example 2, for instance, start with the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & -3 & -2 & -9 \\ 2 & -5 & 1 & 3 \\ -3 & 6 & 2 & 8 \end{array}\right]$$

and carry out the following steps (the instructions are for a TI-83 Plus calculator).

1. To define the matrix,

   press **2nd** **MATRIX**, select EDIT, select matrix **A**, and press **ENTER**.

   To set its dimensions,

   press 3 **ENTER** and 4 **ENTER**.

   To enter its elements,

   press 1 **ENTER**, then -3 **ENTER**, etc., for all twelve elements.

   Then press **2nd** **QUIT** to return to the home screen.

(continued)

2. To put the matrix in reduced row-echelon form,
press [2nd] [MATRIX], select MATH, then cursor down to **B:rref(** and press [ENTER].
To select which matrix to reduce,
press [2nd] [MATRIX], select NAMES, select matrix [A], and press [ENTER].
To complete and execute the instruction,
press [)] and press [ENTER].

The calculator carries out a Gauss-Jordan elimination.

The result should be $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$ You can read off the solution (2, 1, 4) directly.

Exercise 8.5

Part A

Communication

1. Using diagrams, classify the intersections of three planes according to whether the intersection is a point, a line, a plane, or no common points.

Communication

2. State whether the normals to the following planes are collinear, coplanar, or neither.

a. $3x - 4y + 5z = 6$
$5x - 6y + 7z = 8$
$6x - 8y + 10z = 9$

b. $-4x + 9y + 8z = 13$
$5x + 3y - z = 15$
$2x + 5y + 2z = -8$

c. $2x - 2y + z = 6$
$4x - 2y - 7z = 3$
$5x - 4y - 2z = 11$

d. $2x + 2y - 6 = 0$
$5x + 5y - 8 = 0$
$3x + 3y - 10 = 0$

Knowledge/Understanding

3. For each of the following, state the point of intersection of the three planes.

a. $x - 4 = 0$
$6y - 3 = 0$
$2z + 6 = 0$

b. $x = 0$
$x + 3y = 6$
$x + y + z = 2$

c. $x - y - z = -1$
$y - 1 = 0$
$x + 1 = 0$

Knowledge/Understanding

4. Using algebraic elimination, find the point of intersection of these three planes.

$$\begin{aligned} x + y + z &= -1 \\ 2x + 2y + 3z &= -7 \\ 3x - 2y + 7z &= 4 \end{aligned}$$

Part B

5. Write the following linear systems in matrix form.

a. $\begin{aligned} 5x - 2y + z &= 5 \\ 3x + y - 5z &= 12 \\ x - 5y + 2z &= -3 \end{aligned}$

b. $\begin{aligned} -2x + y - 3z &= 0 \\ x + 5y &= 8 \\ 3y + 2z &= -6 \end{aligned}$

c. $\begin{aligned} 4y - 3z &= 12 \\ 2x + 5y &= 15 \\ 4x + 6z &= 10 \end{aligned}$

6. Write the equations that correspond to the following matrices.

a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 3 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & 0 & -6 & 4 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Knowledge/Understanding

7. Using Gaussian elimination, find the point of intersection of these planes.

$$\begin{aligned} 2x - 6y + 4z - 11 &= 0 \\ x - 3y + 4z + 7 &= 0 \\ 8x + 18y - 2z + 1 &= 0 \end{aligned}$$

Application

8. Determine the intersection, if any, of each of the following sets of planes. In each case, give a geometrical interpretation of the system of equations and the solution. Also state whether the system has no solutions, a unique solution, or an infinite number of solutions.

a. $\begin{aligned} x + 2y + z &= 12 \\ 2x - y + z &= 5 \\ 3x + y - 2z &= 1 \end{aligned}$

b. $\begin{aligned} x - y + 2z &= 4 \\ 2x - 2y + 4z &= 7 \\ 3x - 3y + 6z &= 11 \end{aligned}$

c. $\begin{aligned} x + y - z &= 5 \\ 2x + 2y - 4z &= 6 \\ x + y - 2z &= 3 \end{aligned}$

d. $\begin{aligned} -2x + 4y + 6z &= -2 \\ 4x - 8y - 12z &= 4 \\ x - 2y - 3z &= 1 \end{aligned}$

e. $\begin{aligned} x + y + 2z &= 2 \\ x - y - 2z &= 5 \\ 3x + 3y + 6z &= 5 \end{aligned}$

f. $\begin{aligned} x + 3y + 5z &= 10 \\ 2x + 6y + 10z &= 18 \\ x + 3y + 5z &= 9 \end{aligned}$

g. $\begin{aligned} x - 3y - 2z &= 9 \\ x + 11y + 5z &= -5 \\ 2x + 8y + 3z &= 4 \end{aligned}$

h. $\begin{aligned} x + y + 2z &= 6 \\ x - y - 4z &= -2 \\ 3x + 5y + 12z &= 27 \end{aligned}$

i. $\begin{aligned} 2x + y + z &= 0 \\ x - 2y - 3z &= 0 \\ 3x + 2y + 4z &= 0 \end{aligned}$

Part C

Thinking/Inquiry/Problem Solving

9. For what value of k will the following set of planes intersect in a line?

$$\begin{aligned} x - 2y - z &= 0 \\ x + 9y - 5z &= 0 \\ kx - y + z &= 0 \end{aligned}$$

Key Concepts Review

In this chapter, the vector methods used to find the equations of a line have been extended to planes. The resulting equations of a plane are

the **vector equation**	$(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$
the **parametric equations**	$x = x_0 + sa_1 + tb_1$ $y = y_0 + sa_2 + tb_2$ $z = z_0 + sa_3 + tb_3$
the **scalar equation**	$Ax + By + Cz + D = 0$

As with lines, it is essential to memorize these equations and to learn to convert quickly, by inspection when possible, from one form to another.

Make a connection between the algebraic equations and the geometrical position and orientation of a line or plane in space. Draw graphs, diagrams, or sketches to increase your ability to visualize intersections.

Finally, try to invest your solutions to problems with meaning. Look at the equations or numerical values of your answers and ask if they answer the question asked, whether are they consistent, and whether they meet your expectations. In a summary statement, express the solution in words.

Rich Learning Link wrap-up

investigate and apply

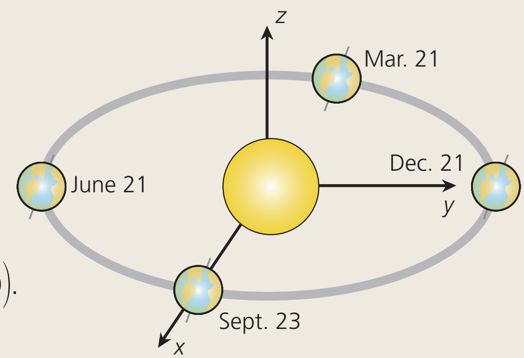
CHAPTER 8: SUN ELEVATION

Like so many astronomers before us and throughout history, we shall determine, through calculations, the angle of elevation of the sun for any given time of any given day at any given place on the surface of the earth.

Investigate and Apply

Let d be the number of days past December 21. Let h be the number of hours (positive or negative) from noon, and let θ be the latitude of the observer.

As previously noted, the vector from the earth to the sun is $\vec{s} = \left(150\sin\left(\frac{360d}{365}\right), -150\cos\left(\frac{360d}{365}\right), 0\right)$.



We want to find the angle between $\vec{s}$ and the observer's plane of tangency to the earth. To do this, we will need the normal, $\vec{n}$, of this plane of tangency.

Pick specific values of d, h, and θ (perhaps the current date and time and your current latitude θ). Use negative values of θ for southern latitudes. Calculate $\vec{s}$.

1. Now to find $\vec{n}$ we start by assuming the earth's axis is not tilted.
   a) Given that $\vec{s} = (s_1, s_2, 0)$, let $\vec{n}_1 = s\cos\left(\frac{360h}{24}\right) + (-s_2, s_1, 0)\sin\left(\frac{360h}{24}\right)$. Why is this the correct normal for a person on the equator?
   b) Let $\vec{n}_2 = \vec{n}_1\cos\theta + (0, 0, |\vec{n}_1|\sin\theta)$. What does $\vec{n}_2$ represent?
2. The earth's axis is tilted $\phi = 23.45°$ away from the z-axis in the direction of the y-axis. If $\vec{n}_2 = (a, b, c)$, then $\vec{n} = (a, b\cos\phi + c\sin\phi, c\cos\phi - b\sin\phi)$. Justify this and then calculate $\vec{n}$.
3. a) Let β be the angle between $\vec{s}$ and $\vec{n}$.
   b) Calculate $\alpha = 90° - \beta$. This is the angle of elevation of the sun.
   c) Why is α the angle of elevation of the sun, and not β?
4. What does it mean if the angle of elevation is negative? (In practice, the angle between a line and a plane will always be between 0° and 90°. Why?)

INDEPENDENT STUDY

Develop a general formula for α in terms of d, h, and θ.

How can we find the positions of the stars and the other planets?

Review Exercise

1. a. Can a plane be perpendicular to the x-axis and contain the line $x = z, y = 0$? Explain.
   b. Can a plane be parallel to the yz-coordinate plane and contain the point $(-4, 0, 5)$? Explain.

2. Find vector and parametric equations of the plane
   a. that passes through the point $(-1, -1, 2)$ and is parallel to the plane $\vec{r} = (2, -1, 0) + s(5, 4, 2) + t(0, 0, 1)$
   b. that passes through the points $(1, 1, 0)$ and $(-2, 0, 3)$ and is parallel to the y-axis
   c. that has intercepts $x = -2$, $y = -3$, and $z = 4$
   d. that contains the point $(1, 1, 1)$ and the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$
   e. that contains the two intersecting lines $\vec{r} = (3, -1, 2) + s(4, 0, 1)$ and $\vec{r} = (3, -1, 2) + t(4, 0, 2)$

3. Find the scalar equation for the plane
   a. that passes through the point $(1, 7, 9)$ and has normal $\vec{n} = (1, 3, 5)$
   b. that passes through the points $(3, 2, 3)$, $(-4, 1, 2)$, and $(-1, 3, 2)$
   c. that passes through the point $(0, 0, 6)$ and is parallel to the plane $y + z = 5$
   d. that contains the point $(3, -3, 0)$ and the line $x = 2, y = 3 + t, z = -4 - 2t$
   e. that contains the line $\vec{r} = (2, 1, 7) + s(0, 1, 0)$ and is parallel to the line $\vec{r} = (3, 0, 4) + t(2, -1, 0)$
   f. that contains the points $(6, 1, 0)$ and $(3, 0, 2)$, and is parallel to the z-axis

4. For what value of k, if any, will the planes $3x + ky + z - 6 = 0$ and $6x + (1 - k)y + 2z - 9 = 0$ be
   a. parallel?
   b. perpendicular?

5. Find the scalar equation of the plane that contains the parallel and distinct lines $x = 1, \frac{y - 3}{4} = \frac{z}{2}$ and $x = 5, \frac{y + 5}{2} = \frac{z - 3}{1}$.

6. Find a vector equation of the plane that contains the origin and the point $(2, -3, 2)$ and is perpendicular to the plane $x + 2y - z + 3 = 0$.

7. Find the scalar equation of the plane that passes through the point $(1, 2, 3)$ and is parallel to the vectors $6\hat{k}$ and $\hat{i} + 2\hat{j} - 3\hat{k}$.

8. A line that passes through the origin intersects a plane at the point $(1, -3, 2)$. If the line is perpendicular to the plane, find the scalar equation of the plane.

9. Find the scalar equation of the plane that contains the intersecting lines $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{-1}$ and $\frac{x-1}{-1} = \frac{y-1}{5} = \frac{z-1}{4}$.

10. Explain why the point $(2, 21, 8)$ and the line $\vec{r} = (-4, -3, -1) + t(2, 8, 3)$ do not determine a plane.

11. Find the distance between
    a. the point $(7, 7, -7)$ and the plane $6y - z + 5 = 0$
    b. the point $(3, 2, 1)$ and the plane $3x+2y+z=10$
    c. the line $\vec{r} = (1, 3, 2) + t(1, 2, -1)$ and the plane $y + 2z = 5$
    d. the planes $x + 2y - 5z - 10 = 0$ and $2x + 4y - 10z - 17 = 0$

12. Find the distance from the point $(1, -2, -2)$ to the plane having an x-intercept of -1, a y-intercept of 2, and a z-intercept of 3.

13. A normal to the plane $4x - 2y + 5z - 9 = 0$ passes through the origin. At what point does this normal intersect the plane?

14. Determine where the plane $4x + 5y - z + 20 = 0$ meets the coordinate axes, and graph the plane.

15. Graph the following planes in an xyz-coordinate system:
    a. $2x + y + z - 3 = 0$
    b. $3y - 4z + 24 = 0$
    c. $3z + 9 = 0$
    d. $\vec{r} = (4, -5, 0) + s(-12, 9, 8) + t(8, -7, -8)$

16. Show that the line $x = -5 - 3t$, $y = 3 - 4t$, $z = 1 + 5t$ lies in the plane $2x + y + 2z + 5 = 0$.

17. For what values of k will the planes $2x - 6y + 4z + 3 = 0$ and $3x - 9y + 6z + k = 0$
    a. not intersect?
    b. intersect in a line?
    c. intersect in a plane?

18. A plane passes through the points $(1, 0, 2)$ and $(-1, 1, 0)$ and is parallel to the vector $(-1, 1, 1)$.

    a. Find the scalar equation of the plane.

    b. Find the equation of the line through the point $Q(0, 3, 3)$ that is perpendicular to the plane.

    c. Find the point at which the perpendicular through Q intersects the plane.

    d. Use a distance formula to check your answer to part **c.**

19. Find the equation of the plane that passes through the point $(3, 0, -4)$ and is perpendicular to the line of intersection of the planes $x + 2y - 7z - 3 = 0$ and $x - 5y + 4z - 1 = 0$.

20. Let l be the line of intersection of the two planes $x + y + z - 1 = 0$ and $2x - 3y - z + 2 = 0$.

    a. Find the scalar equation of the plane that contains the line l and passes through the origin.

    b. Show that the plane found in part **a** makes an angle of 60° with the plane $x - z = 0$.

21. Are the two planes $\vec{r} = (4, 0, 3) + t(-8, 1, -9) + u(-1, 5, 7)$ and $\vec{r} = (-14, 12, -1) + p(1, 1, 3) + q(-2, 1, -1)$ parallel, coincident, or neither?

22. Solve each of the following systems of equations. Give a geometrical interpretation of each system and its solution.

a. $x + 5y - 8 = 0$ $5x - 7y + 8 = 0$	b. $2x - 2y + 4z = 5$ $x - y + 2z = 2$	c. $3x + 2y - 4z + 1 = 0$ $2x - y - z + 3 = 0$
d. $x + 2y - 3z = 11$ $2x + y = 7$ $3x + 6y - 8z = 32$	e. $x - y + 3z = 4$ $x + y + 2z = 2$ $3x + y + 7z = 9$	f. $x + 3y + 3z = 8$ $x - y + 3z = 4$ $2x + 6y + 6z = 16$
g. $x + 2y + z = -3$ $x + 7y + 4z = -13$ $2x - y - z = 4$	h. $3x - 3z = 12$ $2x - 2z = 8$ $x - z = 4$	i. $x + y + z = -3$ $x + 2y + 2z = -4$ $2x + 2y + 2z = -5$

Chapter 8 Test

Achievement Category	Questions
Knowledge/Understanding	2, 5
Thinking/Inquiry/Problem Solving	7
Communication	1, 3
Application	4, 6

1. What can you conclude about the intersection of
   a. two planes, if their normals satisfy $\vec{n}_1 \bullet \vec{n}_2 = 0$?
   b. two planes, if their normals satisfy $\vec{n}_1 \times \vec{n}_2 = 0$?
   c. three planes, if their normals satisfy $\vec{n}_1 \times \vec{n}_2 \bullet \vec{n}_3 = 0$?

2. For each of the following, state whether each line lies in the plane $4x + y - z - 10 = 0$, is parallel to the plane, or intersects the plane at a point. Give your reasons.
   a. $x = -3t, y = -5 + 2t, z = -10t$
   b. $\frac{x-2}{4} = \frac{y-2}{1} = \frac{z}{-1}$

3. Describe with diagrams all the ways that three planes can intersect in one or more common points.

4. The plane $\vec{r} = (0, 0, 5) + s(4, 1, 0) + t(2, 0, 2)$
   a. intersects the x-axis at what point?
   b. intersects the xz-coordinate plane in what line?

5. Find the scalar equation of the plane containing the line $x = y, z = 0$ and the point $(2, -5, -4)$.

6. Solve the following system of equations and give a geometrical interpretation of the result.

$$x + 2y + z + 3 = 0$$
$$x + 7y + 4z + 13 = 0$$
$$2x - y - z - 4 = 0$$

7. a. Find the distance from the origin to the plane $3x + 2y - z - 14 = 0$.

   b. Find the distance from the point $P(10, 10, 10)$ to the plane $3x + 2y - z - 14 = 0$.

   c. Is P on the same side of the plane as the origin? Give evidence to support your answer.

Extending and Investigating

GENERATING RANDOM NUMBERS

Random numbers are used for computer simulations of processes that can be modelled using probability. For example, airlines often sell more seats than exist on a plane because they know that some ticket holders may not show up. Of course, if too many tickets are sold, then the airline will have to provide costly incentives to convince some of the extra passengers to wait for the next flight. Using random numbers as part of a model, the airline can simulate different seat-selling strategies without ever trying one in practice. This is the advantage of simulation.

Your calculator has a function that can generate random numbers, usually between 0 and 1, or random digits within a specified range. How does this work? One way to produce a sequence of random digits from the set {0, 1, 2, ..., 9} is to put 10 identical balls numbered 0 to 9 into a container and shake it vigorously. Then, without looking, reach into the container and choose a ball. Note the digit on the ball you selected, replace the ball in the container, and repeat the process. You might get a sequence such as 7, 5, 4, 0, 5, 2, 2, 8, 1.

Here is a sequence of random digits produced by a TI-83. Use the key strokes MATH → → → PRB 5 to get to the function **randInt**. Then enter 0,9,8) ENTER to get the 8 random digits 9, 2, 3, 0, 9, 1, 0 5. Of course, if you try this you will get a different answer!

How does the calculator produce this sequence since there is no one inside to shake up a container of balls? The answer is that the calculator uses an algorithm that is completely deterministic. The numbers produced will be exactly the same every time if you start with the same initial conditions. This is not true with the container of balls. Hence, the numbers generated by the calculator are far from random. However, the sequence shares many properties with a sequence of random numbers and, if the algorithm is well selected, the numbers produced are good enough for practical purposes.

To see how the calculator generates a sequence of random numbers, we must look (perhaps suprisingly) at how division works. When we divide 37 by 8, the remainder is 5. That is, $37 = 4 \times 8 + 5$. If we divide any integer by 8, we get a remainder of 0, 1, 2, ... or 7. Generally, if we divide any integer x by the integer m, the remainder r is an integer between 0 and $m - 1$, inclusive. We use a fancy notation $x \equiv r\text{mod}(m)$ and say that x is congruent to $r$modulo m". For example, $37 \equiv 5 \text{ mod}(8)$ and $63 \equiv 7$ mod(8). Spreadsheet programs such as EXCEL have a function **mod(*x*, *m*)** that returns the remainder when x is divided by m.

One algorithm for generating a sequence of random numbers is a mathematical equation of the form $xn = ax_{n-1}\text{mod}(m)$, $n = 1, 2, \ldots$ where x_0 is a specified number called the *seed*. The seed can be set by the user or determined in some other way (e.g., from the clock inside the calculator). For example, consider the generator $x_n = 8x_{n-1}\text{mod}(13)$, $n = 1, 2, \ldots$ with seed $x_0 = 1$. If we substitute $n = 1, 2, 3, \ldots$ 12, we get the sequence 8, 5, 12, 1, 8, 5, 12, 1, 8, 5, 12, 1. This sequence does not look very random since it repeats itself every four terms. We say that the sequence has *period* 4. If we change a in the generator to 2 so that the equation is $x_n = 2x_{n-1}\text{mod}(13)$, $n = 1, 2, \ldots$, we get the sequence 1, 2, 4, 8, 3, 6, 12, 11, 5, 9, 10, 7, 1, which then repeats. This looks better since the period is now 12. Could the period be longer than 12 for any choice of a?

In practice, m and a are selected so that the sequence has a very large period and other good properties. For example, one version of Waterloo MAPLE uses a generator with $m = 10^{12} - 11$, $a = 427419669081$, which produces a sequence with period $10^{12} - 12$ (do not try to check this by hand!). For amusement, you can try the following.

1. Explain why you must get a periodic sequence with this generator (try specific values for a and m first).

2. For $m = 2^3, 2^4, 2^5$, investigate different values of a to determine the longest possible period. Can you guess the answer for 2^e for any integer value of e?

3. Suppose m is a prime, for example $m = 17$. What are the possible periods for various choices of a?

On your calculator, the function **rand** returns a rational number between 0 and 1. Since the remainder x_n is always less than m, the number displayed is $\frac{x_n}{m}$.

Cumulative Review

CHAPTERS 4–8

1. Show that the cross product of two unit vectors is not generally a unit vector.

2. Prove that $(\vec{u} \times \vec{v}) \times \vec{u}$ is perpendicular to $\vec{v}$.

3. The points $A(2, 4)$, $B(0, 0)$, and $C(-2, 1)$ define a triangle in the plane. Find the cosine of $\angle ABC$.

4. Write the vector $(0, 8)$ as a linear combination of the vectors $(2, 4)$ and $(-2, 1)$.

5. For the four points $A(2k, 0, 0)$, $B(0, 2k, 0)$, $C(0, 0, 2k)$, and $D(2l, 2l, 2l)$, let W be the midpoint of AB, X the midpoint of BC, Y the midpoint of CD, and Z the midpoint of DA. Prove that W, X, Y, and Z are coplanar.

6. In ΔABC, P is the midpoint of BC. Q is the point that divides AP internally in the ratio 5:2. R is on AC such that $\overrightarrow{AR} = k\overrightarrow{AC}$, for k a real number. For what value of k is BQR a straight line?

7. $P_1, P_2, P_3, \ldots P_{12}$, are consecutive vertices of a regular polygon with 12 sides. If $\overrightarrow{P_1, P_2} = \vec{x}$ and $\overrightarrow{P_1P_3} = \vec{y}$, express the following vectors in terms of $\vec{x}$ and $\vec{y}$:
   a. $\overrightarrow{P_2P_3}$
   b. $\overrightarrow{P_1P_4}$
   c. $\overrightarrow{P_3P_7}$

8. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be linearly independent vectors in space, and let
   $\vec{u} = 3\vec{a} + 2\vec{b} - \vec{c}$
   $\vec{v} = -2\vec{a} + 4\vec{c}$
   $\vec{w} = -\vec{a} + 3\vec{b} + k\vec{c}$
   Determine k so that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are coplanar.

9. Prove that the diagonals of a parallelogram bisect each other.

10. Draw a quadrilateral $ABCD$ with opposite sides AB and DC parallel. Let M be the point of intersection of the diagonals AC and BD. Through M draw a line parallel to AB that intersects AD in P and BC in Q. Prove that M is the midpoint of PQ.

11. Prove that the bisector of the apex angle of an isosceles triangle is perpendicular to the base.

12. Consider the two lines with equations
$\frac{x+8}{1} = \frac{y+4}{3} = \frac{z-2}{1}$ and $(x, y, z) = (3, 3, 3) + t(4, -1, -1)$.
   a. Show that the lines are perpendicular.
   b. Find the point of intersection of the lines.

13. Determine whether the point $O(0, 0, 0)$ lies on the plane that passes through the three points $P(1, -1, 3)$, $Q(-1, -2, 5)$, and $R(-5, -1, 1)$.

14. Determine the equation in the form $Ax + By + Cz + D = 0$ of the plane that passes through the point $P(6, -1, 1)$, has z-intercept -4, and is parallel to the line $\frac{x+2}{3} = \frac{y+1}{3} = \frac{z}{-1}$.

15. Determine a point A on the line with equation
$(x, y, z) = (-3, 4, 3) + t(-1, 1, 0)$, and a point B on the line
$(x, y, z) = (3, 6, -3) + s(1, 2, -2)$, so that $\overrightarrow{AB}$ is parallel to $\overrightarrow{m} = (2, -1, 3)$.

16. The equation $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 9$ defines a sphere in three-dimensional space. Find the equation (in the form $Ax + By + Cz + D = 0$) of the plane that is tangent to the sphere at $(2, 4, 5)$, a point at one end of a diameter of the sphere.

17. Determine the intersection of the line $x = -1 + t$, $y = 3 + 2t$, $z = -t$ with each of the following planes:
   a. $x - y - z + 2 = 0$
   b. $-4x + y - 2z - 7 = 0$
   c. $x + 4y - 3z + 7 = 0$

18. Find the point on the xy-plane that lies on the line of intersection of the planes with equations $4x - 2y - z = 7$ and $x + 2y + 3z = 3$.

19. A plane passes through the points (2, 0, 2), (2, 1, 1), and (2, 2, 4). A line passes through the points (3, 2, 1) and (1, 3, 4). Find the point of intersection of the plane and the line.

20. a. Determine the parametric equations of the line of intersection of the two planes $3x - y + 4z + 6 = 0$ and $x + 2y - z - 5 = 0$.
   b. At what points does the line of intersection intersect the three coordinate planes?
   c. Determine the distance between the xy-intercept and the xz-intercept.

21. The point Q is the reflection of $P(-7, -3, 0)$ in the plane with equation $3x - y + z = 12$. Determine the coordinates of Q.

22. Determine the components of a vector of length 44 that lies on the line of intersection of the planes with equations $3x - 4y + 9z = 0$ and $2y - 9z = 0$.

23. The line through a point $P(a, 0, a)$ with direction vector $(-1, 2, -1)$ intersects the plane $3x + 5y + 2z = 0$ at point Q. The line through P with direction vector $(-3, 2, -1)$ intersects the plane at point R. For what choice of a is the distance between Q and R equal to 3?

24. Consider two lines

$$L_1: (x, y, z) = (2, 0, 0) + t(1, 2, -1)$$

$$L_2: (x, y, z) = (3, 2, 3) + s(a, b, 1)$$

where s and t are real numbers. Find a relationship between a and b (independent of s and t) that ensures that L_1 and L_2 intersect.

25. Determine all values of x, y, and z satisfying the following system of equations.

$$x + 2y - 3z = 1$$

$$2x + 5y + 4z = 1$$

$$3x + 6y - z = 3$$

26. In the following system of equations, k is a real number.

$$-2x + y + z = k + 1$$
$$kx + z = 0$$
$$y + kz = 0$$

a. For what value(s) of k does the system

i) have no solution?

ii) have exactly one solution?

iii) have an infinite number of solutions?

b. For part **a iii**, determine the solution set and give a geometric interpretation.

Chapter 9

PROOF USING DIFFERENT APPROACHES

"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't it ain't. That's logic." (Lewis Carroll)

Mathematics is one of the best ways to teach logic and proof. In mathematical problem solving, you can consider problems using deduction, intuition, working backwards, and trial and error. The ability to consider a problem from many angles is as valuable for business professionals as for researchers in academic environments. In this chapter, you will consider the value of different approaches to solving a problem.

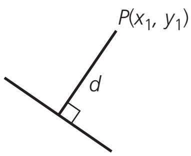
CHAPTER EXPECTATIONS In this chapter, you will

- prove some properties of plane figures algebraically, **Section 9.1, 9.2**
- solve problems by combining a variety of problem-solving strategies, **Section 9.3**
- generate multiple solutions to the same problem, **Section 9.3**
- understand the relationship between formal proof and the use of dynamic geometry software, **Section 9.4**
- use technology in testing conjectures, **Section 9.4**

Review of Prerequisite Skills

1. Recall the properties of congruent triangles, similar triangles, and parallel lines.
2. Recall the properties of circles, chords, and tangents.
3. Recall the properties of vectors and combinations of vectors, and their use in the equations of lines.
4.
   - The distance between the points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
   - The equation of the straight line through (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$.
   - The equation of the straight line through (x_1, y_1) and (x_2, y_2) is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.
   - If two lines are perpendicular and have slopes m_1 and m_2, $m_1 m_2 = -1$.
   - The coordinates of the midpoint of the line segment joining (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
   - If $P(x, y)$ divides the line segment connecting $A(x_1, y_1)$ and $B(x_2, y_2)$ so that $AP = 2PB$, then $(x, y) = \left(\frac{2x_2 + x_1}{3}, \frac{2y_2 + y_1}{3}\right)$.
   - The perpendicular distance from $P(x_1, y_1)$ to the straight line whose equation is $Ax + By + C = 0$ is

     $$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$



   - The equation of a circle having radius r is $x^2 + y^2 = r^2$ if the centre is $(0, 0)$ and is $(x - h)^2 + (y - k)^2 = r^2$ if the centre is (h, k).

Exercise

1. List conditions under which two triangles are
   a. congruent
   b. similar

2. Determine whether the triangles defined by the following sets of points are equilateral, isosceles, or right-angled.
   a. $(1, 0), (-1, 0), (0, 4)$
   b. $(4, 5), (0, -2), (-3, 1)$
   c. $(1, 7), (7, 1), (2, 2)$
   d. $(2, -2), (1, 5), (-1, -1)$

3. The line segment joining $A(-1, 5)$ to $B(5, -3)$ is divided internally by the point C. Determine the coordinates of C if
   a. $AC{:}CB = 3{:}1$
   b. $AC{:}CB = 2{:}3$

4. Determine the coordinates of the point on the y-axis that is equidistant from the points $(5, 7)$ and $(10, 4)$.

5. A circle has its centre on the x-axis and a chord that connects $(-2, 1)$ and $(10, 7)$. What are the coordinates of the centre?

6. What is the equation of a circle having
   a. centre $(0, 0)$ and radius 3?
   b. centre $(-1, 4)$ and radius 4?
   c. centre $(3, 2)$ and passing through $(7, 4)$?

7. The point $(x, -4)$ is twice as far from the point $(-9, 4)$ as it is from the origin. Determine all possible values of x.

8. A triangle has vertices $(1, 3)$, $(7, 5)$, and $(-3, 6)$. What are the coordinates of the midpoints of the sides?

9. What is the equation of the median from the point $(2, 3)$ in a triangle if the other vertices are $(5, 8)$ and $(1, -2)$?

10. A quadrilateral has vertices $A(0, 0)$, $B(6, 0)$, $C(8, 11)$, and $D(3, 7)$. What are the coordinates of E and F, the midpoints of AC and BD? What are the coordinates of G, the midpoint of EF?

Rich Learning Link investigate

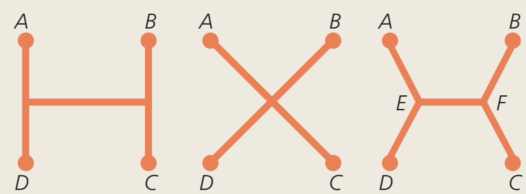
CHAPTER 9: STEINER NETWORKS

Consider the problem of providing a fibre-optic network to three or more sites on a university campus so that any two sites are connected by some fibre-optic path. To minimize cost and maximize communication speed, the network must be as short as possible. How can this be done? Such a shortest-length network is called a **Steiner network,** after the Swiss geometer Jacob Steiner, who lived from 1796 to 1863. Applications of Steiner networks arise in designing transportation networks, production facility layouts, and computer microchip design.

If ΔABC is an acute triangle, then there is a point F inside the triangle, called the Fermat point, for which $AF + BF + CF$ is less than for any other point. It is the centre of the Steiner network for the three points A, B, and C.

Investigate

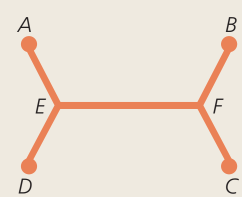
The diagram shows a square $ABCD$ with side length one unit and three possible networks. The H-pattern network has total length of 3 units. A shorter network is formed by the diagonals of the square. Applying the Pythagorean theorem, we find that this



H-Pattern Diagonals Steiner Network

network has length $2\sqrt{2} \approx 2.83$ units. The shortest network, a Steiner network, is formed by locating two points E and F such that EF passes through the centre of the square and the angles $\angle AED$ and $\angle BFC$ are both 120°. This network has a total length of $1 + \sqrt{3} \approx 2.73$ units. Verify this calculation.

The Steiner network for a rectangle $ABCD$ is very similar to that of the square. Points E and F are located such that EF passes lengthwise through the centre of the rectangle and the angles $\angle AED$ and $\angle BFC$ are both 120°.



Steiner Network

DISCUSSION QUESTIONS

1. How many Steiner networks does a square have? How many Steiner networks does a rectangle have?
2. Will a Steiner network connecting five points always be longer than a network connecting four points?
3. Besides length, what other concerns arise when designing communications or transportation networks?
4. Can you think of other applications in which a shortest-length network might be sought?

Section 9.1 — Using Analytic Methods

An equation involving variables x and y represents a restriction of the xy-plane to points whose coordinates satisfy the equation. An equation such as $3x + 2y - 6 = 0$ defines a straight line; one such as $y = 3x^2 - 4x$ defines a parabola; one such as $x^2 + y^2 = 9$ defines a circle with centre at the origin and radius 3; and so on.

In this section, we expand our ability to develop the equations of figures having specific conditions.

EXAMPLE 1

Determine an equation for a circle such that the endpoints of its diameter are determined by the points $A(-5, 1)$ and $B(1, 9)$.

Solution

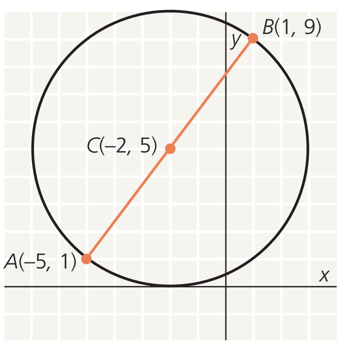
Since the end points of the diameter are $A(-5, 1)$ and $B(1, 9)$, the coordinates of the centre are at $C\left(\frac{1-5}{2}, \frac{9+1}{2}\right) = C(-2, 5)$.

The radius is

$$r = AC = \sqrt{[-5-(-2)]^2 + (1-5)^2} =$$
$$\sqrt{9+16} = 5.$$

The equation of this circle is

$$(x+2)^2 + (y-5)^2 = 25.$$



This answer is given in standard form.
If we expand the squares, we get

$$x^2 + 4x + 4 + y^2 - 10y + 25 = 25$$
$$x^2 + y^2 + 4x - 10y + 4 = 0$$

This equation is in general form. Since the radius and/or the coordinates of the centre might be fractions, we could multiply the equation by a number to eliminate fractions.

EXAMPLE 2

Find the radius and centre of the circle defined by $3x^2 + 3y^2 - 10x + 12y - 13 = 0$.

Solution

Divide the given equation by 3.

$$x^2 + y^2 - \frac{10}{3}x + 4y - \frac{13}{3} = 0$$

Rearranging, $$x^2 - \frac{10}{3}x + y^2 + 4y = \frac{13}{3}$$

Complete the squares, adding the same numbers on the right as on the left.

$$x^2 - \frac{10}{3}x + \frac{25}{9} + y^2 + 4y + 4 = \frac{13}{3} + \frac{25}{9} + 4$$

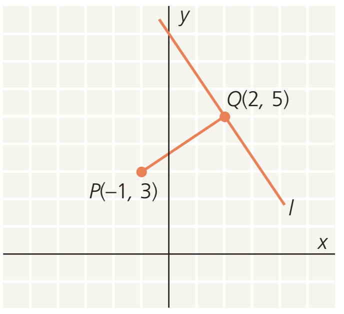
$$\left(x - \frac{5}{3}\right)^2 + (y + 2)^2 = \frac{100}{9}$$

The centre is $\left(\frac{5}{3}, -2\right)$ and the radius is $\frac{10}{3}$.

If the constant had been a large positive number, so that instead of $\frac{13}{3}$ on the right, we got a negative number larger than $\frac{25}{9} + 4$, the final result would have a negative value for r^2, and no circle would exist.

EXAMPLE 3

A perpendicular from the point $P(-1, 3)$ meets a line l at the point $Q(2, 5)$. What is the equation of line l?



Solution

The slope of PQ is $\frac{5 - 3}{2 - (-1)} = \frac{2}{3}$.

Then the slope of l is $-\frac{3}{2}$.

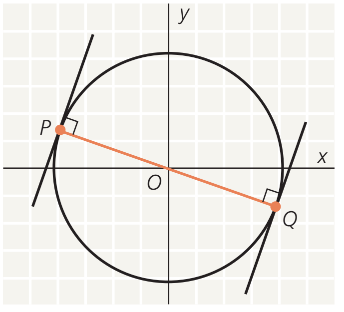
The equation of l is $y - 5 = -\frac{3}{2}(x - 2)$ or $3x + 2y - 16 = 0$.

EXAMPLE 4

Determine the equations of all tangents having slope 4 to the circle $x^2 + y^2 = 17$.

Solution

If the tangents have slope 4, then the line from the tangent contact point to the centre has slope $-\frac{1}{4}$, since these lines are perpendicular. Then, the equation of the diameter with slope $-\frac{1}{4}$ is $y = -\frac{1}{4}x$.



We determine the coordinates of P and Q by solving this equation with the circle equation.

Replacing y by $-\frac{1}{4}x$, we obtain

$$x^2 + \frac{1}{16}x^2 = 17$$

$$\frac{17}{16}x^2 = 17$$

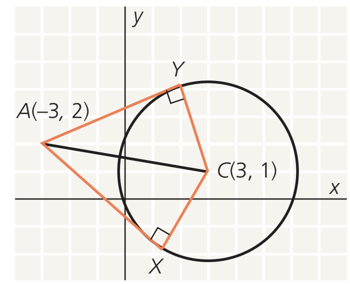
$$x^2 = 16$$

$$x = \pm 4$$

The point Q is $(4, -1)$, and P is $(-4, 1)$.
The tangent through Q has equation $y + 1 = 4(x - 4)$, and the tangent through P has equation $y - 1 = 4(x + 4)$.

EXAMPLE 5 Determine the length of the tangent from $A(-3, 2)$ to the circle with equation $x^2 + y^2 - 6x - 2y = 0$.

Solution



Rewriting $x^2 + y^2 - 6x - 2y = 0$, we have $(x - 3)^2 + (y - 1)^2 = 10$. This is a circle with radius $\sqrt{10}$ and with its centre at $C(3, 1)$. We have two tangents, AY and AX, of equal length.

Since ΔACY is right-angled, $AC^2 = AY^2 + CY^2$.

$CY^2 = r^2$ so $CY^2 = 10$

and $AC^2 = (-3 - 3)^2 + (2 - 1)^2 = 37$

Thus, $AY^2 = 37 - 10 = 27$

$\therefore AY = \sqrt{27}$

Thus, the length of the tangent from A to the circle is $\sqrt{27}$, or $3\sqrt{3}$.

Exercise 9.1

Part A

Communication

1. For each of the following circles, identify the radius and the coordinates of the centre.
   a. $(x - 2)^2 + (y + 1)^2 = 25$
   b. $x^2 + y^2 + 2x + 4y - 4 = 0$
   c. $2x^2 + 2y^2 + 4x + 12y - 29 = 0$

Knowledge/Understanding

2. Identify each of the following.
   a. $x^2 + y^2 - 4x + 10y + 29 = 0$
   b. $3x^2 + 3y^2 = 8y$
   c. $x^2 + y^2 + 4x + 6y + 23 = 0$

3. Find the length of the tangent from $A(5, 7)$ to $x^2 + y^2 - 2x - 8y - 4 = 0$.

4. What is the distance from the origin to the circle with equation $(x - 5)^2 + (y - 12)^2 = 1$?

Part B

Application

5. Find the length of the chord of the circle $x^2 + y^2 - 5 = 0$, which is part of the line $y = 3x + 5$.

6. For the circle $x^2 + y^2 = 16$, determine the length of the longest possible chord passing through $P(1, -2)$.

7. Find the coordinates of the intercepts of the circle whose equation is $x^2 + y^2 + 6x - 2y = 0$.

8. Find the length of the tangent from (5, 7) to the circle with equation $x^2 + y^2 - 6x - 2y + 6 = 0$.

9. Show that the point $A(1, 5)$ is on the circle with equation $x^2 + y^2 + 4x - 2y - 20 = 0$. Find the coordinates of the other end of the diameter through A.

10. Find the equation of the circle that passes through the points (8, 2) and (−2, −4) and which has its centre on the line with equation $y = 2x + 4$.

Thinking/Inquiry/Problem Solving

11. A circle lies in the third quadrant and touches both coordinate axes. If the length of the tangent from the point $A(-4, 2)$ to the circle is 5, determine the equation of the circle.

12. The point $R(3, -4)$ is on a circle with its centre at the origin. The tangent through R intersects the x-axis at the point P and the y-axis at the point Q. Find the length of PQ.

Part C

13. Determine the shortest possible distance between a point on the circle $x^2 + y^2 = 9$ and a point on the circle $x^2 + y^2 - 12x + 6y + 41 = 0$.

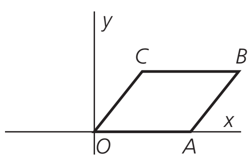
14. Find the length of the common chord of the two circles whose equations are $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x + 2 = 0$.

Section 9.2 — Proof Using Analytic Geometry

In earlier chapters, you saw how geometric facts can be proven using deductive thinking. In Chapter 1, we looked briefly at using analytic methods. If we are to make the best possible use of analytic methods, two steps are necessary. First, determine the usefulness of analytic methods. Second, ensure that the location of figures used is to your advantage. By considering a few examples we can see how this can be done.

EXAMPLE 1 Describe an efficient way of defining a parallelogram in analytic terms.

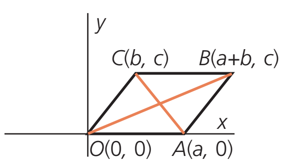
Solution



By making one vertex the origin and a second vertex a point on the x-axis, we require only one variable. Let one vertex be $O(0, 0)$ and a second vertex be $A(a, 0)$. The vertex C in the diagram cannot be defined using the variable a. We let C have coordinates (b, c). Then, since $CB = a$, the coordinates of B are $(a + b, c)$. The parallelogram is now defined using a minimum number of variables.

EXAMPLE 2 Using analytic methods, prove that the diagonals of a parallelogram bisect each other.

Solution



Using the parallelogram defined in Example 1, we have

midpoint of OB is $\left(\frac{a+b}{2}, \frac{c}{2}\right)$

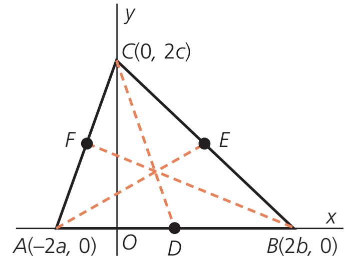
midpoint of AC is $\left(\frac{a+b}{2}, \frac{c}{2}\right)$

Since these are the same, the diagonals bisect each other.

We are free to define coordinates as long as we use properties of the figures. In the examples above, we must ensure that $OA = BC$ and $OC = AB$. We could have avoided fractions in the midpoint coordinates by letting the coordinates be $A(2a, 0)$, $C(2b, 2c)$, and $B(2a + 2b, 2c)$. The strength of the analytic method is that we can choose coordinates for convenience.

EXAMPLE 3 Use analytic methods to prove that the medians of a triangle are concurrent and divide each other in a ratio 2:1 from a vertex to a midpoint.

Solution



Let any triangle have coordinates $A(-2a, 0)$, $B(2b, 0)$, and $C(0, 2c)$. Then D, the midpoint of AB, has coordinates $(b - a, 0)$; E, the midpoint of BC, has coordinates (b, c); and F, the midpoint of AC, has coordinates $(-a, c)$.

If P divides AE in the ratio 2:1, then P has coordinates $\left(\frac{2b - 2a}{3}, \frac{2c}{3}\right)$.

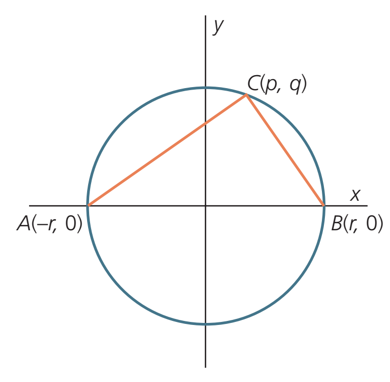
If Q divides BF in the ratio 2:1, then Q has coordinates $\left(\frac{2(-a) + 2b}{3}, \frac{2c}{3}\right)$ or $\left(\frac{2b - 2a}{3}, \frac{2c}{3}\right)$.

If R divides CD in the ratio 2:1, then R has coordinates $\left(\frac{2(b - a) + 0}{3}, \frac{0 + 2c}{3}\right)$ or $\left(\frac{2b - 2a}{3}, \frac{2c}{3}\right)$.

But P, Q, and R are the same point. Therefore, the medians intersect at the same point and divide each other in a ratio 2:1.

EXAMPLE 4 Using analytic methods, prove that the angle in a semicircle is a right angle.

Solution



Let the circle have equation $x^2 + y^2 = r^2$. Then the diameter on the x-axis has end points whose coordinates are $A(-r, 0)$ and $B(r, 0)$. Let $C(p, q)$ be any other point on the circumference. We will prove that $\angle ACB = 90°$.

The slope of AC is $m_1 = \frac{q}{p + r}$

The slope of BC is $m_2 = \frac{q}{p - r}$

Then $m_1 m_2 = \frac{q}{p + r} \bullet \frac{q}{p - r} = \frac{q^2}{p^2 - r^2}$

But C is on the circle, so $p^2 + q^2 = r^2$ or $q^2 = r^2 - p^2$

Then $m_1 m_2 = \frac{r^2 - p^2}{p^2 - r^2} = -1$

Then $AC \perp BC$ and $\angle ACB = 90°$

Exercise 9.2

Part A

Use analytic methods to solve the following problems.

1. Prove that the diagonals of a rectangle are equal.

Knowledge/Understanding

2. Prove that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices of the triangle.

Part B

3. Prove that the midpoints of successive sides of a quadrilateral are the vertices of a parallelogram.

4. Prove that the line connecting the midpoints of two sides of a triangle is parallel to the third side and equal to one-half of it.

Application

5. Prove that if the diagonals of a parallelogram are equal, the parallelogram is a rectangle.

6. Prove that the length l of the tangent from an external point $P(x_1, y_1)$ to a circle with equation $(x - h)^2 + (y - k)^2 = r^2$ is $l = \sqrt{(x_1 - h)^2 + (y_1 - k)^2 - r^2}$.

7. P is any point on the diameter AB of a circle. CD is any chord of the circle parallel to AB. Prove that $PC^2 + PD^2$ is independent of the position of CD.

8. Prove that the lines joining the midpoints of opposite sides of a quadrilateral bisect each other.

9. A convenient way of expressing equations briefly is to give them a name. Hence, by writing C_1: $x^2 + y^2 - r^2$, we can describe the circle $x^2 + y^2 - r^2 = 0$ as $C_1 = 0$. Show that if $P(x_1, y_1)$ is a point that subtends equal tangents to two circles $C_1 = 0$ and $C_2 = 0$, then P lies on the line whose equation is $C_1 - C_2 = 0$.

Section 9.3 — Different Techniques of Proof

Many problems in most branches of mathematics can be solved by a variety of approaches. In the last section, you proved a number of things that you had done earlier by other methods. It is sometimes clear that one method is easier than others, as you will see in the following examples. However, the best method is the one that leads you to a solution.

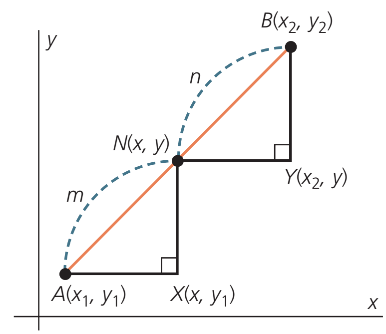
There is value in trying different methods on a problem. By doing so, we strengthen our understanding of concepts and increase our confidence that we can find a solution.

EXAMPLE 1

Find a formula for the coordinates of the point $N(x, y)$ that divides the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m{:}n$.

Solution

You have already seen a development of a formula using vectors. Here we use analytic methods.



In the given diagram, we determine points $X(x, y_1)$ and $Y(x_2, y)$ by drawing lines parallel to the axes through A, N, and B, as shown.

In ΔNXA and ΔBYN,

$\angle NAX = \angle BNY$ (parallel lines)

$\angle NXA = \angle BYN$ (right angles)

Then $\Delta NXA \sim \Delta BYN$

Therefore

$$\frac{AX}{NY} = \frac{XN}{YB} = \frac{AN}{NB} = \frac{m}{n}$$

$$\frac{x - x_1}{x_2 - x} = \frac{m}{n} \text{ and } \frac{y - y_1}{y_2 - y} = \frac{m}{n}$$

$$nx - nx_1 = mx_2 - mx$$

$$(m + n)x = mx_2 + nx_1$$

$$x = \frac{mx_2 + nx_1}{m + n}$$

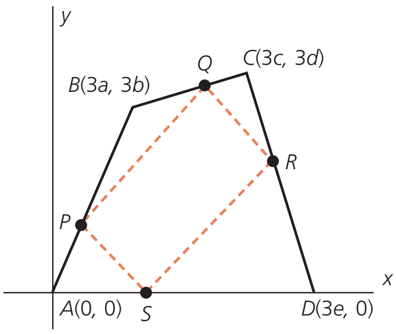
Similarly $y = \dfrac{my_2 + ny_1}{m + n}$

The coordinates of N are $\left(\dfrac{mx_2 + nx_1}{m + n}, \dfrac{my_2 + ny_1}{m + n}\right)$.

EXAMPLE 2

In quadrilateral $ABCD$, point P divides AB in the ratio 1:2; Q divides BC in the ratio 2:1; R divides CD in the ratio 1:2; and S divides DA in the ratio 2:1. Prove that $PQRS$ is a parallelogram.

Solution 1 using analytic methods



Let the coordinates of A, B, C, and D be $(0, 0)$, $(3a, 3b)$, $(3c, 3d)$ and $(3e, 0)$.

The coordinates of P are $\left(\frac{3a + 2(0)}{3}, \frac{3b + 2(0)}{3}\right)$ or (a, b).

The coordinates of Q are $\left(\frac{2(3c) + 3a}{3}, \frac{2(3d) + 3b}{3}\right)$ or $(2c + a, 2d + b)$.

The coordinates of R are $\left(\frac{3e + 2(3c)}{3}, \frac{0 + 2(3d)}{3}\right)$ or $(e + 2c, 2d)$.

The coordinates of S are $\left(\frac{2(0) + 3e}{3}, 0\right)$ or $(e, 0)$.

The slope of PQ is $\frac{(2d + b) - b}{(2c + a) - a} = \frac{d}{c}$

The slope of SR is $\frac{2d - 0}{(e + 2c) - e} = \frac{d}{c}$

Then $PQ \parallel SR$.

The slope of SP is $\frac{b - 0}{a - e} = \frac{b}{a - e}$

The slope of RQ is $\frac{(2d + b) - 2d}{(2c + a) - (e + 2c)} = \frac{b}{a - e}$

Then $SP \parallel QR$

Therefore, $PQRS$ is a parallelogram.

Solution 2 using deductive methods

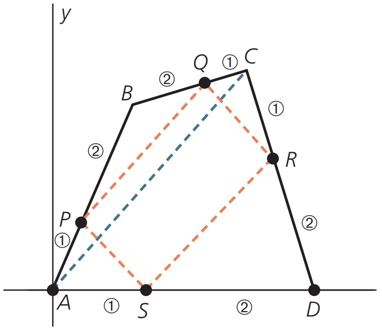
Join AC.

In ΔBAC, P divides BA in the ratio 2:1 and Q divides BC in the ratio 2:1.

Then $PQ \parallel AC$ (line parallel to base)

Similarly in ΔDAC, $RS \parallel AC$

Therefore $PQ \parallel RS$



Join BD.

In ΔABD, $PS \parallel BD$

In ΔCBD, $QR \parallel BD$

Therefore $PS \parallel QR$

Then $PQRS$ is a parallelogram.

Solution 3
using vector methods

From the diagram in Solution 2,

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PB} + \overrightarrow{BQ}\\ &= \tfrac{2}{3}\overrightarrow{AB} + \tfrac{2}{3}\overrightarrow{BC}\\ &= \tfrac{2}{3}\overrightarrow{AC}\\ \overrightarrow{SR} &= \overrightarrow{SD} + \overrightarrow{DR}\\ &= \tfrac{2}{3}\overrightarrow{AD} + \tfrac{2}{3}\overrightarrow{DC}\\ &= \tfrac{2}{3}\overrightarrow{AC}\end{aligned}$$

Then $\overrightarrow{PQ} = \overrightarrow{SR}$

Therefore $PQ = SR$ and $PQ \parallel SR$

Therefore, $PQRS$ is a parallelogram.

These solutions are quite different. Discuss the merits of the different approaches with your classmates.

EXAMPLE 3

In the parallelogram $ABCD$, the point F is chosen on DC such that $DF{:}FC = 3{:}1$. AF intersects DB at the point E. Determine the ratio in which the point E divides AF and DB.

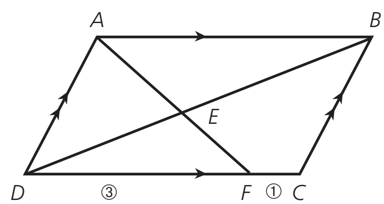
Solution 1

In ΔABE and ΔFDE,

$\angle ABE = \angle EDF$ (alternate angles)

$\angle BAE = \angle DFE$ (alternate angles)

$\therefore \Delta ABE \sim \Delta FDE$ (equal angles)



Since $DF = \frac{3}{4}DC$ and $DC = AB$, then

$DF = \frac{3}{4}AB$ or $DF{:}AB = 3{:}4$

Using similar triangles, $\frac{DF}{AB} = \frac{FE}{AE} = \frac{DE}{BE} = \frac{3}{4}$

Therefore, E divides AF in a 4:3 ratio, and it divides DB in a 3:4 ratio.

Solution 2

Let $\overrightarrow{DA} = \overrightarrow{CB} = \vec{a}$ and $\overrightarrow{AB} = \overrightarrow{DC} = \vec{b}$

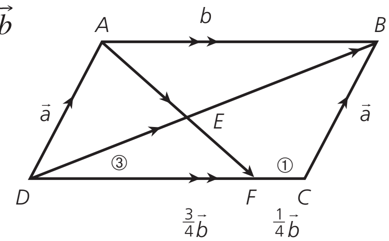
Since $DF{:}FC = 3{:}1$, then $\overrightarrow{DF} = \frac{3}{4}\vec{b}$ and $\overrightarrow{FC} = \frac{1}{4}\vec{b}$

In ΔADF, $\overrightarrow{AF} = \frac{3}{4}\vec{b} - \vec{a}$

Let $\overrightarrow{AE} = m\overrightarrow{AF}$, $m \in R$

Then $\overrightarrow{AE} = \frac{3m}{4}\vec{b} - m\vec{a}$

Let $\overrightarrow{DE} = k(\vec{b} + \vec{a}) = k\vec{b} + k\vec{a}$, $k \in R$



Now $\overrightarrow{AE} = \overrightarrow{DE} - \vec{a}$

or $\frac{3m}{4}\vec{b} - m\vec{a} = k\vec{b} + k\vec{a} - \vec{a}$

or $\left(\frac{3m}{4} - k\right)\vec{b} + (-m - k + 1)\vec{a} = 0$

Since $\vec{a}$ and $\vec{b}$ are sides of a parallelogram, they are linearly independent, so

① $\frac{3}{4}m - k = 0$

② $-m - k + 1 = 0$

Subtracting, $\frac{7}{4}m = 1$

$m = \frac{4}{7}$ and $k = \frac{3}{7}$

Thus, $\overrightarrow{AE} = \frac{4}{7}\overrightarrow{AF}$ and so $|\overrightarrow{AE}|:|\overrightarrow{EF}| = 4:3$

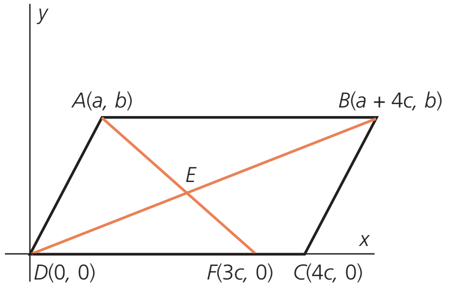
$\overrightarrow{DE} = \frac{3}{7}\overrightarrow{DB}$ and so $|\overrightarrow{DE}|:|\overrightarrow{EB}| = 3:4$

Solution 3

Let the coordinates of the parallelogram be as in the given diagram.

The equation of AF is $y - 0 = \frac{b}{a - 3c}(x - 3c)$

The equation of DB is $y = \frac{b}{a + 4c}x$



For the point E, $\frac{b}{a - 3c}(x - 3c) = \frac{b}{a + 4c}x$

$$(a + 4c)(x - 3c) = (a - 3c)x$$

$$(a + 4c - a + 3c)x = (a + 4c)3c$$

$$7cx = (a + 4c)3c$$

$$x = \frac{3}{7}(a + 4c)$$

Then $$y = \frac{b}{a + 4c}\left(\frac{3}{7}(a + 4c)\right) = \frac{3}{7}b$$

Then the coordinates of E and B are in the ratio of 3:7, so $DE:EB = 3:4$

Also $$AE = \sqrt{\left(a - \frac{3(a + 4c)}{7}\right)^2 + \left(b - \frac{3}{7}b\right)^2} = \frac{4}{7}\sqrt{(a - 3c)^2 + b^2}$$

and $$EF = \sqrt{\left(3c - \frac{3(a + 4c)}{7}\right)^2 + \left(-\frac{3}{7}b\right)^2} = \frac{3}{7}\sqrt{(3c - a)^2 + b^2}$$

Then $AE:EF = 4:3$

Note that in this example the analytic approach is more difficult algebraically and the deductive method is shortest. There is no best approach, although some approaches may be better than others. Experience allows us to choose the most likely approach, but to gain experience, we must do a number of problems.

Exercise 9.3

In this exercise, you should discuss different approaches with classmates, then attempt the questions using the methods you have agreed upon. For some questions, pairs of students can agree to share approaches and then discuss their results with classmates.

Part A

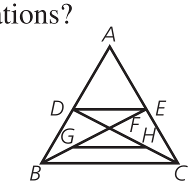
1. Prove that the line joining points that divide sides AB and AC in ΔABC in the ratio 3:1 is parallel to BC and equal to $\frac{3}{4}BC$.

2. In a trapezium, the ratio of the parallel sides is 5:3. Prove that a line through the intersection of the diagonals and parallel to the base divides the non-parallel sides in the same ratio.

Application

3. A town K is 12 km from a straight railroad. Two stations on the railroad are 20 km and 13 km from K. How far apart are the stations?

4. In the given diagram, D bisects AB, E bisects AC, G bisects FB and H bisects FC. Prove that $DE = GH$.



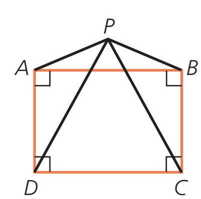
Part B

5. Square $ABCD$ has sides of length 2. E is the midpoint of BC. AE and BD intersect at F.
   a. What is the height from F of ΔBFE?
   b. What is the ratio of ΔBFE:ΔFAD?

6. Draw a quadrilateral $ABCD$. On each of the four sides of the quadrilateral select a point (not the midpoint) so that when these four points are joined in succession, a parallelogram is formed. Verify that your selection of points is correct by the use of analytic and deductive methods.

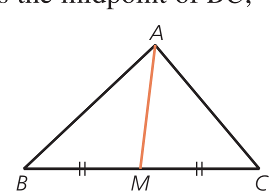
Part C

7. Previously, we proved that if P is a point in the interior of rectangle $ABCD$, then $PA^2 + PC^2 = PB^2 + PD^2$. Prove that this result is also true if the point P is not contained in the plane of rectangle $ABCD$ but is above the rectangle.



Thinking/Inquiry/ Problem Solving

8. The Theorem of Apollonius states that in ΔABC, if M is the midpoint of BC, then $AB^2 + AC^2 = 2AM^2 + 2MC^2$.
   a. Prove this theorem using analytic methods.
   b. By letting $\angle AMB = \alpha$ and $\angle AMC = 180 - \alpha$, use the cosine law in ΔAMB and derive the result.



Section 9.4 — Locus

Locus is a fundamental concept in mathematics. From earlier work you are familiar with examples of locus, like the path of a point that moves such that it is always a constant distance from a fixed point (a circle), or the set of points that are equidistant from two fixed points (the right-bisector of a line segment).

A locus is a set of points that satisfy a given condition, or the path traced out by a point that moves according to a stated geometric condition.

technology

The purpose of this investigation is to examine the path of a moving point in differing situations. Where possible, use technology in the study.

INVESTIGATION

1. AB is a fixed horizontal straight-line segment. A point moves so that it is 3 cm above AB. What is its locus?
2. What is the locus of an airplane flying at constant height from the equator to the north pole?
3. A quarter is rolled around the circumference of a nickel, always in contact with it. What is the locus of the centre of the quarter?
4. A set of points has the property that every point is equidistant from two fixed intersecting lines. What is the locus of the set of points?
5. A stick moves so that one end is in contact with the wall and the other end is in contact with the floor. What is the locus of the midpoint of the stick?
6. AB is a chord in a circle. A is fixed, and B is allowed to move along the circumference. If AB is extended to C so that $AB = BC$, what is the locus of C as B moves?
7. Chords are drawn from a fixed point on the circumference of a circle. What is the locus of their midpoints?

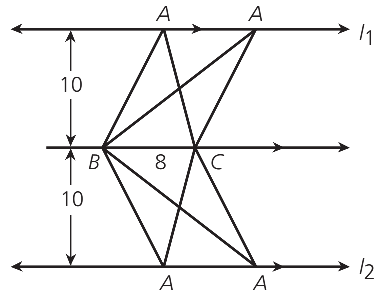
In this section, we examine some methods of identifying loci (the plural of locus). In problems of this type, technology can be helpful, and in some examples we employ it.

EXAMPLE 1

A triangle ABC has a base $BC = 8$. Describe the locus of the vertex A so that the triangle has an area of 40.

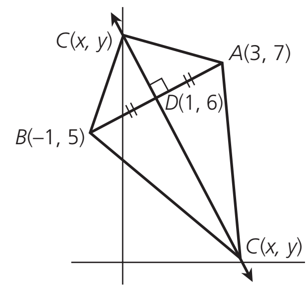
Solution

Since the area of ΔABC is 40, the height of ΔABC must be 10. The locus of A consists of all those points such that the distance from A to BC, or BC extended, is 10. Since the distance between parallel lines is constant by definition, then A must lie on a line parallel to BC, distance 10 away from it. The locus of A is a pair of parallel lines, one on each side of BC, and 10 away from BC.



EXAMPLE 2

The points $A(3, 7)$ and $B(-1, 5)$ are the vertices of the base of an isosceles triangle ABC. Determine the locus of vertex C.



Solution

Let the coordinates of C be (x, y).
Since ΔABC is isosceles, then $CB = CA$.

Thus, $\sqrt{(x+1)^2 + (y-5)^2} = \sqrt{(x-3)^2 + (y-7)^2}$

$x^2 + 2x + 1 + y^2 - 10y + 25$

$= x^2 - 6x + 9 + y^2 - 14y + 49$

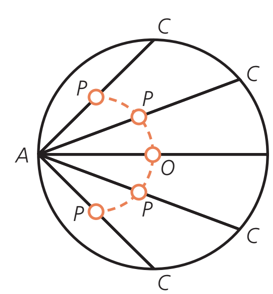
$2x + y - 8 = 0$

The locus of vertex C is the line having equation $2x + y - 8 = 0$.
We must exclude point $D(1, 6)$, the midpoint of AB, from our definition because A, B, and C must form the vertices of a triangle. If we had said C *is equidistant from* A *and* B, then D would be included.

EXAMPLE 3

A circle with a radius r units has diameter AB and its centre at O. From A, chords of the circle are drawn to a moving point C on the circle. Determine the locus of P, the midpoint of AC.

technology

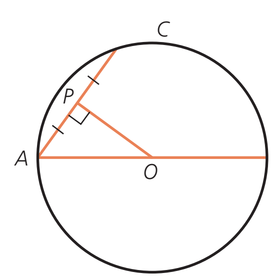


In order to see what the locus of P will look like, we use Geometer's Sketchpad to draw the locus. You can also draw a number of chords manually, if you prefer.

The locus of P appears to be a circle. This is only a strong hint; it does not constitute a proof. It does, however, tell us what we are aiming for, and gives us confidence that we are proceeding properly.

Solution 1

$\angle OPA = 90°$ (P is the midpoint of AC).
Then OA is the diameter of a circle for all positions of P (angle in a semicircle). The locus of P is a circle with AO as diameter; that is, it has radius $\frac{r}{2}$ and its centre at the midpoint of AO.



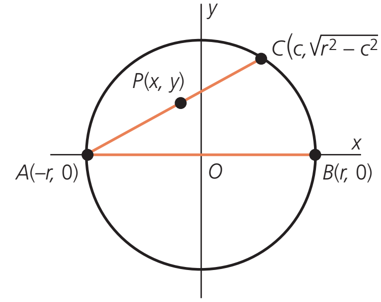
Solution 2

Let the circle have equation $x^2 + y^2 = r^2$, and let A have coordinates $(-r, 0)$.
Let $C(c, \pm\sqrt{r^2 - c^2})$ be any other point on the circumference.
If $P(x, y)$ is the midpoint of AC, then

$$x = \frac{c - r}{2} \quad \text{or} \quad c = 2x + r$$

and

$$y = \frac{\pm\sqrt{r^2 - c^2}}{2} \quad \text{or} \quad 4y^2 = r^2 - c^2$$



Substitute $c = 2x + r$ to obtain

$$4y^2 = r^2 - (2x + r)^2$$

Then

$$(2x + r)^2 + 4y^2 = r^2$$

$$4\left(x + \frac{r}{2}\right)^2 + 4y^2 = r^2$$

$$\left(x + \frac{r}{2}\right)^2 + y^2 = \left(\frac{r}{2}\right)^2$$

This is the equation of a circle with centre $\left(-\frac{r}{2}, 0\right)$ and radius $\frac{r}{2}$.

EXAMPLE 4

Determine the equation of the locus of a point P such that the ratio of its distances from $X(2, 1)$ and $Y(-1, -2)$ is 1:2, and identify the locus.

Solution

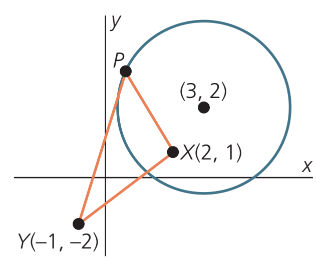
Let the coordinates of P be (x, y).

Since $\frac{PX}{PY} = \frac{1}{2}$

$$2PX = PY$$

Now $PX = \sqrt{(x - 2)^2 + (y - 1)^2}$ and

$$PY = \sqrt{(x + 1)^2 + (y + 2)^2}$$



Then $2\sqrt{(x - 2)^2 + (y - 1)^2} =$

$$\sqrt{(x + 1)^2 + (y + 2)^2}$$

$$4(x^2 - 4x + y^2 - 2y + 5) =$$

$$x^2 + 2x + y^2 + 4y + 5$$ (squaring both sides and expanding)

$$3x^2 + 3y^2 - 18x - 12y + 15 = 0$$

or

$$x^2 + y^2 - 6x - 4y + 5 = 0$$

Rewriting as $(x - 3)^2 + (y - 2)^2 = 8$.

The locus is a circle with centre $(3, 2)$ and radius $2\sqrt{2}$.

We know that the locus contains the point N, which divides XY in the ratio 1:2. Our formula from Section 9.2 tells us that N is (1, 0). Substituting N into our equation provides a simple way to check the accuracy of the work.

EXAMPLE 5

Triangle ABC is variable such that BC is fixed and $\angle BAC$ is constant. For any position of A, $\angle ABC$ and $\angle ACB$ are bisected, with their bisectors meeting at P. What is the locus of P?

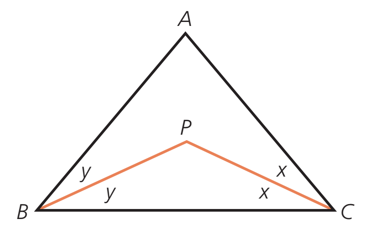
Solution

$\angle ABC + \angle ACB = 180° - \angle BAC$ (angle sum)

Then $\angle PBC + \angle PCB = 90° - \frac{1}{2}\angle BAC$

But $\angle BPC + \angle PBC + \angle PCB = 180°$ (angle sum)

Then $\angle BPC = 90° + \frac{1}{2}\angle BAC$



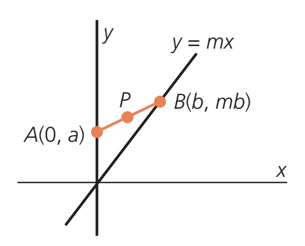
Since $\angle BAC$ is constant, $\angle BPC$ is constant for every position of A. Then BC subtends a constant angle at P. Therefore, P lies on a circle segment on BC as chord (angles on a chord).
The locus of P is a circle segment with BC as chord.

EXAMPLE 6

The ends of a line segment of length 4 move along two intersecting lines. Determine the equation of the locus of the midpoint of the line segment.

Solution

Let the intersecting lines be $x = 0$ (the y-axis) and $y = mx$. Let A be a point on $x = 0$ and B be a point on $y = mx$ such that $AB = 4$. We represent the coordinates of A by $(0, a)$ and the coordinates of B by (b, mb). If P is the midpoint of AB, then its coordinates (x, y) are given by $x = \frac{b}{2}$ and $y = \frac{a + mb}{2}$.



Then $b = 2x$ and $y = \frac{a + 2mx}{2}$

or $\quad a = 2y - 2mx$

Now $AB = 4$, so $(AB)^2 = 16$.

Then $b^2 + (mb - a)^2 = 16$

$$4x^2 + (2mx - 2y + 2mx)^2 = 16$$
$$4x^2 + 4(2mx - y)^2 = 16$$
$$x^2 + 4m^2x^2 - 4mxy + y^2 = 4$$
$$(4m^2 + 1)x^2 - 4mxy + y^2 = 4$$

The equation of the locus is $(4m^2 + 1)x^2 - 4mxy + y^2 = 4$.

Note that if $m = 0$, so that the lines are perpendicular, the equation becomes $x^2 + y^2 = 4$, a circle with centre (0, 0) and radius 2.

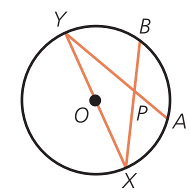
Exercise 9.4

Part A

1. A and B are points on the lines $y = x$ and $y = 2x$, respectively. If AB has a length of 6 units, determine the equation for the locus of P where P divides AB in a 2:1 ratio.

Part B

2. If A and B are fixed points on a given circle not collinear with centre 0 of the circle and if XY is a variable diameter, find the locus of P (the intersection of the line through A and Y and the line through B and X).



3. A point moves so that the sum of the squares of the lengths of the perpendiculars from it to the four sides of a square is constant. Find an equation for the locus and show that it represents a circle. (Use a unit square and a constant k.)

4. A and B are fixed points on a circle and XY is a diameter. XA and YB extended meet at P. As XY revolves around the centre, the point P moves. Determine, with proof, the locus of P.

Part C

5. From an external point, two equal tangents to a circle are drawn. Determine the locus of the set of points such that these equal tangents are always perpendicular.

Key Concepts Review

CHAPTER 9

In a problem where you wish to use analytic geometry, keep the following ideas closely in mind:

- position figures so as to minimize the number of variables needed; place one side of a figure on an axis
- choose coordinates for vertices to simplify your work as much as possible
- seek a solution that minimizes your work

In problems where you can consider different approaches, consider how you might approach the question using deductive, vector, or analytic methods, and try to determine which will yield the easiest approach for you. Remember that it is not necessarily true that the easiest approach for one person is also easiest for you. When possible, discuss different approaches with classmates, and develop different approaches to a given problem yourself.

In locus problems, setting the problem up so that you can simplify necessary algebra is of great importance.

Rich Learning Link wrap-up

investigate and apply

CHAPTER 9: STEINER NETWORKS

A Steiner network for a given set of points is a set of line segments joining the points in such a way that the total length of all the line segments is minimal. Steiner networks are the topic of a significant amount of contemporary mathematical research because of their application to data communications and computer microchip design. Some of this research considers Steiner networks in three dimensions, or in cases when only two perpendicular directions are permissible (if the network must follow the grid of city streets, for example), or circumstances in which the shortest total length is not necessarily the most cost effective.

Investigate and Apply

We will find a Steiner network for the points $A(0, 0)$, $B(-2, 5)$, $C(7, 8)$, and $D(6, 0)$. This can be done by using analytic geometry, but it will be much easier to use Geometer's Sketchpad (using the GRID feature).

1. Draw the quadrilateral $ABCD$ in a Cartesian coordinate system. What is the sum of the lengths of the diagonals?

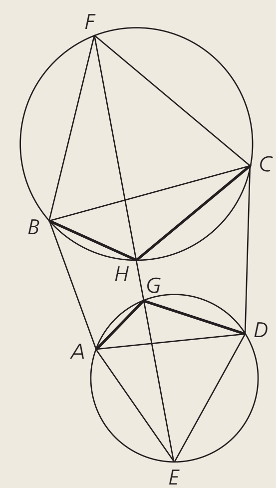
2. a. Construct equilateral triangles on the edges AD and BC pointing away from $ABCD$. Label the two new vertices E and F. Draw the line segment connecting E and F.

   b. Draw the circumcircles of the two equilateral triangles.

   c. Find the points G and H where the two circumcircles intersect $\overline{EF}$.

   d. Draw the network AG, DG, GH, HBH, and HC. It should look a bit like the diagram shown. Find $AG + DG + GH + HB + HC$.

   e. What are $\angle AGD$ and $\angle BHC$?



3. Repeat the steps of Question **2** with equilateral triangles on the edges AB and CD. Either this network or the one produced in Question **2** is the Steiner network. Which one is it?

INDEPENDENT STUDY

The Fermat point of an acute triangle is the point for which the sum of the distances to the vertices is as small as possible. How do you find the Fermat point of an acute triangle? What are some of its other properties?

What are some other geometric problems that Jacob Steiner considered?

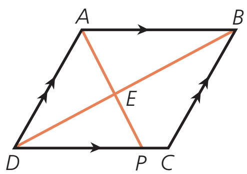
Review Exercise

1. Describe the following loci algebraically.
   a. the set of points equidistant from the x- and y-axes
   b. the set of points equidistant from $A(0, 4)$ and $B(0, 6)$
   c. the set of points 1 unit away from $O(0, 0)$

2. Determine the equation of the locus of a point P that moves so that the line joining P to $A(2, 5)$ always has an inclination of 45º.

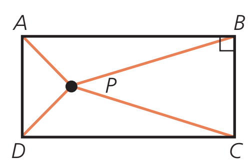
3. The line segment AB has end points $A(-3, 5)$ and $B(9, 16)$. Determine the coordinates of the point C where C divides AB in a 3:1 ratio.

4. In the parallelogram $ABCD$, the point P divides DC in a 5:1 ratio. The line joining A to P and the diagonal BD intersect at E. Determine
   a. $AE:EP$
   b. $DE:EB$



5. Find an equation for the locus of points $P(x, y)$ such that P is always equidistant from the line $x = -3$ and the point $X(3, 1)$.

6. In the rectangle $ABCD$, P is a point in the interior of the rectangle as shown. If $PA^2 + PC^2 = 85$ and we are told that the lengths PB and PD are integers, determine the possible values for PB and PD.



7. Find the equation of a locus such that its distance from the origin is numerically twice its distance from the line $y = 6$.

8. The point P is such that the difference between the squares of its distances from the two points $A(-3, 7)$ and $B(4, -2)$ always equals 4. Find the equation of its locus.

9. Ian has a square garden that measures 10 m × 10 m. He constructs a pathway around the square garden so that the pathway is always 1 m away from the garden. What is the area of the pathway, given that the pathway is 1 m in width?

10. The slope of the line joining $P(x, y)$ and the point $(3, -2)$ is always 2 greater than the slope of the line joining P to the point $(0, 4)$. Find the equation of its locus.

11. Show that the locus of the vertex of a right-angled triangle whose hypotenuse is the line joining the points $A(3, 5)$ and $B(7, 11)$ is a circle.

12. Determine the coordinates of the point that is equidistant from the points $A(2, 5)$, $B(2, 7)$, and $C(-6, 3)$.

13. Show that each of the following circles passes through the centre of the other. $x^2 + y^2 + 4x - 18y + 60 = 0$ and $x^2 + y^2 - 2x - 10y + 1 = 0$.

14. In the circle with equation $x^2 + y^2 - 6x - 8y = 24$, a chord is bisected by $A(5, -1)$. Determine the equation of the chord.

15. Find the equation of the circle that has its centre on the y-axis and passes through $O(0, 0)$ and the point $A(2, 1)$.

16. a. Show that the set of points that bisects chords drawn through an end of the horizontal diameter of the circle represented by $x^2 + y^2 = a^2$ is a circle.

    b. Find the coordinates of the centre and the length of the radius in **a.**

17. A point moves so that the sum of the squares of its distances from the vertices of a triangle is constant. Identify this locus.

18. In parallelogram $ABCD$, AC is the shorter diagonal and a point P moves so that $AP^2 + CP^2 = BP^2$. Show that the locus of P is a circle with centre at D. (Let the origin be at the centre of the parallelogram and the diagonal AC lie on the x-axis.)

Chapter 9 Test

Achievement Category	Questions
Knowledge/Understanding	all
Thinking/Inquiry/Problem Solving	6, 7
Communication	1, 2
Application	all

1. Describe the following loci in algebraic terms.
   a. the set of points equidistant from the lines at $x = 3$ and $x = 5$
   b. the set of points equidistant from $A(-3, 5)$ and $B(1, 3)$
   c. the set of points 5 units away from $A(-3, 2)$

2. Describe each of the following loci in one sentence.
   a. $x^2 + y^2 - 2x + 6y - 3 = 0$
   b. $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 9$

3. Determine each of the following.
   a. The coordinates of a point A that divides the line segment joining $A(-3, 7)$ and $B(2, 17)$ in a 1:4 ratio.
   b. The locus of a point P that moves so that P is the midpoint of the line segment joining the origin to any point on $x = 4$.

4. Determine the coordinates of a point that is equidistant from the points $A(1, 4)$, $B(1, 8)$, and $C(5, 4)$. Determine the equation of the circle passing through A, B, and C.

5. OX and OY represent two straight rulers placed at right angles to each other. A and B are the ends of a rod of length 10 units with A on OX and B on OY. The rod is allowed to move so that the ends slide along the axes. Find the locus of P, the point that divides AB in a 1:4 ratio.

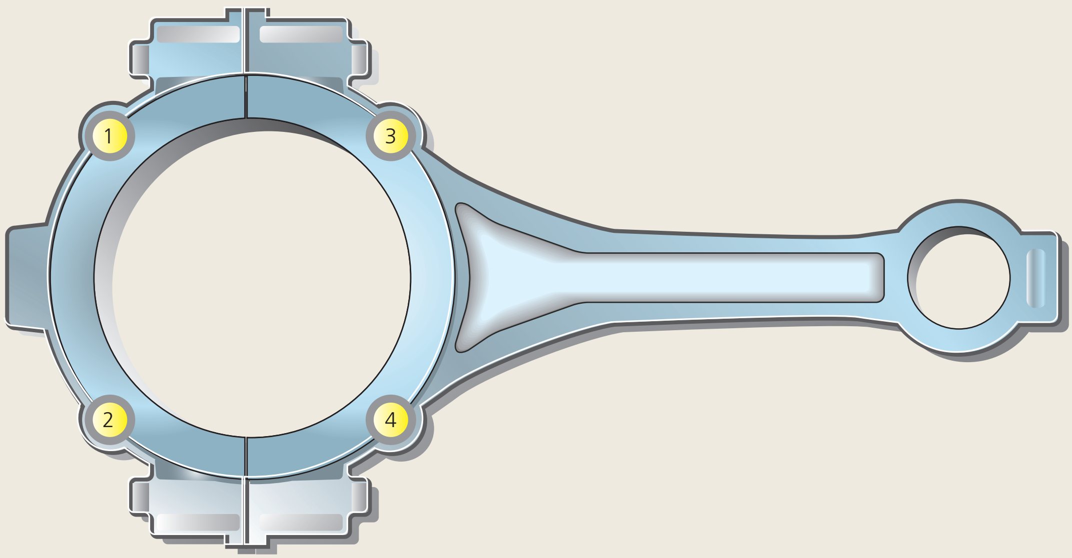
6. An equilateral triangle has a side length of 2 units. A point P moves so that the sum of the squares of the distances from P to the three vertices is 11. Determine the equation of the locus of P and show that it is a circle.

7. Find the equation of the locus of the centre of a circle that passes through the point $(0, 0)$ and cuts off a length $2\,k$ from the line with equation $x = c$.

Extending and Investigating

COORDINATE MEASURING MACHINES

Modern cars are made by mass production. Thousands of components are assembled to get a finished vehicle. One necessary criterion for mass production to work is the ability to manufacture particular components so that they are all essentially the same. For example, a rod (more fully called a connecting rod) is part of the engine. A V6 engine requires one rod for each of its six cylinders.



Thousands of rods are manufactured every day. When the engine is assembled, the rods must be interchangeable. This means that the dimensions of each rod must be very tightly controlled. For the engine to run properly, the distance from the centre of the hole on the left (the larger end) and the centre of the hole on the right (the small end) can vary by no more than a few microns (1 micron $= 10^{-6}$ metres). How can this distance be measured to such a high degree of precision?

One common tool for measuring dimensions is a coordinate measuring machine, often called a CMM. Use a search engine to find a picture and a description of a CMM on the Web. To measure geometrical properties, a part is placed on the bed of the CMM. A computer-controlled arm with a stylus on the end can be moved so that the stylus touches the rod at any point. The computer then calculates the x-, y- and z-coordinates of the point on the part relative to a fixed origin. For example, to measure the thickness of the rod at one of the yellow dots on the illustration, the rod can be mounted in a clamp. The CMM determines the coordinates of the yellow dot and the corresponding coordinates of the point on the opposite side of the rod. Then the standard geometric formula is used to find the distance between the two points.

For a harder problem, suppose we wanted to find the coordinates of the centre of the hole at the larger end. There is no place to touch! Now we need to use some mathematical thinking. One algorithm is to first determine the coordinates of three of the four yellow dots or, more precisely, the coordinates of three points A, B, C on the edge of the circle at the larger end. We know that these three points form a triangle on a plane. The centre of the circle is the point of intersection of the right bisectors of the edges of the triangle. Instruct the computer to carry out the following calculations:

1. Find the equation of the plane π through A, B, C.
2. Find the coordinates of the midpoints P and Q of AB and AC.
3. Find the equation of the line l_1 in π through P perpendicular to the line AB.
4. Find the equation of the line l_2 in π through Q perpendicular to the line AC.
5. Find the point of intersection of the lines l_1 and l_2.

This gives us the coordinates of the centre of the circle. It is a good thing that we have computers to do all this work. If we use the same procedure at the small end, we can then determine the critical distance between the two centres.

This algorithm is relatively easy to describe and implement. What can go wrong? There is a long list. For example, if we repeat the measurement, we will not get the same answer, because it is not possible for the CMM to move the stylus to exactly the same three points on the edge of the circle. You might think that this would not matter since we know that the centre of the circle is the same for any three points on the edge. The problem is that the shape of the hole is not a perfect circle! In fact, we do not know the exact shape, which will vary somewhat from one rod to the next. To deal with this uncertainty, the above algorithm is modified. First, the coordinates of several points (say five) on the edge of the circle are measured. Then, the algorithm is repeated for each of the ten subsets of three points. Finally, the average coordinates of the ten centres are determined.

There are many interesting mathematical problems associated with this algorithm. Where is the best place to put the five points? Would it be better to use six or seven or more points? How much variation will there be in the coordinates of the centre if the algorithm is repeated on the same rod?

People who run CMMs are experts at three-dimensional geometry. They know all about points, lines, planes, cylinders, and other geometric shapes. They exploit this knowledge to measure, often indirectly, critical features of manufactured parts.

Chapter 10

INTRODUCTION TO COUNTING

Counting is wonderful, counting is marvelous, counting's the best thing to do…

The Count from Sesame Street had the right idea. Counting is the oldest mathematical operation. Ancient herdsmen counted their herds by matching each animal with a stone. When evening came and the herds returned, the herdsmen would know if any animals were missing, even though they did not use a number system. Today, the Census of Canada counts people so that resources may be allocated fairly. The properties of combinations are used in statistics and probability. In this chapter, you will be introduced to the mathematics of counting or *combinatorics.*

CHAPTER EXPECTATIONS In this chapter, you will

- express the answers to permutation and combination problems, **Section 10.1**
- solve problems using counting principles, **Section 10.1, 10.3, 10.4**
- solve problems involving permutations and combinations, **Section 10.2, 10.5**

Review of Prerequisite Skills

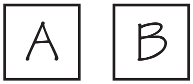
Counting is a fundamental mathematical operation. However, with a moment's reflection, you will realize that there is much more to counting than pointing at objects and saying *one*, *two*, *three*, …. For example, how can we count the number of combinations of six different balls that are drawn from the drum in the 6/49 lottery? In this lottery, the balls are numbered from 1 to 49 and you win a very large prize if you hold a ticket that matches the set of six numbers drawn. Your chance of winning the grand prize depends directly on the number of possible combinations. We also might be interested in knowing how many ways the balls can be selected so that at least three of them match the number on our ticket. In trying to answer these questions, since the numbers are obviously very large, we must be careful not to miss any possibilities or to count any combination more than once. In other words, we must be very organized. The mathematics of counting is called **combinatorics.**

The following activity will help you understand the usefulness of what you will learn in the next two chapters.

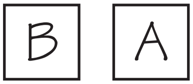
ACTIVITY

With a partner, cut out six small squares of paper (approximately 5 cm × 5 cm) and label them *A*, *B*, ... , *F*. Use the squares as a physical model to answer the following questions. You will have to invent a way to record all of the possibilities so that you do not miss any.

1. Make a list of all possible ordered pairs of squares. One example is shown below.



Note that in any pair there are two different letters and that



is a different pair than the first. How many pairs do you have in your list? How many pairs in your list have *A* in the first position?

2. Repeat Question **1** using an ordered triple of squares. Since the list is long, you will want to find a convenient way of writing it.

3. A set of two squares is not ordered. The squares A and B can be used to make two different ordered pairs, but only one set. Make a list of all sets of two squares that can be formed from the six letters. How many sets are in the list? How many of the sets contain square A? Can you explain the relationship between the answers here compared to those in Question **1**?

4. List all sets of three squares that can be formed from your six squares. How many sets can you form? How does this number compare to the answer in Question **2**? Can you explain the relationship?

In the next two chapters, we will look at ways to count lists of objects. As a result of the activity, you may have discovered some of the fundamental rules that we will use to solve such counting problems.

To provide some organizational tools, we start this chapter with some ideas from **set theory. Sets** are mathematical objects that describe how groups of things or elements can be organized in a formal way. Here, we use sets to specify the elements that we are trying to count. In the following sections, we use the notation and properties of sets to develop some rules and strategies for counting the number of elements in a set.

Rich Learning Link investigate

CHAPTER 10: LOTTERIES AND EXPECTED VALUES

In March 1998, the holders of one Canadian lottery ticket collected $22.5 million. The possibility of winning such a large sum of money entices millions of people to try their luck at lotteries. One of the most popular lotteries in Canada is Lotto 6/49. Players choose six numbers and hope to match the six numbers selected on draw night. Many times there is no winning ticket. Sometimes there are four or five winning tickets. If nobody wins, the prize amount keeps growing. If more than one ticket matches the six numbers drawn, the prize is divided equally among all the winners. With only a small chance of winning, are lotteries worth playing? One way to answer this is through the concept of *expected value*.

Investigate

The expected value of a lottery ticket is the amount of money you can expect to lose, per ticket, by playing over a long period of time.

Consider the following simple lottery:

You buy ticket #17 for $5.00. A number between 1 and 100 is drawn at random. If the number drawn is 17, you win a $200.00 prize.If the number drawn is 16 or 18, you win a $50.00 prize. Otherwise, you lose. The probability of winning $200.00 is $\frac{1}{100}$ and the probability of winning $50.00 is $\frac{1}{50}$. Hence, if you play this game about 100 times at a cost of $500.00, you expect to win $200.00 once and $50.00 twice. Your total lossess are $500.00 − $200.00 − (2 × $50.00) = $200.00. Divide this by the 100 times that you played, and you will find that the expected loss per play is $2.00. In other words, you lose on average $2.00 every time you play this lottery.

DISCUSSION QUESTIONS

1. Some people only play Lotto 6/49 when the prize is very large. Does this strategy reduce their expected loss?
2. Approximately one-half of Lotto 6/49 numbers are *quick picks* (i.e., randomly selected). The other half are people's own choices, and they are not evenly distributed among all 49 possible numbers. Lottery statistics show that people favour lower numbers, especially numbers between 1 and 12. Numbers less than 31 are also chosen more frequently than numbers larger than 31. Why do you think this is the case? What does this suggest about strategies for picking lottery numbers?

Section 10.1 — Sets

In this section, we look at some basic properties of sets.

A set is a collection of elements. It is identified by a rule for deciding whether or not a particular element is in the set.

The collection of numbers $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is the set A with elements that are integers from 1 to 10. We name the set using a capital letter and enclose the elements or members of the set in braces $\{\}$. If there is no possibility of misunderstanding, we specify the rule by the pattern of the elements listed. For example, the set of integers from 1 to 49 can be written $B = \{1, 2, \ldots, 49\}$. We often use a partial listing such as this, even if the rule is not obvious and must be stated separately, so that we can improve our understanding of what the elements are.

There are two further considerations when we are specifying a set. First, the order in which we list the elements does not matter. The sets $\{1, 8, 15, 22, 29, 36\}$ and $\{36, 8, 15, 1, 22, 29\}$ are the same. Second, the elements of a set are unique; no two can be the same. Therefore, $\{1, 8, 15, 22, 29, 29\}$ is not a set, since 29 appears twice. Here are some examples.

EXAMPLE 1

A spreadsheet program uses cells that are labelled by their column and row position. The columns are designated by letters A, B, C, ... and the rows by numbers 1, 2, 3, ... Write down the set of all cells that are in rows 1, 2, and 3 and columns A, B, C, D.

Solution

We can label a cell uniquely using its column letter and row number so $A1$ is the cell in the first column and first row. The set of all cells in the specified region is

$$U = \{A1, A2, A3, B1, B2, B3, C1, C2, C3, D1, D2, D3\}$$

Here, the 12 elements of the set U are cells.

In Example 1, we do not need to write out the rule that defines U because U is small enough that we can list all of its elements. The next example is more complicated.

EXAMPLE 2

Postal codes are used to help sort mail automatically. An optical character reader scans the postal code on each letter to help direct the letter to a particular location. A Canadian postal code is a sequence of length six; for example, N2L 3G1.

The first, third, and fifth terms are upper-case letters, and the second, fourth, and sixth terms are digits from 0 to 9. Not all letters are used. The letters D, F, I, O, Q, U, and W are left out because they look too much like other digits or letters to the character reader. Specify the set of possible postal codes.

Solution

The collection of all postal codes is a set U (a very large one).

$$U = \{A0A\ 0A0, A0A\ 0A1, ..., Z9Z\ 9Z9\}$$

Each element in the set is a postal code. We can imply the rule by writing the postal codes in alphabetic/numeric order. That is, we list the terms that are letters in alphabetic order and the other terms in numerical order. In the next sections, we will learn how to count the number of postal codes.

In counting problems, we usually start with the set of all the possible elements of the type being counted. This set is called the **universal set.** We will consistently name the universal set U. The set of all postal codes in Example 2 is the universal set if we are interested in counting postal codes with various properties. It is helpful to specify the universal set to be sure that we understand the objects that we are counting.

Another important idea is a **subset of the universal set.** A subset A of U is a set whose elements are all elements of U. If the universal set is the integers from 1 to 10, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then $A = \{1, 3, 5, 7, 9\}$ is a subset of U. $B = \{0, 1, 2\}$ is not a subset of U. Note also that $C = \{1, 3\}$ is a subset of U and also a subset of A. It is important to distinguish between the elements of a set and the number of elements. If we wish to refer only to the number of elements, we use the notation $n(A)$. For this example, $n(A) = 5$ and $n(C) = 2$.

To make the idea of a universal set and its subsets clearer and also to demonstrate some games that we can play with subsets, consider the next example.

EXAMPLE 3

Suppose three coins are tossed. Each coin can come up either heads (H) or tails (T). We can describe the eight possible outcomes by the universal set U.

$U = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

where, for example, the sequence HTH indicates that the first coin came up heads, the second tails, and the third heads. Define the subsets corresponding to the following statements, and state the number of elements in each.

a. E: the second coin has come up heads
b. F: exactly one coin has come up heads
c. G: two out of the three coins have come up the same

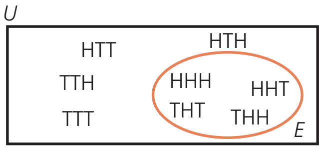
Solution

a. All the sequences in E have an H as the second term. Hence, $E = \{HHH, HHT, THH, THT\}$, and $n(E) = 4$.

b. The sequences in F are made up of one H and two Ts, so $F = \{HTT, THT, TTH\}$, and $n(F) = 3$.

c. The sequences in G have two Hs and one T or one H and two Ts. Hence, $G = \{HHT, HTH, THH, TTH, THT, HTT\}$, and $n(G) = 6$.

We can display the universal set and selected subsets in a picture called a **Venn diagram.** We display a subset by enclosing the appropriate elements within a closed curve. Here we show a Venn diagram for subset E, given in Example 3. A Venn diagram is useful for showing the relationships among subsets.



In Example 3, the object was to specify subsets based on a verbal description. Sometimes we need to play the game in reverse. We assume the most logical rule to describe a set when not all the elements are listed. The answers to the following example might be different for different people.

EXAMPLE 4 Consider the universal set U of integers from 1 to 100. That is, $U = \{1, 2, \ldots, 100\}$. Describe the following subsets of U in a simple sentence.

a. $A = \{1, 2, 3, \ldots, 10\}$

b. $B = \{1, 4, 9, \ldots, 81, 100\}$

c. $C = \{1, 10, 11, 12, \ldots, 19, 21, 31, \ldots, 91, 100\}$

Solution

a. A is the subset of U consisting of all integers from 1 to 10.

b. B is the subset of U consisting of all perfect squares up to 100.

c. C is the subset of U consisting of all integers from 1 to 100 that contain at least one digit 1. (Notice how many elements had to be listed to make this rule obvious!)

If A is a subset of U, then *the complement of A is the subset of U containing all the elements of U not in A*. We usually denote the complement of A by $\overline{A}$, that is, A with a bar over the top. For instance, in Example 4, we have

$\overline{A} = \{11, 12, \ldots, 100\}$ is the subset of integers in U greater than 10;

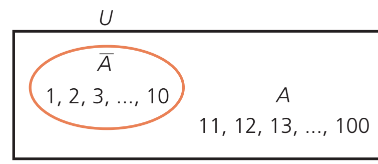
$\overline{B} = \{2, 3, 5, \ldots, 99\}$ is the subset of integers in U that are not perfect squares;

and $\overline{C}$ is the subset of integers in U that do not contain the digit 1.

By definition, the complement of a subset A contains all the elements of U not in A. When we are counting the number of elements, $n(A) + n(\overline{A}) = n(U)$, or more usefully,

$$n(A) = n(U) - n(\overline{A})$$

If it is difficult to count the elements in A, we can determine this number indirectly by finding $n(\overline{A})$ and $n(U)$. A Venn diagram demonstrates the rule clearly. All the elements in U are divided into two distinct groups, those in A and those in $\overline{A}$.



In Example 5, it is difficult to count the elements in A but easy to count those in $\overline{A}$.

EXAMPLE 5 How many integers from 1 to 1000 are not divisible by 5?

Solution

Let $U = \{1, 2, 3, \ldots, 1000\}$ be the universal set so that $n(U) = 1000$. If A is the subset of integers not divisible by 5, then the complement of A is the set of integers divisible by 5.

$$\overline{A} = \{5, 10, 15, \ldots, 1000\} = \{1 \times 5, 2 \times 5, 3 \times 5, \ldots, 200 \times 5\}$$

We can see that $n(\overline{A}) = 200$, so it follows that
$n(A) = n(U) - n(\overline{A}) = 1000 - 200 = 800$. There are 800 integers from 1 to 1000 not divisible by 5.

This simple rule is surprisingly useful. If it looks difficult to count the number of elements in a subset directly, try instead to count the elements of the complement. You will also have to define U and find $n(U)$.

Exercise 10.1

Part A

1. Suppose that U is the set of all two-digit integers and A is the subset of U of integers containing the digit 7. List the elements of U and A.

Application

2. A restaurant serves four main courses and three desserts. If a meal consists of a main course and a dessert, list the set U of all possible meals.

3. A school library has five different calculus books. A student is allowed to sign out two books at one time. List the set of all possible ways that two books can be signed out.

4. The universal set U is made up of the letters of the alphabet. If V is the subset of vowels, list the elements of V and its complement and find $n(V)$ and $n(\overline{V})$.

Knowledge/Understanding

5. The three letters of the word *cat* can be re-arranged to form other three-letter words, most of which are not real words. Let U be the set of all such words.

a. List all the elements in the universal set U.

b. What is $n(U)$?

c. List the elements in the following subsets of U.

A: all words beginning with the letter a

B: all words ending with t

d. Show A, B, and U on a Venn diagram.

e. Describe the complements $\overline{A}$ and $\overline{B}$ in words.

Part B

Knowledge/Understanding

6. A binary sequence is a sequence of 0s and 1s. Consider the universal set U of binary sequences of length 3.

a. List all the elements in U.

b. What is $n(U)$?

c. List the elements in the following subsets of U.

E: the sequence has exactly one 1

F: the sequence has at least one 1

d. If $\overline{F}$ is the subset of all elements in U that are not in F, describe $\overline{F}$ in words.

e. Verify directly that $n(F) + n(\overline{F}) = n(U)$.

7. Consider the set of integers $U = \{1, 2, 3, \ldots, 100\}$.

a. Describe the following subsets in words.

$R = \{5, 10, 15, \ldots, 100\}$, $S = \{53, 54, 55, \ldots, 75, 76\}$, $\overline{R}$

b. Indicate the elements in the following subsets of U.

P: all integers that are greater than 70

Q: all integers that end in the digit 7

c. For P and Q given in **b,** describe $\overline{P}$ and $\overline{Q}$ in words.

8. Suppose U is the set $\{1, 2, 3, \ldots, 1000\}$ with subset $S = \{10, 20, 30, \ldots, 1000\}$. In words, describe S and its complement. What are $n(S)$ and $n(\overline{S})$?

Communication

9. Two subsets of the set $U = \{1, 2, \ldots, 100\}$ are X: the integers from 1 to 100 that are prime, and Y: the integers from 1 to 100 that are perfect squares.

a. Do X and Y have any elements in common? Explain. Note that 1 is not a prime number.

b. Is Y the complement of X? Explain.

10. Let U be the set of all subsets of size 2 with elements selected from the six letters a, b, c, d, e, f.

a. Write out all of the elements of U.

b. Find $n(U)$.

c. Find the following subsets of U.

W: each subset contains the letter a

V: each subset contains at least one of a or f

d. Show V and $\overline{V}$ on a Venn diagram.

Application

11. In a simple lottery, three balls are selected to form a three-digit number from a set of nine balls numbered from 1 to 9. Balls are not replaced once they have been selected.

a. Let U be the universal set of all possible outcomes. Develop an appropriate notation and rule to describe the elements of U.

b. Determine the number of elements of the following subsets of U.

A: the ball labelled 4 is selected first

B: the ball labelled 4 is selected

Communication

12. Repeat Question **11** if the balls are replaced on each draw. Suppose U_1 is the set of all possible outcomes. Explain why $n(U_1) > n(U)$.

13. Let $U = \{1, 2, 3, \ldots, 1000\}$ be the set of positive integers less than or equal to 1000. Find the size of the following subsets of U.

a. E: the subset of integers divisible by 7

b. F: the subset of perfect squares

c. G: the subset of integers not divisible by 3

d. H: the subset of integers not ending in 9

Part C

Thinking/Inquiry/ Problem Solving

14. Three letters that include the name of the recipients are folded and put into three envelopes. Suppose we label the letters A, B, and C and the corresponding envelopes a, b, and c. Construct a notation for the universal set to describe all the possible assignments of letters to envelopes. List the elements of the subset that corresponds to no letter being sent to the correct person.

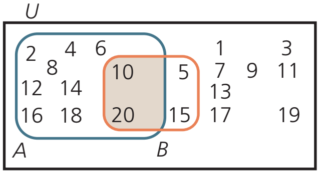
15. Suppose U is the set of points in the plane defined by $U = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$. $V = \{(x, y): x + y \leq 1\}$ is a subset of U. Draw a Venn diagram that shows U, V, and $\overline{V}$.

Section 10.2 — Combining Subsets

In this section, we look at two operations for combining subsets that will later be helpful in solving counting problems. To demonstrate these operations, suppose the universal set is the positive integers from 1 to 20.

$$U = \{1, 2, \ldots, 20\}$$

The subset $A = \{2, 4, 6, \ldots, 20\}$ corresponds to the even integers in U (i.e., those divisible by 2), and the subset $B = \{5, 10, 15, 20\}$ corresponds to the integers in U divisible by 5. We can display A, B, and U in a Venn diagram.



The fact that A and B overlap in the picture indicates that they have one or more elements in common. The elements 10 and 20 are found in both subsets. The subset corresponding to the common elements of A and B is called the **intersection** of A and B and is denoted by $A \cap B$. The intersection is represented by the shaded area on the Venn diagram. In this case,

$$A \cap B = \{10, 20\}.$$

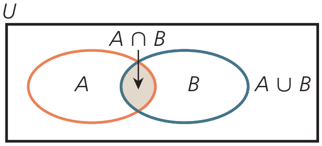
In words, $A \cap B$ contains all positive integers less than or equal to 20 that are divisible by both 2 and 5.

The total area covered by A and B represents all the elements in U that are found in either A or B or both. This subset is called the **union** of A and B and is denoted by $A \cup B$. Here

$$A \cup B = \{2, 4, 5, 6, 8\ 10, 12, 14, 15, 16, 18, 20\}.$$

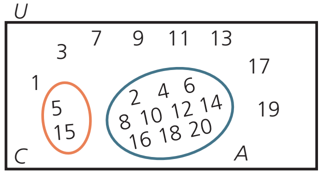
Remember the fundamental rule that we do not repeat elements in a set. Even though 10 is found in both A and B, it appears only once in the union $A \cup B$. In words, $A \cup B$ is the subset of U of integers divisible by either 2 or 5.

The general case is displayed in the Venn diagram. Note that this Venn diagram has been simplified by omitting the individual elements.



In some cases, two subsets have no elements in common. We call these subsets **disjoint.**

In the above example, let $C = \{5, 15\}$ be the subset of odd integers divisible by 5. The subsets A and C have no common elements, so these two subsets are disjoint. On the Venn diagram, A and C have no overlapping area.



EXAMPLE 1

Consider the set U of binary sequences of length 4.

$$U = \{0000, 0001, 0010, \ldots, 1111\}$$

Let W be the subset of binary sequences starting with 11, and V be the subset of sequences that include three or more 1s.

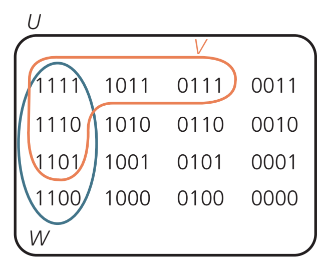
a. Show U, W, and V on a Venn diagram.

b. Find the elements in $W \cap V$ and describe this subset in words.

c. Find the elements in $W \cup V$ and describe this subset in words.

d. On a test, a student explained that the complement of W is the subset of sequences that end with 11. Is this answer correct? Explain.

Solution



a. We have $W = \{1111, 1110, 1101, 1100\}$ and $V = \{1111, 1110, 1101, 1011, 0111\}$. The Venn diagram is shown to the right.

b. $W \cap V = \{1111, 1110, 1101\}$ and $W \cap V$ is the subset of binary sequences with at least three 1s starting with 11.

c. $W \cup V = \{1111, 1110, 1101, 1011, 0111, 1100\}$ and $W \cup V$ is the subset of binary sequences that have at least three 1s or start with 11.

d. The complement of $\overline{W}$ contains all of the sequences in U that are not in W. For example, 0000 is in $\overline{W}$, so the complement of W is not the subset of binary sequences ending in 11.

We can build many subsets by using the union, intersection, and complement. For example, if A, B, and C are three subsets, then $A \cap B \cap C$ is the subset containing all those elements that are common to A, B, and C. Similarly, $A \cup B \cup C$ is the subset containing all elements found in at least one of A, B, or C. These definitions can be extended to any number of subsets.

Exercise 10.2

Part A

1. Four books labelled A, B, C, and D are placed on a shelf. The universal set U is the set of all possible arrangements. The subset R is all arrangements in which A is to the left of B. The subset S is all arrangements in which B is at the right end of the shelf. List the elements of U, R, S, $R \cup S$ and $R \cap S$.

Application

2. A store stocks jeans and shirts in three colours: blue, black, and grey. You decide to buy a pair of jeans and a shirt. The universal set U is the set of all possible selections you can make. If A is the subset of choices with a blue shirt, find $n(A)$ and $n(\overline{A})$.

3. If U is the set of all binary sequences of length 4, find $n(E \cap F)$, where E is the subset of sequences that start with 1 and F is the subset of sequences that have at least two 1s.

Part B

Communication

4. A pizza special can be ordered with any three of nine toppings. Let P be the subset of pizzas with pepperoni and M be the subset of those with mushrooms. Are P and M disjoint? Explain.

Thinking/Inquiry/Problem Solving

5. For any two subsets A and B, can $n(A \cup B) > n(A) + n(B)$? Explain.

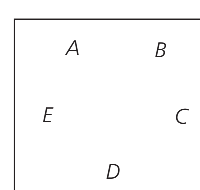
6. Let U be the set of positive integers from 1 to 16, A be the subset of such integers divisible by 3, and B be the subset of such integers greater than 10.
   a. Draw a Venn diagram to display U, A, and B. Do not include the individual elements of U.
   b. Determine $A \cap B$ and describe this subset in words.
   c. Are A and B disjoint?
   d. Determine $A \cup B$ and describe this subset in words.
   e. Determine $n(A)$, $n(B)$, $n(A \cap B)$, and $n(A \cup B)$.

Communication

7. Suppose the 5 letters p, p, p, q, q are arranged in a sequence. Let U be the universal set of all such sequences, C be the sequences in U starting with p, and D be those sequences ending with p.
   a. Write all the elements of U, C, and D.
   b. Find $C \cup D$ and describe this subset in words.
   c. Find $C \cap D$ and describe this subset in words.
   d. Find a subset of U that is disjoint with C.

8. Suppose A and B are two disjoint subsets of U. Explain in words why $n(A \cup B) = n(A) + n(B)$.

9. Five points in the plane, labelled A, B, C, D, E, are shown on the diagram. No three points fall on a line.
   a. Construct the set U of all possible triangles that can be formed using these five points. Note that triangle ABC and triangle BCA are the same.



b. List the elements of the subset F of triangles that have the line segment AB as one side and the subset G of triangles that contain C as a vertex.

c. List the elements of $F \cup G$ and $F \cap G$.

10. The digits 1, 2, 3, 4 are arranged to make a four-digit number. Let A be the subset of those numbers that start with 1, B the subset with 2 in the second place, C the subset with 3 in the third place, and D the subset with 4 in the fourth place.

a. Is any pair of these subsets disjoint? Explain.

b. Does $A \cup B \cup C \cup D = U$, where U is the universal set of all of the possible arrangements? Explain.

c. Find $A \cap B \cap C \cap D$.

Application

11. We want to select a committee of two people from six candidates, three girls named A, B, C and three boys D, E, F. Let U be the set of all such committees. By listing the elements in each subset, show that U can be written as the union of three subsets corresponding to a committee with two girls, a committee with one boy and one girl, and a committee with two boys.

12. The letters of the word *tree* are scrambled. The universal set U is the set of all possible arrangements. If E is the subset of arrangements in which the two *e*s are side by side and F is the subset of arrangements in which the *t* comes before the *r*, find $n(E)$, $n(F)$, $n(E \cup F)$ and $n(E \cap F)$.

13. Consider the set P of points in the Cartesian plane defined by (i, j), $1 \leq i, j \leq 4$, where i and j are integers.

a. Draw a set of axes with a dot at each of the points in P.

b. On the plot, show the subsets corresponding to $i + j = c$, where c takes on values 2 to 8.

c. Is any pair of these subsets disjoint?

d. Does their union include all the elements of P?

Part C

Thinking/Inquiry/
Problem Solving

14. Suppose A and B are two subsets of a set U.

a. In words, describe the subset $A \cup B$ in terms of the elements of A and B.

b. In words, describe the subset $\overline{A \cup B}$, the complement of $A \cup B$.

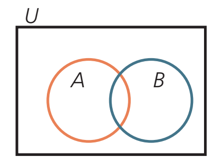
c. In words, describe the subset $\overline{A} \cup \overline{B}$.

d. Explain why $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

e. Show the result in **d** by illustrating each set on a Venn diagram.

15. Use the results from Question **14** to show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

16. A and B are two subsets of a set U as shown on the Venn diagram. Show that U can be written as the union of four pair-wise disjoint subsets defined in terms of A, B, and their complements.



Venn Diagram

(The results in Questions **14** and **15** are known as De Morgan's Rules, after Augustus De Morgan (1806–1871), the first Professor of Mathematics at University College, London.)

Section 10.3 — The Sum Rule

In the previous section, we combined two subsets using the union and intersection operations. Many of the problems we encounter require only that we determine the number of elements in a set. For example, if we wish to know our chances of winning a lottery, we need to determine the number of possible outcomes, but we do not need a listing of them all. For this reason we will now focus on the number of elements in sets. Here, for any two subsets E and F of some set U, we look at the relationship between $n(E)$, $n(F)$, $n(E \cap F)$, and $n(E \cup F)$. We can demonstrate this relationship with a simple example.

EXAMPLE 1

Four wooden blocks are arranged in a row. Two of the blocks are red, one is yellow, and one is blue. Let U be the set of all possible arrangements.

$$U = \{RRYB, RRBY, RYRB, RBRY, RYBR, RBYR, BRRY, YRRB, BRYR, YRBR, BYRR, YBRR\}$$

where, for example, $RRBY$ represents the arrangement with two red blocks first, then the blue block, and finally the yellow block. Let E be the subset of U corresponding to the second block in the arrangement being yellow and F the subset corresponding to all arrangements in which the two red blocks are side by side. Then we have

$$E = \{RYRB, RYBR, BYRR\}$$
$$F = \{RRBY, RRYB, BRRY, YRRB, BYRR, YBRR\}$$
$$E \cup F = \{RRBY, RRYB, BRRY, YRRB, BYRR, YBRR, RYRB, RYBR\}$$
$$E \cap F = \{BYRR\}$$

and $n(E) = 3,\ n(F) = 6,\ n(E \cup F) = 8,\ n(E \cap F) = 1$

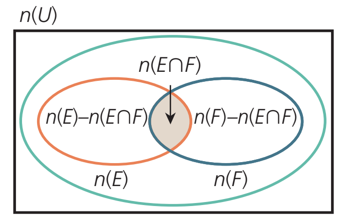
We observe that $n(E \cup F) = n(E) + n(F) - n(E \cap F)$.

This rule is true in general and simply states that to count the number of elements that are in one or both of E or F, we first count the number in E plus the number in F. Since we have counted the elements in $E \cap F$ twice, we must subtract the number of elements that appear in both E and F. We call this result the **sum rule.**

Sum Rule

If E and F are two subsets, the number of elements found in one or both of E or F is $n(E \cup F) = n(E) + n(F) - n(E \cap F)$.

A Venn diagram can help to demonstrate the sum rule. In the diagram, the number of elements that are in E but not in F is shown as $n(E) - n(E \cap F)$. Similarly, the number of elements in F but not in E is $n(F) - n(E \cap F)$.



The number of elements in the overlapping area is $n(E \cap F)$. In total, the number of elements in E or F is

$$\begin{aligned} n(E \cup F) &= [n(E) - n(E \cap F)] + [n(F) - n(E \cap F)] + n(E \cap F) \\ &= n(E) + n(F) - n(E \cap F) \end{aligned}$$

We use the sum rule to count the number of elements in the union of two subsets, as illustrated below.

EXAMPLE 2 What percent of the integers from 1 to 100 are divisible by either or both of 2 or 3?

Solution

$U = \{1, 2, \ldots, 100\}$ is the set of integers from 1 to 100. Let E be the subset of integers that are divisible by 2 and F the subset of integers divisible by 3. Then

$$E = \{2, 4, 6, \ldots, 100\} = \{1 \times 2, 2 \times 2, 3 \times 2, \ldots, 50 \times 2\}$$
$$F = \{3, 6, 9, \ldots, 99\} = \{1 \times 3, 2 \times 3, 3 \times 3, \ldots, 33 \times 3\}$$

The subset $E \cap F$ is all integers in U that are divisible by both 2 and 3. That is, this subset contains those that are divisible by 6.

Hence $E \cap F = \{6, 12, 18, \ldots, 96\} = \{1 \times 6, 2 \times 6, 3 \times 6, \ldots, 16 \times 6\}$.
The integers divisible by 2 or 3 are all those in $E \cup F$.

We find $n(E) = 50$, $n(F) = 33$, $n(E \cap F) = 16$ by counting the elements directly. Applying the sum rule, we get

$$n(E \cup F) = n(E) + n(F) - n(E \cap F) = 50 + 33 - 16 = 67$$

Hence, $\frac{67}{100}$ or 67% of the integers from 1 to 100 are divisible by 2 or 3.

In many counting problems, the subsets E and F are disjoint. That is, they have no common elements, so that $n(E \cap F) = 0$. Then we can state the sum rule for disjoint sets. Of course, we can always use the first rule, remembering that for disjoint sets $n(E \cap F) = 0$.

Sum Rule for Disjoint Subsets

If E and F are disjoint, the number of elements found in E or F is $n(E \cup F) = n(E) + n(F)$.

EXAMPLE 3

We want to form a committee of two people. There are six people available, three boys and three girls. How many committees can we form so that both members are the same sex?

Solution

Let the three girls be denoted by a, b, and c. If G is the subset of committees with two girls, then

$$G = \{\{a, b\}, \{a, c\}, \{b, c\}\}$$

where, for example, $\{a, b\}$ is the committee of a and b. Then $n(G) = 3$. Similarly, if B is the subset of committees with only boys, $n(B) = 3$. Note that G and B are disjoint. The subset $G \cup B$ contains all committees with both members the same sex, so we have $n(G \cup B) = n(G) + n(B) = 6$. The possible number of committees is six.

Exercise 10.3

Part A

Knowledge/Understanding

1. Suppose that $U = \{1, 2, 3, \ldots, 100\}$ is the set of integers from 1 to 100. If A is the subset of perfect squares and B is the subset of integers divisible by 4, verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ by directly counting the number of integers in each subset.

Application

2. Six trees—three cedars and three pines—can be planted in a row in 20 different ways. Considering the complement, determine the number of arrangements that have at least two trees of the same type side by side.

3. Two dice, one red and one green, are rolled. The set U of possible outcomes is given by all ordered pairs of the form (r, g) where $1 \leq r, g \leq 6$. For example, the pair (3, 4) indicates that the red die came up 3 and the green die came up 4. Consider the following subsets of U.

   A: both dice have the same value

   B: the sum of the values is 7

   a. List all the pairs in these two subsets.

   b. List all pairs that are in $A \cap B$.

   c. Find $n(A \cup B)$.

Part **B**

Knowledge/Understanding

4. Consider the set U of arrangements of the four letters a, b, c, d. That is

$$U = \{abcd, abdc, \ldots, dcba\}.$$

Let A be the subset of arrangements that start with the letter a, and B the subset of arrangements that have b in the second position. By listing the elements, verify the sum rule for the subset $A \cup B$.

Communication

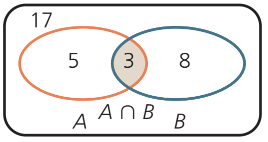
5. A student was asked to count the number of two-digit positive integers that start or end with 7. She decides that there are 10 such numbers that start with 7 (i.e., 70 to 79) and 9 that end with 7 (i.e., 17, 27, …, 97). She concludes that there are 19 such two-digit numbers. Explain why this answer is not correct.

6. Let U be the set of two-digit integers. V is the subset of such integers that contain at least one 5 and W is the subset of integers that contain at least one 6. Describe the complement of $V \cup W$ in words and, hence, find $n(V \cup W)$.

7. How many integers in the set $U = \{1, 2, 3, \ldots, 30\}$ are not divisible by 3? (*Hint:* First look at the subset of U corresponding to integers that are divisible by 3.)

8. A first-year calculus class has 90 students, of whom 42 are girls. Of the 90 students, 37 take Business 101. If 19 of the girls do not take Business 101, how many of the boys do?

9. The Venn diagram shows the subsets A and B of the set U and gives the number of elements in each non-overlapping area.



   a. Find $n(A)$, $n(B)$, $n(A \cup B)$.

   b. What is $n(U)$?

10. Suppose U is the set of binary sequences of length 3 and E_i, $i = 0, 1, 2, 3$ is the subset of such sequences with exactly i 1s. Explain why $n(U) = n(E_0) + n(E_1) + n(E_2) + n(E_3)$.

11. Prove that $n(A \cup B) \leq n(A) + n(B)$ for any subsets A and B.

Thinking/Inquiry/Problem Solving

12. What is the maximum number of days in a non-leap year that can fall on the weekend?

13. How many numbers from 1 to 1000 are

   a. divisible by 5

   b. divisible by 7

c. divisible by both 5 and 7

d. divisible by neither 5 nor 7

e. divisible by 5 but not divisible by 7

14. How many integers between 1 and 1000 are divisible by 7 or 13?

Part C

15. Suppose A and B are two subsets of a set U. Use De Morgan's Rules (see Questions **14** and **15** in Section 10.2) to show that $n(A \cup B) = n(U) - n(\overline{A} \cap \overline{B})$. Show this result on a Venn diagram.

16. Consider the set U of all binary sequences of length 4; that is, $U = \{0000, 1000, \ldots, 1111\}$.

    Three subsets of U are

    A: sequences that start with 1

    B: sequences that end in 1

    C: sequences that contain exactly two 1s.

    a. By listing the sequences, find the number of elements in the following subsets.

    $A, B, C, A \cap B, A \cap C, B \cap C, A \cap B \cap C$

    b. Verify that

    $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C)$
    $- n(B \cap C) + n(A \cap B \cap C)$.

    c. Show this result, the sum rule for three events, on a Venn diagram.

17. Use the sum rule for three events to find the number of integers between 1 and 1000 that are divisible by 2 or 3 or 5.

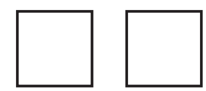
18. Suppose we have n subsets $E_1, E_2, \ldots, E_n$. State a formula for $n(E_1 \cup E_2 \cup \ldots \cup E_n)$ in terms of $n(E_i)$, $n(E_i \cap E_j)$, $n(E_i \cap E_j \cap E_k)$, …, $1 \le i, j, l, \ldots \le n$. (This development is known as the Principle of Inclusion and Exclusion.)

Section 10.4 — The Product Rule

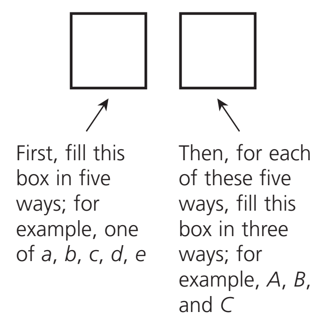
Suppose Eric has five shirts and three sweaters. How many different shirt-sweater combinations can he make? He doesn't worry about whether the colours match; however, he does know that sweaters can be worn only over shirts.

We solve this problem with the product rule, the most useful and powerful counting tool. It will be used in almost every counting problem.

We label the shirts a, b, c, d, e and the sweaters A, B, C. The ordered pair bC corresponds to shirt b and sweater C. We will always list the shirt first to avoid confusion. To answer the question posed, we want to find $n(U)$ where the universal set $U = \{aA, aB, aC, \ldots, eC\}$ represents all the possible shirt-sweater combinations. We count the elements of U constructively. That is, we look at how many ways we can build an element of U by filling in the two boxes shown to the right. Each way of filling the boxes will correspond to a unique element of U.



The box on the left is the shirt choice and can be filled in five ways, each way corresponding to a different choice of shirt. Once we have chosen the shirt, there are then three ways to fill the second box corresponding to the choice of sweater. Since there are three choices of sweater for each choice of shirt, there are $5 \times 3 = 15$ different combinations.



Note that we build the combination in order, first the shirt and then the sweater. The set U has 15 elements listed below so that you can confirm the logic of the argument.

$$U = \{aA, aB, aC, bA, bB, bC, cA, cB, cC, dA, dB, dC, eA, eB, eC\}$$

Product Rule

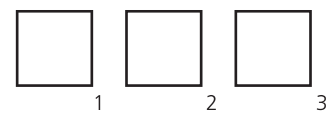
If the first of the two tasks can be done in p ways and, for each of these ways, the second task can be done in q ways, then together the two tasks can be done in $p \times q$ ways.

In the shirt and sweater example, there were only two tasks: choose the shirt and then the sweater. In most applications, there are more than two tasks. For these applications, we simply use the product rule repeatedly.

EXAMPLE 1 Suppose we have three copies of each of the letters of the alphabet and we want to make a three-letter acronym such as IBM (which stands for International Business Machines). The acronym IBM is different from BMI (an American performing rights organization that represents more than 140 000 U.S. songwriters and composers and over 60 000 U.S. publishers), even though both acronyms use the same three letters. The order of the letters in the acronym is important.

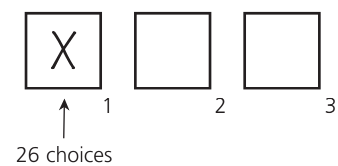
The set of all such acronyms is $U = \{AAA, AAB, \ldots, ZZZ\}$

U has a large number of elements and we will need a good strategy to find $n(U)$. Note that we can construct any three-letter acronym by placing one of the 26 letters in each of the boxes.

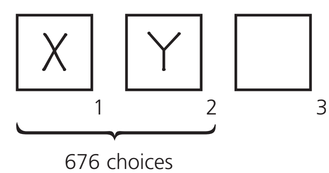


The subscript on each box indicates which letter in the acronym it represents.

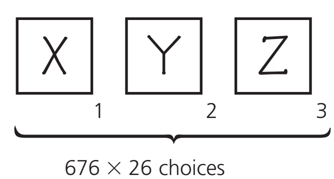
We find $n(U)$ by looking at how we construct the acronyms. To build an acronym, the first box can be filled with any one of the 26 letters. Suppose for example, we place an X in the first box.



Now let's choose the second letter of the acronym. For each possible choice for the first letter, there are 26 choices for the second box. Using the product rule, there are $26 \times 26 = 676$ ways to construct the first two letters of the acronym. We might, for example, choose Y for the second letter.



Finally, we complete the acronym by choosing the third letter. For every possible choice for the first two letters, there are 26 choices for filling the third box, so any letter, including X and Y, can be used.



There are 676×26 ways to construct the three-letter acronym. Hence, there are 17 576 three-letter acronyms.

The product rule was used twice in the above example. We can write out a general form if there is a series of tasks to perform in sequence.

Generalized Product Rule

If the first of a number of tasks can be done in p ways and, for each of these ways, the second task can be completed in q ways and, for each of these ways, the third task can be completed in r ways, and so on, then the entire sequence of tasks can be done in $p \times q \times r \times \ldots$ways.

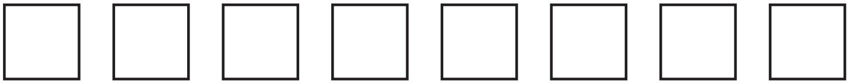
EXAMPLE 2 A computer codes information in a binary sequence of length 8, using 0 or 1 for each term in the sequence. Each such sequence is called a byte. How many different bytes can be formed?

Solution

It is always a good idea to identify the objects being counted. A typical byte is 01001110 and the set of all bytes is

$U = \{00000000, 00000001, 00000010, \ldots, 11111111\}$.

To count the bytes, we look at how we can construct them. Here we have eight boxes to fill, corresponding to the eight terms of the sequence.



Each box can be filled in two ways and the conditions for the product rule apply, so there are

$$2 \times 2 \times 2 \times \ldots \times 2 = 2^8$$

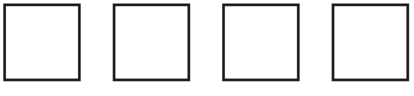
binary sequences of length 8.

We can make counting problems more interesting by looking at subsets of the universal set.

EXAMPLE 3 A *word* is formed by arranging the four letters a, b, c, d with no repetition. U is the set of all such words. B is the set of all words in U with last term a, and C is the set of all words in U with third term b or c. List two elements of each set. Then determine $n(U)$, $n(B)$, and $n(C)$.

Solution

We have $U = \{abcd, abdc, \ldots\}$, $B = \{bcda, bdca, \ldots\}$, and $C = \{abcd, acbd, \ldots\}$ To count the elements of U, we can construct a word by filling the four boxes as shown.



We can select any one of the four letters for the first letter of the word. For any one of these choices, there are three ways to select the second letter. There are then 4×3 ways to choose the first two letters. For each of these, the third letter can be selected in two ways from the remaining unused letters. Finally, for each of these selections there is only one remaining letter and, hence, only one way to choose the fourth letter. Applying the product rule, there are $4 \times 3 \times 2 \times 1 = 24$ words in U.

To count the words in B, we can again count the ways of filling the four boxes. This time we start with the fourth letter in the word, which must be an a. There is only one choice. For this choice, there are three ways to select the first letter, then two ways to select the second. For each of these choices, there is one way to select the third letter. Hence $n(B) = 1 \times 3 \times 2 \times 1 = 6$.

To count the elements of C, we start with the third letter of the word, which is either b or c. There are two choices. For each of these choices, the first letter can be selected in three ways, then the second in two ways. For each of these choices, the fourth letter can be selected in one way. Hence, using the product rule, we have $n(C) = 2 \times 3 \times 2 \times 1 = 12$.

In the above example, we see that we can construct sequences by starting with different terms. A very good strategy is to start with the terms that are most restricted. To find $n(C)$, if we had started with the first letter (four choices) and then selected the second (three choices), we would have a problem with the third letter, since the number of possible letters would vary depending on whether we had already selected b, or c, or both.

In the product rule, stated once again, the phrase *and for each of these ways* cannot be ignored. When using the product rule in its simple or general form, you must always check that this condition is satisfied. The next example shows what can go wrong if the condition does not apply.

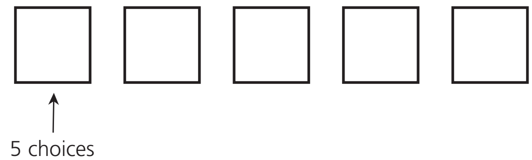
Product Rule
If the first task can be done in a ways and, for each of these ways, the second task can be done in b ways, then together the two tasks can be done in $a \times b$ ways.

EXAMPLE 4 The digits 1, 2, 3, 4, 5 can be arranged to form five-digit numbers. How many of these numbers are even?

Solution

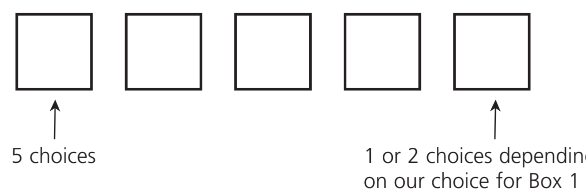
A five-digit number is a sequence of length 5. Let E be all those five-digit numbers that are even. Then $E = \{12354, 14532, \ldots\}$.

There are five boxes to fill. Suppose we start with the first. The first term can be any one of 1, 2, 3, 4, or 5. There are five choices.



The last term must be even, so we consider it next. There are two possibilities, either 2 or 4. Then the second term can be selected in three ways, the third in two ways, and the fourth in one way. Hence, there are $5 \times 3 \times 2 \times 1 \times 2 = 60$ elements in E.

This is the wrong answer because our analysis is faulty. There are five choices for the first term. If that term is 1, 3, or 5, there are two choices for the last term. However, if we use 2 or 4 for the first term, there is only one possibility for the last term. Once the first digit is chosen in five ways, it is not true that *for each of these ways* there are two ways to select a digit for the fifth place.



Rethinking the solution, we fill the final position first and have two choices. For each of these, there are four choices for position 1, three for position 2, two for position 3, and one for position 4. Then $n(E) = 4 \times 3 \times 2 \times 1 \times 2 = 48$.

When you are using the product rule to count sequences and arrangements,

1. List some of the sequences you are counting by defining the universal set and appropriate subsets. This helps to develop a good notation and to clarify any necessary restrictions.
2. Start by constructing the most restricted terms.
3. Make sure that the condition for each of these ways is met. Sometimes it is not possible to do this. We will discuss these situations in the next section. You can avoid mistakes by writing a brief explanation of the counting process.

Exercise 10.4

Part A

Knowledge/ Understanding

1. A four-digit PIN number can be represented as a sequence with four terms. If each term can be any digit from 1 to 9,

a. list three different elements in the set U of all possible four-digit PIN numbers.

b. Using the product rule, write a clear explanation of how to find $n(U)$.

2. Four different calculus books are arranged on a shelf. Explain why there are 24 different arrangements possible.

Communication

3. A restaurant menu has three appetizers, four main courses, and three desserts. If you decide to order one appetizer, one main course, and one dessert, how many different meals can you order? Explain.

Application

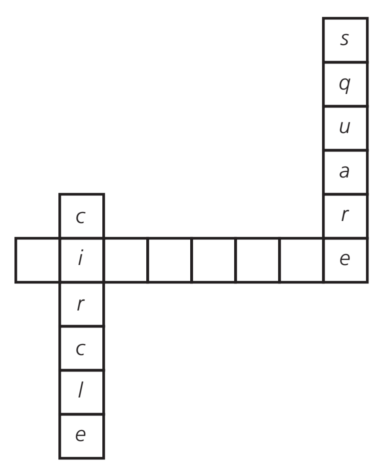
4. A Canadian postal code has the form $XxX\ xXx$ where X is an uppercase letter and x is a digit from 0 to 9. How many postal codes are there if the letters D, F, I, O, Q, U, and W are not used?

5. How many possible seven-digit telephone numbers are there within a given area code such that the first digit is 3, 5, or 6?

Part B

6. Six people arrange themselves in a row for a photograph. How many arrangements are possible if the two tallest people are at opposite ends of the row?

7. The solution to a clue in a crossword puzzle is an anagram (a rearrangement) of the word *alerting*. Based on the clues already solved, you know that the second letter is *i* and the last letter is *e*, as shown. A computer program is available to look at all the possible arrangements of the remaining letters. How many possibilities are there?



Communication

8. The letters of the word *cat* can be rearranged to form six different words. What is wrong with the following argument?

a. The first letter can be any one of the three letters; for example, A, C, or T.

The second letter can be any one of the three letters; for example, A, C, or T.

The third letter can be any one of the three letters; for example, A, C, or T.

Hence, there are $3 \times 3 \times 3 = 27$ possible arrangements of the letters of cat.

b. Write a correct argument using the product rule.

Communication

9. A pizza restaurant has a special deal: for only \$6.99, you get a large pizza with any three toppings chosen from mushrooms, extra cheese, pepperoni, sausage, pineapple, onions, green pepper, anchovies, or olives. A student uses the product rule to determine that there are $9 \times 8 \times 7 = 504$ different pizzas possible. Is this correct? Explain.

10. There are ten questions on a true/false test. Students attempt all questions.

a. Show how an answer sheet can be represented by a sequence of length 10.

b. How many different answer sheets are possible?

11. A mathematics contest has seven questions, each with four possible answers A, B, C, D. As well, a student may choose not to answer any particular question. Suppose that 30 000 students enter the contest. Is it possible that every answer sheet is different? Explain.

Knowledge/Understanding

12. A three-letter acronym is a sequence of length 3 with terms that are letters of the alphabet. Let U be the set of all such sequences. Consider the following subsets.

A: acronyms that start with a vowel

B: acronyms using only the letters from the set P, Q, R, S, T.

C: acronyms made up of three different letters

a. List two elements in each of U, A, B, C.

b. Find the size of each of these sets.

c. Find $n(A \cap B)$.

d. In words, describe the set of acronyms $A \cup B$.

e. Find $n(A \cup B)$.

f. Find the number of three-letter acronyms that use one letter at least twice.

Thinking/Inquiry/Problem Solving

13. The letters a, b, c, d, e, f can be rearranged to form a number of words or sequences of length 6 (no repeated letters are allowed). Let U be the set of all such words. Find $n(U)$ and the number of such words that do not begin with a.

14. Repeat Question **13** assuming the word formed has only four letters selected from the given six.

15. Consider the set of four-digit integers {1000, 1001, 1002, …, 9999}.

    a. Use the product rule to explain why there are 9000 such integers.

    b. How many of these integers end in 7 or 8?

    c. How many of these integers have no repeated digits?

    d. How many of these integers have repeated digits?

    In these problems and elsewhere, "repeated" means that the same digit can be used more than once, regardless of position.

16. A computer password must have eight symbols. These symbols can be either upper- or lower-case letters or digits from 0 to 9, and any symbol can be used repeatedly.

    a. How many passwords can be formed?

    b. How many passwords can be formed that start and end with a digit?

    c. How many passwords can be formed with no repeated symbols?

    d. How many passwords can be formed that have at least one 9?

17. We can always write a positive integer as the product of prime factors. For example, $12 = 2^2 3$. Every integer divisor of 12 can then be written in the form $2^a 3^b$ where $0 \le a \le 2$, $0 \le b \le 1$.

    a. Show that every divisor is equivalent to a sequence of length 2 where the first term is 0, 1, or 2 and the second term is 0 or 1.

    b. How many such sequences can be formed?

    c. How many divisors of 12 are there?

    d. Use the same method to count the divisors of 144.

    e. How many divisors of 144 are odd?

18. The integer 64 800 can be factored as $2^5 \times 3^4 \times 5^2$.

    a. How many divisors does 64 800 have?

    b. What fraction of these divisors is even?

Part C

Thinking/Inquiry/Problem Solving

19. An integer n can be factored as $n = 2^a 3^b 5^c$, where $a \ge 1$, $b \ge 1$, $c \ge 1$.

    a. How many divisors does n have?

    b. What fraction of the divisors is even?

20. Suppose we have m symbols available. How many sequences of length 3 can be formed if

  a. each symbol can be used at most once?

  b. each symbol can be repeated up to three times?

21. Repeat Question **20** assuming the sequence is of length r. In part **b**, each symbol can be repeated up to r times.

22. The days of the year (not including leap years) can be labelled 1 to 365. If we have five people in a room, we can create a sequence of length 5 to represent their five birthdays. The first term gives the first person's birthday, and so on.

  a. How many sequences are possible?

  b. What percent (to two decimals) of the sequences have five distinct birthdays?

  c. What percent (to two decimals) of the sequences have two or more birthdays the same?

technology

23. Suppose there are n people in a room.

  a. Repeat Question **22** to find an expression for the fraction of sequences with two or more birthdays the same.

  b. Using a calculator or spreadsheet program, find the smallest value of n so that the fraction in part **a** exceeds $\frac{1}{2}$.

Section 10.5 — The Use of Cases

To this point, we have discovered several useful tools such as the sum and product rules for solving counting problems. In this section, we look at the use of **cases,** another simple but handy strategy. When we use cases, we divide a problem into smaller (and hopefully easier) sub-problems and solve each of these separately. Then we combine the results to get the answer to our original problem.

For a simple example, consider the following question. In the list of numbers 1, 2, 3, …, 99, how many digits are there in total? We consider two cases.

Case	**Number of digits**
single digit numbers 1, 2, 3, …, 9	9
two digit numbers 10, 11, …, 99	$90 \times 2 = 180$
Total	189

For each case, we can count the number of digits easily. Note that the two cases are completely separate; there is no overlap. In the language of sets, to find the number of elements in a set A, we divide A into disjoint subsets and count the number of elements in each. This is simply a special case of the sum rule applied to two or more disjoint sets. In any problem, the challenge is to identify the disjoint subsets where we can easily count the number of elements.

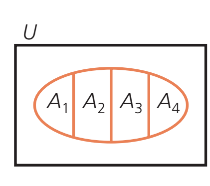
EXAMPLE 1 From year 1 to year 2000, inclusive, how many years started with 1?

Solution

Let A be the subset of years that start with 1. We partition A into four mutually disjoint subsets of years.

A_1: all years with a single digit starting with a 1
A_2: all years with two digits starting with a 1
A_3: all years with three digits starting with a 1
A_4: all years with four digits starting with a 1

Every pair of these subsets is disjoint and $A = A_1 \cup A_2 \cup A_3 \cup A_4$. The Venn diagram is shown at the right.



Using a simple extension of the sum rule, we have

$n(A) = n(A_1) + n(A_2) + n(A_3) + n(A_4)$.

Next, we count the number of years in each of the simpler cases. We have $n(A_1) = 1$ (1 year starting with 1; i.e., year 1), $n(A_2) = 10$ (10 years starting with 1; i.e., years 10 to 19), $n(A_3) = 100$ (100 years starting with 1; i.e., years 100 to 199), and $n(A_4) = 1000$ (1000 years starting with 1; i.e., years 1000 to 1999).

Since the subsets are disjoint, $n(A) = n(A_1) + n(A_2) + n(A_3) + n(A_4) = 1111$. There are 1111 years between 1 and 2000, inclusive, starting with the digit 1.

The formal extension to the sum rule is shown below.

Extended Sum Rule

The subsets $E_1, E_2, \dots, E_k$ of a universal set U are pairwise disjoint if every pair of subsets is disjoint. In this case, $E_1 \cup E_2 \cup \dots \cup E_k$ is the subset that contains all elements in any one of $E_1, E_2, \dots, E_k$.
Then $n(E_1 \cup E_2 \cup \dots \cup E_k) = n(E_1) + n(E_2) + \dots + n(E_k)$.

EXAMPLE 2 A sequence of length 3 is formed from the digits 1, 2, 3, …, 9 with no repetition allowed. What fraction of these sequences contains the digit 4?

Solution

Let $U = \{123, 124, \dots\}$ be the set of all sequences of length 3 formed from the 9 digits. Filling three boxes and using the product rule, we see that U has $9 \times 8 \times 7 = 504$ elements. Let A be the subset of U in which each sequence contains a 4. Let A_1 be the subset of A in which the 4 is the first term, A_2 be the subset in which 4 is the second term, and A_3 the subset in which 4 is the third term. Now A_1, A_2, and A_3 are pairwise disjoint and $A = A_1 \cup A_2 \cup A_3$.

Again filling boxes (in each case place the 4 first in one way), we have $n(A_1) = 1 \times 8 \times 7$, $n(A_2) = 8 \times 1 \times 7$, and $n(A_3) = 8 \times 7 \times 1$. Hence, $n(A) = n(A_1) + n(A_2) + n(A_3) = 168$.

The fraction of sequences containing 1 is $\frac{n(A)}{n(U)} = \frac{168}{504} = \frac{1}{3}$.

There are other solutions. For example, it is easy to find $n(\overline{A})$ here or, alternatively, especially after you see the answer, to see that the number of sequences containing 4 is the same as the number of sequences containing 2 and so on. Since each sequence contains three out of nine digits, exactly $\frac{1}{3}$ must contain 4.

EXAMPLE 3 User identifications for a local e-mail system must be formed from upper-case letters and can be five to eight letters long. How many such identifications are possible?

Solution

Let U be the set of possible identifications. To find $n(U)$, look at four cases in which the length of the sequence is fixed. That is, define B_5, B_6, B_7, B_8 to contain the identifications of length 5, 6, 7, and 8, respectively. These subsets are pairwise disjoint and $U = B_5 \cup B_6 \cup B_7 \cup B_8$. We have $B_5 = \{AAAAA, AAAAB, \ldots\}$. Filling five boxes with 26 symbols, $n(B_5) = 26 \times 26 \times 26 \times 26 \times 26 = 26^5$
$B_6 = \{AAAAAA, AAAAAB, \ldots\}$. Filling six boxes with 26 symbols, $n(B_6) = 26^6$.
$B_7 = \{AAAAAA, AAAAAB, \ldots\}$. Filling seven boxes with 26 symbols, $n(B_7) = 26^7$.
$B_8 = \{AAAAAAAA, AAAAAAAB, \ldots\}$. Filling eight boxes with 26 symbols, $n(B_8) = 26^8$.
Now using the extended sum rule, we have
$n(U) = 26^5 + 26^6 + 26^7 + 26^8 \cong 2.2 \times 10^{11}$.

Sometimes a problem that seems like a straight forward application of the product rule might be counted using cases because there are two restrictions, each of which affects the other.

EXAMPLE 4

A four-digit number is formed using the digits 0, 1, 2,…, 9 without repetition. How many such numbers are divisible by 25?

Solution

Let A be the set of four-digit numbers divisible by 25. Let A_1 be the subset of A ending in 25, A_2 be the subset ending in 50, A_3 be the subset ending in 75, and A_4 be the subset ending in 00. Then $n(A) = n(A_1) + n(A_2) + n(A_3) + n(A_4)$. For A_1 there are seven choices for the first digit, since we cannot use 0, 2, or 5; there are then seven choices for the second digit, since we cannot use the one chosen for the first digit but can now use 0.

Hence, $n(A_1) = 7 \times 7 \times 1 \times 1 = 49$. Similarly, $n(A_3) = 7 \times 7 \times 1 \times 1 = 49$.
For A_2 there are eight choices for the first digit and seven choices for the second digit, so $n(A_2) = 8 \times 7 \times 1 \times 1 = 56$.
Clearly, $n(A_4) = 0$, since repetition of a digit is not allowed.
Then, $n(A) = 49 + 56 + 49 + 0 = 154$.

Exercise 10.5

Part A

Knowledge/Understanding

1. By defining appropriate disjoint cases based on the number of days in a month, count the number of days in the year with a date of either 30 or 31.

Application

2. A special password is a sequence of length 6 that uses the digits 0 to 9 and the letters *a* to *z* (lower case only). How many such passwords can be constructed that contain exactly one letter and no repeated digits?

3. What is the total number of binary sequences with length no greater than 5?

Knowledge/Understanding

4. How many binary sequences of length 6, 7, or 8 begin with 0?

Part B

5. The symbols *a*, *b*, *c*, and *d* can be used to make words of length 1 to 4 if no repetition is allowed.
   a. How many such words can be created?
   b. How many of these words end with *a*?
   c. How many of these words contain the letter *a*? (*Hint:* Look at the complement.)

6. Repeat Question **5** assuming repetition of the symbols is allowed.

Thinking/Inquiry/Problem Solving

7. The numbers 1 to 10 are stored in a spreadsheet in a rectangular array. This means that the number of rows times the number of columns must be 10. How many ways can the numbers be stored in the spreadsheet?

8. How many words with three, four, or five letters can be formed if each word must contain at least one of *a*, *e*, *i*, *o*, *u*, and *y*.

Thinking/Inquiry/Problem Solving

9. The number of 5s in an integer is the highest power of 5 that divides evenly into the integer, so, for example, the number of 5s in 10, 12, and 25 are 1, 0, and 2, respectively. What is the total number of 5s in the set of integers {1, 2, 3,…, 1000}?

10. Six letters, *A*, *B*, *C*, *D*, *E*, and *F*, are arranged into a sequence of length 6. How many such sequences start with *A* or end with *F*?

11. How many positive integers between 1 and 2000 inclusively have distinct digits? What fraction of these integers is odd?

12. A sequence of length 3 is formed by first selecting one of the three words *cat*, *mouse,* or *goldfish*, then using three letters (without repetition) from the chosen word.
    a. How many sequences are there?
    b. How many sequences end in the letter *s*?
    c. How many sequences start with a vowel?
    d. How many sequences contain an *o*?

13. A sequence has terms selected from the digits 0, 1, 2, …, 9. Consider the set U of all such sequences with length at most 10.

a. Find $n(U)$.

b. A is the subset of U corresponding to sequences with unique digits. Find $n(A)$.

c. B is the subset of U corresponding to sequences that contain at least one 0. Find $n(B)$.

Part C

14. A sequence of length 3 is formed from r symbols that include the letter A. How many of the sequences contain at least one A if

a. repetition of symbols is allowed?

b. no repetition of symbols is allowed?

15. U is the set of all sequences of length 3 that can be formed using the digits 1, 2, 3, …, 9 without repetition. Let A be the subset of U in which the terms of the sequence form an arithmetic progression. Find $n(A)$.

16. Prove that the number of binary sequences with length less than k is 2 less than the number of binary sequences of length k.

Key Concepts Review

This chapter introduced you to some tools and strategies for solving counting problems. Once you are given a counting problem, how should you proceed?

First, write down a few of the objects you are counting. Sometimes this is obvious (e.g., all the arrangements of the letters of the word *dog*). In other problems, you will have to invent a notation to carry out this task. For example, if the problem is to count the number of committees of three that can be formed from six people, you might label the people A, B, C, D, E, F and denote a committee by a set such as $\{A, B, C\}$.

If the objects you are counting are sequences or arrangements, then you should consider counting the objects by filling in boxes and using the product rule. Always start with the terms that are most restricted.

If the problem as posed seems difficult, then specify a set A so that the counting problem is to find $n(A)$ and consider the complement of the set. Note that you need to define a universal set in order to specify the complement. We hope that it will be easier to determine the size of the complement and the universal set and then apply the rule of the complement.

Another good strategy for difficult problems is to use cases. That is, divide A into disjoint subsets, determine the size of each subset as a smaller and possibly easier problem, and use the sum rule to combine the results.

As a last resort, we can sometimes write the set A in the form $A = B \cup C$, with B and C not disjoint. To find $n(A)$ we can use the sum rule if we can first find $n(B)$, $n(C)$, and $n(B \cap C)$.

When you work through the Review Exercise, remember to write complete solutions. It is important that you be able to explain your solution to others. It is not enough to just get the correct answer. When you study mathematics, you are learning to communicate in a way that can be applied to all sorts of problems, not just specified mathematical ones. Solving counting problems is a useful way to practise these skills.

Rich Learning Link wrap-up

investigate and apply

CHAPTER 10: LOTTERIES AND EXPECTED VALUES

People who buy tickets for lotteries offering multi-million-dollar prizes are attracted to the possibility of changing their lives. Psychological theories suggest that in order to stay attracted to lotteries, many people need the occasional rewards associated with smaller prize amounts. You do not need to match all six numbers to win some money playing Lotto 6/49. The expected loss of a Lotto 6/49 ticket is lessened by the possibility of winning on five, four, or even three numbers.

Investigate and Apply

Lotto 6/49 draws involve randomly selecting six numbers and one bonus number from the numbers 1 through 49. In the year 2000, Lotto 6/49 paid out prizes as follows.

	1st Prize: Match all 6 draw numbers	2nd Prize: Match any 5 draw number and the bonus number	3rd Prize: Match any 5 draw numbers	4th Prize: Match any 4 draw numbers	5th Prize: Match any 3 draw numbers
Number of winners:	114	749	31 324	1 640 264	N/A
Average amount won:	$2 532 204.99	$93 300.75	$1 798.53	$65.80	$10

1. After the six numbers and bonus number have been drawn, how many different tickets could win each of the prizes?
2. Use the results from question **1** and the average amounts won to determine the expected loss of a $1.00 Lotto 6/49 ticket purchased in the year 2000.
3. Approximately how many people do you think won fifth prize in the year 2000?

INDEPENDENT STUDY

Research recent Lotto 6/49 payouts. Has the expected loss of a Lotto 6/49 ticket changed since 2000?

Lotteries have been called a regressive form of taxation. Investigate this statement.

Review Exercise

1. We have looked at four basic tools for solving counting problems. Write a brief description of each tool. Use an example to demonstrate its use.
   a. the rule of the complement
   b. the sum rule
   c. the product rule
   d. cases

2. The letters of the word *goldfinch* can be rearranged into a large number of sequences of length 9. Let U be the set of all such sequences and let

   G: all sequences in U that start with *gold*

   F: all sequences in U that end with *finch*

   a. Find $n(U)$, $n(G)$, and $n(F)$.
   b. Are G and F disjoint? Explain.
   c. In words, describe the subsets $G \cup F$ and $G \cap F$.
   d. Find $n(G \cap F)$ and $n(G \cup F)$.

3. Canadian postal codes are a sequence of length 6. The first, third, and fifth terms are upper-case letters (D, F, I, O, Q, U, W are not used) and the second, fourth, and sixth terms are digits from 1 to 9. Let U be the set of all postal codes. Consider the following subsets.

   A: all codes in U starting with N

   B: all codes in U ending with 8

   C: all codes in U that use the letter A

   D: all codes in U that use the letter N exactly once

   a. List two elements in each of U, A, B, C and D.
   b. Find $n(U)$, $n(A)$, $n(B)$, $n(C)$, and $n(D)$.
   c. Calculate $n(A \cup B)$.

4. A sequence of length 3 can be formed from the digits 0, 1, 2, ..., 9 without repetition. Let U be the set of all such sequences and let

A: all such sequences starting with an even digit

B: all such sequences ending with an even digit

C: all such sequences containing an even digit

a. List two sequences in each of U, A, B, and C.

b. Find $n(U)$, $n(A)$, $n(B)$, and $n(C)$

c. List two sequences in the intersection of A and B and then find $n(A \cap B)$.

5. A student is trying to count the fraction of binary sequences of length 6 that start or end with 1. He cannot remember the product rule but cleverly deduces that since the first term is either 0 or 1, exactly $\frac{1}{2}$ of the sequences start with 1 and, similarly, exactly $\frac{1}{2}$ of the sequences end with 1. He then concludes that the fraction of sequences that start or end with 1 must be $\frac{1}{2} + \frac{1}{2} = 1$. In other words, all the sequences start or end with 1. Being well trained, he had originally listed some examples of binary sequences, one of which was 011110, which does not start or end with 1, so that he knew his answer was wrong. Help your poor fellow counter and show him the error in his thinking. It would be friendly to give him a correct solution too. (If you are wondering why we have included so many binary string problems, it is because they are of fundamental interest in computer science.)

6. Local telephone numbers (within an area code) are a sequence of seven digits. The first digit in an area code is selected from 2 to 9. How many usable numbers are there? What fraction of the numbers end in 99?

7. How many integers between 1 and 1000 include the digit 7?

8. A PIN number for a photocopier can be any sequence of three or four digits.

   a. How many such PIN numbers are there?

   b. How many such numbers start with 2?

   c. How many such numbers contain at least one 2?

Chapter 10 Test

Achievement Category	Questions
Knowledge/Understanding	all
Thinking/Inquiry/Problem Solving	8, 9
Communication	2, 3, 4, 6
Application	all

1. $U = \{1,2,3,\ldots,999\}$ is the set of positive integers less than 1000.
   a. In words, describe the complement of the subset $A = \{5,10,15,\ldots,995\}$.
   b. What is $n(\overline{A})$?

2. What is the product rule?

3. The set U is all six-letter words that can be formed by rearranging the letters of *euclid*. Let A be the subset of such words that end *id* and B the subset of such words that end *ic*.
   a. In words, describe the subset $A \cup B$ and $A \cap B$.
   b. Find $n(A \cup B)$.

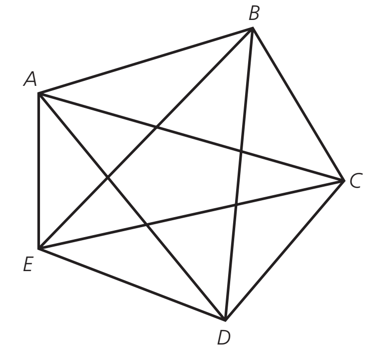
4. How many ways can the three letters selected from A, B, C, D, E, F, G (with no repeats allowed) be arranged in a row if A or B must come first? Explain your reasoning.

5. The set of binary sequences of length 5 is $U = \{00000, 00001, \ldots., 11111\}$. How many of these sequences start and end with the same digit?

6. A student is asked to count the number of integers in the set $U = \{1, 2, \ldots, 50\}$ that are exactly divisible by 2 or 5. He first notes that the subset $A = \{2, 4,\ldots,50\} = \{2 \times 1, 2 \times 2, \ldots, 2 \times 25\}$ has 25 elements all divisible by 2, and the subset $B = \{5,10, \ldots, 50\} = \{5 \times 1, 5 \times 2,\ldots, 5 \times 10\}$ has 10 elements all divisible by 10. He then concludes that there are $25 + 10 = 35$ integers in U divisible by 2 or 10.
   a. Explain the error in the student's argument.
   b. Provide a correct solution to the question.

7. A password for a small computer system is a sequence of any four letters from the alphabet with repeated letters allowed.
   a. How many such passwords are there?
   b. How many have unique letters?
   c. How many have at least one *a*?

8. The lines on the given diagram represent connections between five locations labeled *A*, *B*, *C*, *D*, and *E*. How many different ways can you go from *A* to *E* if you cannot pass through the same location twice?



9. A four-digit number is formed by selecting digits from the set {1, 2, 3, …, 9} with no repetition allowed. How many of these numbers are both even and greater than 5000?

10. How many integers between 1 and 1000 do not contain a digit 7?

Extending and Investigating

COINCIDENCE AND CHANCE

Life is full of strange coincidences — two people in your class have the same birthday or a newspaper reports that the same lucky person has won a lottery twice. More importantly, we sometimes notice clusters where a number of children living near a power station or some high voltage lines all develop the same unusual disease. Is this a coincidence or does the observed pattern indicate that something about the power station or high voltage lines is causing the disease?

One way to examine such events is to use the mathematical theory of probability. Probability (or chance) is a tool that allows us to see if what appears to be unusual is really so. The so-called birthday problem is a good example. Suppose we assume that every student in a class is equally likely to be born on any one of the 365 days in the year. To simplify the calculations, we omit babies born on February 29 in leap years, since they hardly ever reach the age of 25 in any case. It is also true that more babies are born in some periods of the year than others, but we ignore this point to keep matters simple. Suppose that there are 20 students in your class. We can represent the set of all possible birthdays by a sequence of length 20 with terms selected from the set of digits {1, 2, ..., 365}. For example, the sequence 332, 23, 124, ..., 243 indicates that the first student's birthday is on day 332, the second's is on day 23, and so on. With a class of 20 students, there are 365^{20} possible sequences. Based on our assumptions, all these sequences are equally likely to occur. We are interested in knowing about the subset of sequences in which two (or more) of the birthdays are the same. We consider the complementary subset of sequences in which every term is different. There are $P(365, 20) = 365 \times 364 \times ... \times 346$ such sequences. Hence, the probability that two or more people have their birthday on the same day is

$$1 - \frac{P(365, 20)}{365^{20}} = 0.411.$$

That is, with 20 students in a class, the chance that two or more have the same birthday is more than 40 percent. What appeared unusual is in fact quite likely to happen. You might ask how likely it is that someone in the class has the same birthday as you. This is a very different question. With a little thought, you should be able to show that for a class of 20 students, the answer is just over 0.05, so it is in fact quite unlikely for you to find someone who matches your birthday.

We can perform the same type of calculations for lottery winners. For the 6/49 lottery, the chance of picking the 6 correct balls is $\frac{1}{\binom{49}{6}}$ or approximately 1 in 14 million, so winning is very unlikely. To date, there have been about 1800 runs of the 6/49. The chance that any particular person wins twice, even a person who bought a ticket at every opportunity, is incredibly small. However, given the large number of people who play every time, the chance that someone wins twice is much, much larger. If about 2 million people bought one ticket on all 1800 plays, then the chance of a double winner is about 3 percent. As far as we know, this has not happened yet!

We can use similar arguments to show that a cluster of cases of an unusual disease is not as rare by chance as you might expect. The chance of finding a few children with the same rare condition at a particular site is very small. However, if you look at all the opportunities for clusters across the large number of such sites then the probability of one such cluster occurring just by chance is much larger. This is just like the birthday problem. Among your classmates, the chance of finding a match to your birthday is small. The chance of finding a match of some two birthdays is surprisingly large.

The fact that we can explain the occurrence of a cluster of cases by chance does not rule out the possibility that there is a connection between the power plant and the disease. However, it can make the decision to investigate further much more difficult.

Chapter 11

COUNTING METHODS

How many cars, trucks, and trailers are there in Ontario? If each license plate is of the form *XXXnnn*, where *X* is any one of the 26 letters of the alphabet and *n* is any digit from 0 to 9, do we have enough plates available? Since the new plates are in the *XXXXnnn* format, how many vehicles can now be accommodated for licensing? Problems such as this one, as well as the design of social insurance numbers, postal codes, health card numbers, and computer passwords require careful counting techniques. In Chapter 10, you learned how to develop rules and strategies for solving certain counting problems. In this chapter, you will expand your combinatorics knowledge to derive general formulas that can be applied to a wide variety of problems.

CHAPTER EXPECTATIONS In this chapter, you will

- solve problems using counting principles, **Section 11.1, 11.2, 11.5**
- express the answers to permutation and combination problems, **Section 11.1, 11.3, 11.4, 11.5**
- evaluate expressions involving factorial notation, **Section 11.2, 11.3, 11.5**
- solve problems involving permutations and combinations, **Section 11.4, 11.5**
- explain solutions to counting problems, **Section 11.5**
- solve problems by combining a variety of problem-solving strategies, **Section 11.5**

Rich Learning Link

investigate

CHAPTER 11: VOTING SYSTEMS

In Canada, governments are elected by the people. Eligible voters choose from a slate of candidates in their riding. The candidate with the most votes in a riding wins a seat in the House of Commons. The political party with the most seats forms the government. Provincial governments may be elected similarly. One problem with this system is that governments may be formed by parties that do not have the support of the majority of Canadians. For example, in the 2000 Federal election, the Liberals won 40% of the popular vote but gained 57% of the seats in Parliament. There are many voting systems used throughout the world, each with its own strengths and weaknesses. One of these is a system of proportional representation.

Investigate

Consider the simplified, albeit extreme, election results tabulated below. There are 359 eligible voters in seven riding voting for one of three political parties.

	Ridings							Total Votes
Parties	A	B	C	D	E	F	G	
Blue	29	24	25	18	39	34	8	177
Green	28	22	23	17	38	31	1	160
Red	1	2	2	1	3	2	11	22
Winner	Blue	Blue	Blue	Blue	Blue	Blue	Red	

Using the current system in Canada, Blue would win 6 out of 7 seats with 49.3% of the votes, Green would win no seats even though it had about 44.6% of the votes, and Red would win one seat despite earning only 6.1% of the votes.

Proportional representation would award seats in the ratio 177:160:22. For seven seats, the unrounded allocation is 3.45, 3.12, 0.43. Dealing with these decimals is one of the issues generated by proportional representation. Typical methods would award four seats to Blue and three seats to Green. (Should Red get the seat in riding G?) The next question is, which party gets which particular seats? Regional concerns may raise the importance of this question.

Make a list of all the possible ways of assigning four seats to Blue and three seats to Green. In this chapter, we will see that there are 35 different ways to allocate the seats. Which of these 35 is the best way to allocate the seats? We will also see just how big the problem of seat allocation can become.

DISCUSSION QUESTIONS

1. When the number of seats earned by each party is determined, what criteria should be used for selecting the different ways of allocating them?
2. Do you think mathematical methods can determine a fairer system for allocating seats?

Section 11.1 — Counting Sequences With Distinct Elements

In Chapter 10, we developed rules and strategies for solving certain counting problems. In doing so, we looked at each problem and example from first principles. In this chapter, we derive general formulas that can be applied to a wide variety of problems.

Consider this question: *if there are* n *different symbols available, how many sequences of length* r *can be formed if no symbol can be used more than once?*

In Example 1, we review how to count such sequences.

EXAMPLE 1 How many sequences of length 4 can be formed using the seven letters a, b, c, d, e, f, g if no symbol can be used more than once?

Solution

Let U be the set of all such sequences. Some typical elements in U are

$$U = \{abcd, abce, \ldots, gfed\}$$

We can count the number of sequences by using the product rule.

The first term can be selected in seven ways. No matter which letter we choose, there are six letters left. Since we can use a letter only once, the second term can be selected in six ways. The third can then be selected in five ways and the fourth in four ways. Hence, there are $7 \times 6 \times 5 \times 4 = 840$ elements in U. To get the answer, we find the product of four factors, one for each term in the sequence.

We use exactly the same approach for the general problem. Suppose we have n symbols and we want to count the sequences of length r if we can use each symbol at most once. Note that $r \leq n$. If $r > n$, then the answer is 0 because in building the sequence we run out of symbols before we finish the construction.

In general, the first term can be chosen in n ways. For each of these ways, the second term can be selected in $n - 1$ ways, the third in $n - 2$ ways, and so on. The hard part here is to see how many ways there are for selecting the final term, which is the r^{th} term. There are as many factors in the product as there are terms in the sequence. For example, if $r = 2$, then the last (second) term can be selected in $n - 1 = n - 2 + 1$ ways. If $r = 4$, then the last (fourth) term can be selected in $n - 3 = n - 4 + 1$ ways. Following the pattern, the r^{th} term can be selected in $n - r + 1$ ways. Look at it another way. We started with n possibilities. By the r^{th} term, we have used $r - 1$ of these, so we have left $n - (r - 1) = n - r + 1$ possibilities. Hence, the total number of sequences is $n \times (n - 1) \times \ldots \times (n - r + 1)$.

We denote this quantity by $P(n, r)$, which is the product of exactly r factors.

The number of sequences of length r that can be formed using n different symbols, where each symbol can be used at most once, is

$$P(n, r) = n \times (n-1) \times \ldots \times (n-r+1)$$

A sequence formed from symbols is sometimes called a **permutation** of the symbols. The P in $P(n, r)$ refers to permutation.

The formula for $P(n, r)$ has some special cases. First, look what happens if $r = n$. Here we are counting sequences of length n that can be built from n different symbols using each symbol once and only once. In this case, the first factor in $P(n, r)$ is n and the last is 1. The formula for $P(n, r)$ is the product of all of the integers from 1 to n. We call this product n **factorial** and use the notation $n! = n \times (n-1) \times \ldots \times 3 \times 2 \times 1$.

Many calculators have a function to calculate n! For example, your calculator gives $10! = 3\,628\,800$ and $60! \approx 8.321 \times 10^{81}$. Some calculators will go higher; e.g., $100! \approx 9.333 \times 10^{157}$. As n gets larger, n! increases rapidly and soon is larger than the display of a calculator, so this function is useful only for small values of n if exact values are desired. Your calculator may also have a function for $P(n, r)$; it may use the symbol ${}_nP_r$.

Calculating *n*! on a TI-83

1. Enter n
2. Go to MATH menu
3. Scroll across to PRB sub-menu
4. Enter 4 (gives the factorial function)
5. Hit ENTER (produces the value of n!)

technology

Calculating *P*(*n*, *r*) on a TI-83

1. Enter n
2. Go to MATH menu
3. Scroll across to PRB sub-menu
4. Enter 2 (gives the $P(n, r)$ operator)
5. Enter r
6. Hit ENTER (produces the value of $P(n, r)$)

Another special case occurs for $r > n$; in other words, the length of the sequence is greater than the number of symbols. In this instance, one of the factors in the product is 0, so the formula for $P(n, r)$ gives the correct answer of 0.

It is sometimes convenient to express $P(n, r)$ in terms of the factorial notation. To see how this works, consider, for example $P(9, 4) = 9 \times 8 \times 7 \times 6$. The right-hand side is the product of the first four factors in 9! We can multiply top and bottom by the remaining five factors without changing the value of $P(9, 4)$.

$$P(9, 4) = 9 \times 8 \times 7 \times 6 = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{9!}{5!}$$

In general terms,

$$P(n, r) = n \times (n - 1) \times (n - 2) \times \ldots \times (n - r + 1)$$
$$= n \times (n - 1) \times (n - 2) \times \ldots \times (n - r + 1) \times \frac{(n - r) \times (n - r - 1) \times \ldots \times 1}{(n - r) \times (n - r - 1) \times \ldots \times 1}$$
$$= \frac{n!}{(n - r)!}$$

Be careful not to abuse this result. For example, the formula tells us that $P(100, 2) = \frac{100!}{98!}$, which is correct but not very useful. In this instance, it is much better to go back to the original form $P(100, 2) = 100 \times 99 = 9900$. The original definition can also be useful in algebraic situations. $P(n, 3) = n(n - 1)(n - 2)$ is much easier to work with than $P(n, 3) = \frac{n!}{(n - 3)!}$.

One final point about this notation. We know that the number of sequences of length n that can be formed using n symbols is $P(n, n) = n!$. Using the formula given in the box above, $P(n, n) = \frac{n!}{0!}$. We can define $0! = 1$ so that the formula in the box gives the correct answer. This definition is one of convenience; many of the formulas we develop will work in special cases if we define $0! = 1$.

We now use the formula for $P(n, r)$ to solve some problems. Other strategies and tools, such as the sum rule, are also still important.

EXAMPLE 2

A five-digit integer is formed from the digits 1, 2, 3, ... , 9, with no digit used more than once.

a. How many integers can be formed?
b. What fraction of these integers begin with 3?
c. How many of these integers contain the digit 6?

Solution

a. The number of integers is the number of sequences of length 5 formed from 9 symbols using each at most once. There are $P(9, 5) = \frac{9!}{4!} = 15\,120$ integers.

b. If the integer begins with 3, the first position is fixed and the remaining four positions form a sequence of length 4 chosen from the remaining eight digits. There are $P(8, 4) = 1680$ such integers. Hence, $\frac{1680}{15\,120} = \frac{1}{9}$ of the integers begin with 3.

technology

Using a calculator, this calculation can be done directly. The fraction of integers beginning with 3 is $\frac{P(8, 4)}{P(9, 5)} \approx .1111$, or $\frac{1}{9}$.

c. If the integer contains a 6, then the 6 can be in any of five positions, with the remaining four digits forming a sequence of length 4. The number of integers is $5 \times P(8, 4) = 8400$. Alternatively, the number of five-digit integers that do not contain 6 is $P(8, 5)$, so the number that contains 6 is $P(9, 5) - P(8, 5) = 8400$. Part **c** of Example 2 demonstrates an important principle. We determine the number of integers in two different ways. Equating the results gives a relationship among the formulas we have developed. In this case, we can use the same method to establish the general relationship $P(n, r) - P(n - 1, r) = r \times P(n - 1, r - 1)$.

Proof 1

Suppose n different symbols $a, b, \ldots$ are available. There are two ways to count the number of sequences of length r that contain a particular symbol, say a. First, the a can be placed in any one of r places and then the rest of the sequence, an arrangement of $n - 1$ symbols in $r - 1$ positions, can be constructed in $P(n - 1, r - 1)$ ways. Hence, using the product rule, there are $r \times P(n - 1, r - 1)$ sequences that contain a.

Alternately, we consider the complement—those sequences that do not contain a. Since we can now use only $n - 1$ symbols to build the sequences, there are $P(n - 1, r)$ sequences that do not contain a. As there are $P(n, r)$ sequences in total, the number of sequences containing a is $P(n, r) - P(n - 1, r)$. Equating the two results gives the required result $P(n, r) - P(n - 1, r) = r \times P(n - 1, r - 1)$.

Proof 2

Now we have an algebraic proof. Starting from the left side, we have

$$P(n, r) - P(n - 1, r) = \frac{n!}{(n - r)!} - \frac{(n - 1)!}{(n - 1 - r)!}$$

multiply the second fraction by $\frac{n - r}{n - r}$ to make the denominators the same

$$= \frac{n(n - 1)!}{(n - r)!} - \frac{(n - r)(n - 1)!}{(n - r)(n - r - 1)!}$$

$$= [(n - (n - r)]\frac{(n - 1)!}{(n - r)!}$$

$$= r\frac{(n - 1)!}{[(n - 1) - (r - 1)]!}$$

since $n - r = n - 1 - r + 1 = (n - 1) - (r - 1)$

$$= rP(n - 1, r - 1)$$

Note that we can always use $m! = m \times (m - 1)!$ when it is convenient.

EXAMPLE 3

A ten-letter word is formed from the 26 letters of the alphabet. No letter may be used more than once. How many such words are there if

a. there are no restrictions?
b. the first letter is a vowel, (*a, e, i, o, or u*)?
c. either the first or the last letter is a vowel?

Solution

a. If there are no restrictions, the number of words is the number of sequences of length 10 formed from 26 symbols using each at most once.
Hence, there are $P(26, 10) = \frac{26!}{16!} \approx 1.9275 \times 10^{13}$ words. Using a spreadsheet program, we can evaluate $P(26, 10)$ as exactly 19 275 223 968 000.

b. If the first letter is a vowel, then the first term can be chosen in five ways. The remaining nine terms are a sequence of length 9 formed from 25 symbols with no repeats. The remaining nine terms can be formed in $P(25, 9)$ ways. Hence there are $5 \times P(25, 9)$ words starting with a vowel.

c. This question is more difficult. Let E be the set of words with the first letter a vowel and F the set with the last letter a vowel. We want to find $n(E \cup F)$, the number of words that start or end with a vowel. E and F are not disjoint. Typical examples of words in $E \cap F$ are {*abcdefghio, abcdefghij*, … }.
From part **b,** we know that $n(E) = 5 \times P(25, 9)$. By symmetry, $n(F) = n(E)$. To calculate $n(E \cap F)$, there are five choices (one of the vowels) for the first letter and then four choices for the tenth letter. The middle eight letters can then be arranged in $P(24, 8)$ ways, so we have $n(E \cap F) = 5 \times 4 \times P(24, 8)$. Using the sum rule, we have $n(E \cup F) = 5 \times P(25, 9) + 5 \times P(25, 9) - 5 \times 4 \times P(24, 8)$. This is the number of words that begin or end with a vowel. Using a calculator, $n(E \cup F) = 6.820 \times 10^{12}$.

Exercise 11.1

Part A

Knowledge/Understanding

1. Evaluate the following expressions. Explain in each case whether or not using a calculator would be effective.

   a. $7!$ b. $8! - 7!$ c. $\frac{8!}{7!}$ d. $P(8, 2)$

   e. $P(8, 6)$ f. $\frac{11!}{9!}$ g. $\frac{31!}{28!}$ h. $\frac{79!}{76!}$

2. Simplify $P(n, 2)/P(n + 1, 2)$, where $n \geq 1$ is an integer.

Part B

3. Each of the following questions involves forming sequences using the letters of the name *Euclid* at most once. Express the answer in terms of $P(n, r)$ for the appropriate choices of n and r; do not calculate.

   a. the number of sequences of length 6

   b. the number of sequences of length 4

c. the number of sequences of length 5 that end with d

d. the number of sequences of length 6 that start with a vowel

e. the number of sequences of length 4 that start and end with a vowel

Communication

4. Which is larger, $P(10, 5)$ or $P(10, 6)$? Explain.

Knowledge/Understanding

5. A six-digit integer is formed by selecting digits from the set $\{1, 2, \ldots, 9\}$ without replacement. What fraction of the possible numbers

a. start with 6?

b. are even?

c. start with an odd digit?

d. start or end with an odd digit?

e. contain the digit 9?

f. contain both digits 8 and 9?

g. contain the digit 8 or 9 or both?

h. are less than 460 000? (*Hint:* Consider cases.)

6. Twenty-four students arrange themselves in three rows of eight for a class picture. How many different arrangements are possible if the eight tallest are in the back row?

Application

7. A sorting algorithm puts lists of numbers into increasing order. To check the algorithm, a programmer prepares all possible lists of the integers 1, 2, …, 100. Of these lists, how many begin with an integer less than 10 and end with an integer greater than 90? Leave your answer in terms of $P(n, r)$.

8. If the digits 0, 1, 2, …, 9 are formed into a sequence of length 6 with each digit used at most once, how many of the sequences start with 0 or end with 9?

9. Twelve different books—six mathematics text books and six novels—are arranged on a shelf. How many arrangements are there in each of the following? Leave your answers in terms of $n!$ for appropriate n.

a. the six novels are to the left of the six mathematics books

b. the novels and the mathematics books alternate

c. a novel is on the left end of the shelf

d. a novel is on the left end of the shelf and a math book is on the right end

e. a novel is on the left end of the shelf or a math book is on the right end

10. Sequences of length 4 are formed from n symbols, each used at most once. Using the $P(n, r)$ notation, count the number of sequences in the following sets.

U: all possible sequences

A: all sequences that start with a particular symbol α

B: all sequences that have two particular symbols α and β side by side in the order $\alpha\beta$

C: all sequences that have two particular symbols α and β side by side in either order

11. A sequence of length r is formed from n symbols using each symbol at most once. Using the $P(n, r)$ notation, repeat the calculations in question 10.

Thinking/Inquiry/ Problem Solving

12. Show that $\frac{P(n, r+1)}{P(n, r)}$ is an integer for r an integer, $1 \leq r \leq n$.

13. A set of 40 cards consists of cards numbered 1, 2, 3,...,10 in red, yellow, green, and blue. Suppose the set is shuffled and the first five cards are set down in a sequence. Find the number of elements in the following sets.

U: all possible sequences

A: all sequences with cards all the same colour

C: all sequences with two or more cards having the same number

B: all sequences that start with 2 and end with 8

14. The numbers 1, 2, …, 144 are arranged in a square 12×12 array. What fraction of all possible arrangements have the perfect squares in increasing order on the main diagonal?

15. Ten children's blocks have the letters *A*, *B*, ..., *J* on one face. How many of the different arrangements of the blocks do not have the blocks with *A* and *B* adjacent?

Thinking/Inquiry/ Problem Solving

16. What is the largest power of 10 that divides 20!?

Part C

17. By counting in two different ways the number of sequences of length r that can be formed from n symbols, prove that $P(n, r) = n \times P(n-1, r-1)$. *Hint:* Consider building the sequence by selecting the first element and then all the others.

18. Using algebraic manipulation, prove that $P(n, r) = n \times P(n-1, r-1)$.

19. Find two proofs that $P(n, r) = P(n, k) \times P(n-k, r-k)$ for any integer $1 \leq k \leq r-1$.

20. Find the largest value of k so that 10^k divides evenly into 100!.

Section 11.2 — Sequences With Unlimited Repeated Values

In this section, we look at the following question: *If there are* n *symbols available, how many sequences of length* r *can be formed, if each symbol can be used as often as we like?* If we change the question slightly so that each symbol can be used only once then, from the previous section, we know that the answer is $P(n, r) = n \times (n - 1) \times \ldots \times (n - r + 1)$. To see what difference allowing repeats makes, reconsider the example from Section 11.1.

EXAMPLE 1 Let T be the set of sequences of length 4 formed from the seven letters $\{a,b,c,d,e,f,g\}$, where we can use a letter as often as we like. Find $n(T)$.

Solution

We can count the elements of T using the product rule. The first term can be chosen in seven ways, the second term can be selected in seven ways, the third in seven ways and the fourth in seven ways.
Then $n(T) = 7 \times 7 \times 7 \times 7 = 7^4 = 2401$.

There are four factors in the answer, one for each term in the sequence. For the general question, we proceed in the same way as in the example. To build a sequence of length r, the first term can be chosen in n ways. For each of these ways, the second term can be selected in n ways, the third in n ways, and so on until the r^{th} term, which can also be selected in n ways. Hence, there are $n \times n \times \ldots \times n = n^r$ sequences of length r that can be formed using n symbols as often as we like.

If there are n symbols available, the number of sequences of length r that can be formed if each symbol can be used as often as we like is $n \times n \times \ldots \times n = n^r$.

EXAMPLE 2 The set U of binary sequences of length r has elements such as 01001…0, a string of r 0s and 1s. In computing language, a binary sequence is called a **bit string,** so U is the set of bit strings of length r. Let A be the subset of these strings that has at least one 0 bit. Let B be the subset of strings that start with 0 and end with 1. Find $n(U)$, $n(A)$, and $n(B)$.

Solution

Here we have two symbols with unlimited repeats in a string of length r, so $n = 2$ and $n(U) = 2^r$. To find $n(A)$, consider the complement, the subset of strings with no 0s. This set has exactly one element 111…1, so

$$n(A) = n(U) - n(A)$$
$$= 2^r - 1$$

A typical element of B is 0100…01. The first and last terms of these elements are fixed by the definition of B. The $r - 2$ middle terms of the elements of B are all the bit strings of length $r - 2$. Hence, $n(B) = 2^{r-2}$.

EXAMPLE 3 A standard die has its faces numbered 1, 2, 3, 4, 5, 6. Suppose that six standard dice, coloured red, yellow, blue, white, green, and orange, respectively, are rolled simultaneously. In what fraction of the possible outcomes will six different values occur?

Solution

Each die comes up with an outcome 1 to 6. To relate the question to sequences, we can let the first term of the sequence be the outcome of the red die, the second term the outcome of the yellow die, and so on. Thus the possible outcomes of rolling six dice can be described by a sequence of length 6 made using the digits 1 to 6. For example, the sequence 222333 corresponds to the red, yellow, and blue dice coming up 2 and the remaining three dice coming up 3. The total number of possible outcomes is the number of such sequences, which is 6^6. One example in which all six different values occur is the sequence 654321. The outcomes in which all six values occur correspond to sequences of length 6 in which each digit from 1 to 6 is used once. From the previous section we know there are 6! such sequences. The fraction of sequences with distinct outcomes is $\frac{6!}{6^6} = \frac{5}{324}$.

EXAMPLE 4 We can indicate the day of the year by a number between 1 and 365 (ignoring leap year). Suppose there are n people in a room. Of all the possible arrangements of birthdays, what fraction has two or more people with their birthday on the same day? For what values of n does this fraction exceed 0.5?

Solution

We can proceed by finding what fraction of the possible cases have all the people in the room with birthdays on different days. This corresponds to a sequence of length n, using 365 possible symbols in which the terms are all different. There are $P(365, n)$ such sequences. The number of possible cases is found by counting the same sequences with repetition allowed. There are $P(365, n)$ such sequences. The fraction of sequences corresponding to all people having their birthdays on different days is

$$t(n) = \frac{P(365, n)}{365^n} = \frac{365 \times 364 \times \ldots \times (365 - n + 1)}{365^n}$$

The fraction we require is $1 - t(n)$. We could evaluate $t(n)$ recursively by noting that $t_1 = \frac{365}{365} = 1$; $t_2 = \frac{365 \times 364}{365 \times 365} = t_1 \times \frac{364}{365}$; $t_3 = \frac{365 \times 364 \times 363}{365 \times 365 \times 365} = t_2 \times \frac{363}{365}$ and, in general, $t_n = t_{n-1} \times \frac{365 - n + 1}{365}$. A spreadsheet program will give $t_2 \doteq 0.997$, $t_3 \doteq 0.992$, $t_4 \doteq 0.984$, and so on. The first value of $t(n) < 0.5$ is $t(23)$.

technology

The fractions we just calculated can be interpreted as probabilities. For example, if you are in a group of 23 people (perhaps one of your classes) and everyone in the group is equally likely to have a birthday on any day of the year (no twins), the probability that there are two people in the group who have their birthday on the same day is about $\frac{1}{2}$. If there are more people, the probability is greater, since $t(n)$ is decreasing. In a group of 60, the probability is greater than 0.99.

Exercise 11.2

Part B

Application

1. In a plan for North American telephone numbers, each number is a sequence of 10 numbers of the form *xyy xyy yyyy*, where $2 \leq x \leq 9$ and $0 \leq y \leq 9$. How many different telephone numbers can be formed?

2. A sequence of length 7 is formed from the digits 0, 1, …, 9. Each digit can be used as often as you like. What fraction of these sequences
   a. begins with 1?
   b. begins and ends with 1?
   c. uses only even digits?
   d. begins and ends with an even digit?
   e. does not contain a 0?

Knowledge/Understanding

3. A sequence of length 12 is formed using n different symbols including A. Each symbol can be used repeatedly. How many of these sequences
   a. begin with A?
   b. begin with AA?
   c. include at least one A?

4. A sequence of length 10 is formed using the letters a, b, c with unlimited repetition. How many of these sequences use only two symbols?

Knowledge/Understanding

5. Let U be the set of bit strings (binary sequences) of length $r \geq 2$. How many of these strings

a. begin with 1?

b. begin and end with 1?

c. begin or end with 1?

6. How many binary sequences of length 12 start with 1 or end with 0?

7. Eight plain and eight blue tiles are available to cover the rectangular table top that is shown. How many different patterns can be made if the spaces labelled x must have the same type of tile, the spaces labelled y must have the same type of tile, and the tiles on x and y must be different?

x		x	
	y		y

Application

8. In a series of licence plates, the first three symbols are any of the 26 letters in the alphabet and the last three are any of the 10 digits from 0 to 9.

a. How many license plates can be formed?

b. How many plates can be formed with all symbols different?

c. How many plates can be formed in which at least one symbol is repeated?

d. How many plates can be formed in which at least one of the digits and at least one of the letters are repeated?

9. In a programming language, **variable names** are sequences of length 1 or 2 that use lower-case letters, upper-case letters, or digits. The name must start with a letter. The second symbol, if used, can be any letter or digit. How many different variable names can be constructed?

10. My password is a seven-symbol sequence formed from upper- and lower-case letters and the digits from 0 to 9. Yesterday, I forgot the last three symbols in my password. It takes me 20 seconds to try to log onto my computer. Approximately how many hours will it take for me to check all the possible passwords?

Thinking/Inquiry/Problem Solving

11. A voice-mail system has 1253 users. A password to open a mailbox is a sequence of length r formed from the digits 0 to 9. To ensure confidentiality, there should be at least 1000 possible passwords for every user. What is the smallest possible value of r?

12. In a simple lottery there are 100 tickets numbered 1 to 100. You hold ticket number 1. Three tickets are drawn one after the other. After each draw, the ticket is replaced. In how many ways can the tickets be selected so that you win at least one prize?

13. A coin is flipped eight times to produce a sequence of heads and tails. What fraction of these sequences have the same result on the first and last flip?

14. Four switches each have three positions, *up, middle,* and *down*. How many different ways can the switches be arranged?

Thinking/Inquiry/Problem Solving

15. A sequence of bits such as 101001 represents a six-digit binary number, one that is expressed in terms of powers of 2. For example,

$$101001 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0.$$

We can convert the binary number to a decimal by expanding the powers of 2 in decimal form so that

$$1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 32 + 8 + 1 = 41.$$

Conversely, we can represent any decimal number uniquely as a sequence of bits by expanding in powers of 2. For example,

$$\begin{aligned} 27 &= 1 \times 16 + 1 \times 8 + 0 \times 4 + 1 \times 2 + 1 \times 1 \\ &= 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \end{aligned}$$

a. What is the largest decimal number that can be represented by a sequence of length 6?

b. What decimal numbers can be represented by bit sequences of length 6 that end with 1?

c. What decimal numbers can be represented by bit sequences of length 6 that start and end with 1?

16. Suppose we want to represent all decimal numbers less than or equal to 1000 as bit sequences of length r, as in Question **15.** How large must r be?

17. A subset is formed from the integers 1, 2, 3, …, 9.

a. How many possible subsets can be formed?

b. How many of these subsets contain 1?

c. How many of these subsets contain 1 or 2?

Application

18. Six balls are drawn consecutively from a barrel that has 49 numbered balls labelled 1, 2, …, 49. After each draw, the ball is replaced. What percentage of the possible sequences have six different ball numbers?

19. A sequence of length r is formed using n symbols with unlimited repetition. What fraction of these sequences have all terms different?

Part C

20. Suppose n symbols are available to construct a sequence of length $r \geq 2$. One of the symbols is A. Show that the fraction of all sequences that contain exactly one A is greater if each symbol can be used at most once than if there is unlimited repetition.

21. A sequence is a **palindrome** if it looks the same read from either end. For example, the word *level* is a palindrome. Suppose that a sequence of length r is built from n distinct symbols with unlimited repetition. How many of these sequences are palindromes?

22. Suppose $n \geq 2$ symbols are available to construct a sequence of length less than or equal to r where each symbol can be used an unlimited number of times. Show that the total number of sequences with length less than r is smaller than the number of sequences of length r.

23. A function from a set S to a set T assigns exactly one element of T to each element of S. For example, if $S = \{a, b, c, d\}$ and $T = \{0, 1\}$, then one function f from S to T is $f(a) = 0, f(b) = 1, f(c) = 1, f(d) = 0$. In general, how many different functions can we construct from a set S to a set T?

24. In attempting to count the number of sets of any size that can be formed from the six letters A, B, C, D, E, F, we can define a sequence of length 6 using the symbols I (in) and O (out) to indicate whether a specific letter is included. Then *IOIOOI* corresponds to the set $\{A, C, F\}$ and *OOOOOO* corresponds to the empty set $\varnothing$.

    a. Is there a one-to-one correspondence between the possible sets and the sequences of length 6?

    b. Determine the number of sets that can be formed from the six letters.

Section 11.3 — Counting Subsets

Many problems involve counting subsets. For example, the results of the 6/49 lottery depend only on the balls drawn, not their order. That is, the results depend only on which subset of six balls is chosen. We can use the methods that we have developed for counting sequences to count the number of subsets. In this section, we look at systematic ways to count subsets. Remember that the fundamental difference between a subset and a sequence is that the order of the terms in a sequence is important. For example, suppose a set of six letters $\{a, b, c, d, e, f\}$ is available. Two different sequences of length 3 are *abc* and *bca*. Note they are made up of the same three letters. However, the subset $\{b, c, a\}$ is the same as the subset $\{a, b, c\}$. Two distinct subsets are $\{a, b, c\}$ and $\{b, c, d\}$. Two distinct subsets must have at least one element that is different.

EXAMPLE 1

How many subsets of size 3 can be selected from the six letters $\{a, b, c, d, e, f\}$?

Solution

Let x represent the unknown number of subsets. Our strategy is to use two different methods to count the number of sequences of length 3 that can be formed using the six letters at most once. First, we count the number of sequences directly, so that the answer is $P(6, 3)$.

Alternatively, we can select a particular subset. Since there are x subsets, we can do this in x ways. Suppose we choose $\{a, b, c\}$. The three elements of the subset can be used to generate 3! sequences of length 3. In this case, the sequences are *abc*, *acb*, *bac*, *bca*, *cab*, *cba*. If we choose another subset, say $\{a, b, d\}$, then these three letters can be used to generate another set of 3! sequences that are all different from those listed above because they include a term d. In other words, each of the x subsets can be used to generate 3! different sequences. Using the product rule, there are a total of $x \times 3!$ sequences that can be generated in this way. But the terms of every sequence of length 3 make up exactly one subset of size 3. Hence, we have counted all of the sequences of length 3. It follows that $x \times 3! = P(6, 3)$ or, solving,

$$\begin{aligned} x &= \frac{P(6, 3)}{3!} \\ &= \frac{6 \times 5 \times 4}{3!} \\ &= 20 \end{aligned}$$

There are 20 subsets of size 3 that can be selected from $\{a, b, c, d, e, f\}$.

The key to this result is that we have counted the number of possible sequences in two ways and noted that the two answers must be equal. The same strategy works in general. Suppose we have a set of n elements and we want the number of subsets of size r. Again, let x represent this unknown number. As in Example 1, we use two different methods to count the number of sequences of length r that can be formed from the n elements, using each at most once.

Using the direct method, there are $P(n, r)$ such sequences. If we have x possible subsets of size r, then each of these will generate $r!$ different sequences and so there are $x \times r!$ sequences in total. Equating the two counts gives $x \times r! = P(n, r)$ and solving for the unknown number of subsets, we have

$$\begin{aligned} x &= \frac{P(n, r)}{r!} \\ &= \frac{n(n-1)\ldots(n-r+1)}{r!} \end{aligned}$$

A common but rather curious notation for the number of subsets of size r selected from n elements is $\binom{n}{r}$. We read this expression as *n choose r*. We can also use the alternative form of $P(n, r)$ in the numerator. Recall that $P(n, r) = \frac{n!}{(n-r)!}$, so we can write n choose r as $\binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!}$.

The number of possible subsets of size r that can be selected from a set of n different elements is

$$\binom{n}{r} = \frac{n \times (n-1) \times \ldots (n-r+1)}{r!}$$

or

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

The expression $\binom{n}{r}$ can also be written $C(n, r)$ or ${}_nC_r$. Here the letter C stands for *combinations,* just as P used earlier stands for *permutations*. Your calculator probably has a function for $\binom{n}{r}$. Note also from the final expression in the box that

$$\binom{n}{r} = \binom{n}{n-r}$$

There are thus several choices for evaluating $\binom{n}{r}$, with the best choice depending on the relative sizes of n and r. We can use the calculator function directly, or evaluate the first expression in the box by calculator or by hand.

Alternatively, we can first replace $\binom{n}{r}$ by $\binom{n}{n-r}$ and then proceed.

Calculating $\binom{n}{r}$ on a TI-83

1. Enter n
2. Go to MATH menu
3. Scroll across to PRB sub-menu
4. Enter 3 (gives the $C(n, r)$ operator)
5. Enter r
6. Hit ENTER (produces the value of $\binom{n}{r}$)

technology

EXAMPLE 2

Evaluate a. $\binom{10}{5}$ b. $\binom{100}{5}$ c. $\binom{100}{96}$

Solution

technology

a. $\binom{10}{5} = 252$ (using a calculator)

b. $\binom{100}{5} = \frac{100 \times 99 \times 98 \times 97 \times 96}{5 \times 4 \times 3 \times 2 \times 1} = 75\ 287\ 520,$

(this time using a calculator on the simpler form)

Note that the number of factors in the numerator and denominator is the same in the fundamental definition. This is a useful check to avoid silly mistakes.

c. Using the last method,

$$\binom{100}{96} = \binom{100}{4} = \frac{100 \times 99 \times 98 \times 97}{4 \times 3 \times 2 \times 1} = 3\ 921\ 225$$

Note also that if the numerical answer is not required, it is acceptable to leave the answer in the form $\binom{n}{r}$. To complete the notation, note that there is one subset of size n that can be selected from n elements. That is, $1 = \binom{n}{n} = \frac{n!}{0!n!}$. Since we have defined $0! = 1$, the formula makes sense. It will also be useful to define $\binom{n}{0} = 1$.

EXAMPLE 3

Eight people—six students and two teachers—are available to serve on a committee. How many different committees of size 4 can be formed if

a. there are no restrictions?
b. Bob, one of the students, must be on the committee?
c. the committee has exactly one teacher?
d. at least one teacher must be on the committee?

Solution

a. A committee is a subset of size 4 selected from the eight available people. There are

$$\binom{8}{4} = \frac{8!}{4!4!} = 70 \text{ such committees.}$$

b. A typical committee that includes Bob is {Bob, a, b, c}, where a, b, c represent three people other than Bob. To form such a committee, we select the three other people from the seven available in $\binom{7}{3} = \frac{P(7, 3)}{3!} = 35$ ways. There are 35 committees that include Bob.

c. Here we build the possible committees in two steps. First, we can select the one teacher from the two available in $\binom{2}{1}$ ways. For each of these selections, we can then choose the three students in $\binom{6}{3}$ ways. Using the product rule, the number of committees with exactly one teacher is $\binom{2}{1} \times \binom{6}{3} = 40$. You should verify this calculation.

d. This problem is more difficult. One approach is to consider two disjoint cases.
Case 1: The committee contains exactly one teacher. From part **c,** there are 40 such committees.
Case 2: The committee contains exactly two teachers. Constructing the possible committees in two steps as above, the number of committees in this case is $\binom{2}{2} \times \binom{6}{2} = 15$. There are then $40 + 15 = 55$ committees that contain at least one teacher. An alternative approach is to use the complement, since we know that there are 70 committees in total. The committees that do not contain at least one teacher are made up entirely of students. There are $\binom{6}{4} = 15$ such committees, so the number of committees with at least one teacher is $70 - 15 = 55$.

Be careful! It is very easy to make a mistake. Suppose we had used the following argument.

Since we know there is at least one teacher on the committee, start by choosing the teacher in $\binom{2}{1}$ ways. Now pick the other three committee members from the remaining seven people in $\binom{7}{3}$ ways. Thus there are $\binom{2}{1} \times \binom{7}{3} = 70$ such committees. This answer is wrong because we know that there are only 70 committees altogether and some of them have no teachers at all. What went wrong?

The error occurred because, when we separated the selection into two steps, we had to select a teacher in the first step, and we could select the second teacher in the second step. Every committee containing both teachers has been counted twice: once with teacher A selected separately and teacher B as one of the three other members, and once with the selection reversed. There are 15 committees containing both teachers, so our answer was too large by 15.

Whenever you see a problem like this—one containing the key words *at least* or *at most*— divide the problem into cases so that each case corresponds to *exactly* …. Remember that it may be easier to count the complement.

Exercise 11.3

Part A

Knowledge/Understanding

1. Evaluate the following.

a. $\binom{6}{3}$ b. $\binom{60}{3}$ c. $\binom{600}{3}$

2. Evaluate $\dfrac{\binom{10}{3}}{\binom{11}{4}}$

3. A subset of five numbers is chosen from the set $\{1, 2, \ldots, 10\}$.

a. How many subsets can be selected?

b. How many of these subsets contain only numbers less than or equal to 7?

c. How many of the subsets contain two even and three odd numbers?

d. How many of the subsets contain at least two even numbers?

e. How many of the subsets contain the number 10?

f. How many of the subsets contain 9 or 10?

4. A subset of size 3 is formed by selecting three letters from the set $\{A, B, C, D, E, F, G\}$. What fraction of the possible subsets

a. contain the letter A?

b. contain exactly one vowel?

c. do not contain either E or F?

Part B

Thinking/Inquiry/ Problem Solving

5. Suppose a sequence of length 4 is formed by choosing four digits from the set $\{0, 1, \ldots, 9\}$.

a. How many such sequences can be formed if no term in the sequence can be repeated?

b. For any subset of size 4 selected from the above set of digits, how many different sequences of length 4 can be formed?

c. By counting the sequences in two ways, explain why $\binom{10}{4} = \frac{P(10, 4)}{4!}$.

6. How many ways can two girls and two boys be selected from a class of 12 girls and 10 boys?

Knowledge/Understanding

7. In how many ways can a subset of six letters be selected from $\{A, B, C, \ldots, Z\}$ so that both A and Z are included?

Communication

8. Explain without any calculation why $P(n, r) \geq \binom{n}{r}$. When are the two quantities equal?

9. A set of 12 distinct wooden blocks has three that are red, five that are blue, and four that are yellow. The blocks are labeled $R1$, $R2$, $R3$, $B1$, …., $B5$, $Y1$, …,$Y4$, where the letter matches the colour. Consider the following subsets of U, the set of all possible subsets of two blocks.

A: all subsets of two red blocks

B: all subsets with one red and one yellow block

C: all subsets with two blocks the same colour

a. Find $n(U)$, $n(A)$, $n(B)$, $n(C)$.

b. How many subsets have two blocks of different colours?

10. How many committees of size 5 can be selected from 11 people—five men and six women—if

a. there are no restrictions?

b. the committee has three women and two men?

c. the committee must contain at least one man and one woman?

d. Ron and Enzo refuse to serve on the same committee?

11. In a lottery, six balls are selected from 49 balls numbered 1, 2, …, 49. For the following questions, express your answers in terms of $\binom{n}{r}$ for various choices of n and r.

a. How many possible subsets of six balls are there?

b. How many of these subsets contain ball 49?

c. How many of these subsets contain only even-numbered balls?

d. How many of these subsets contain three even-numbered and three odd-numbered balls?

Application 12. To learn how students feel about a proposed dress code, the principal decides to survey a sample of 60 students from the school population of 1200 students.

a. How many different samples can be selected?

b. If there are 300 students in each grade from 9 to 12, how many samples can be selected that have 15 students in each grade?

c. How many samples can be selected that have 60 grade 12 students?

Application 13. A box of 100 electronic components contains three that are defective. If a sample of five components is tested, what fraction of the possible samples contain at least one defective item?

14. How many sequences of length 4 can be constructed using the digits $\{1, 2, \ldots, 9\}$ if two of the terms are even and two are odd?

Communication 15. A subset of three blocks is selected from a population of six blocks, of which three are red. A student is trying to count the number of subsets with at least two red blocks. She reasons that she can select the two red blocks in $\binom{3}{2}$ ways and then the remaining block in $\binom{4}{1}$ ways. Using the product rule, the student calculates that there are $\binom{3}{2} \times \binom{4}{1} = 12$ subsets with at least two red blocks. Label the blocks $R1$, $R2$, $R3$, A, B, C, where the first three blocks are red.

a. List all subsets that have at least two red blocks and count them directly.

b. Explain why the student got the wrong answer.

Part C

16. A carton of 100 light bulbs (labelled 1, 2, …, 100) has three that are defective (labels 1, 2, 3). A sample of five bulbs is selected.

a. How many different samples are possible?

b. How many of the samples have no defects?

c. How many of the samples have exactly one defect?

d. How many of the samples have exactly two defects?

e. How many of the samples have exactly three defects?

f. Explain how the results of parts **a** to **e** show that

$$\binom{100}{5} = \binom{97}{5}\binom{3}{0} + \binom{97}{4}\binom{3}{1} + \binom{97}{3}\binom{3}{2} + \binom{97}{2}\binom{3}{3}$$

17. A diagonal of a regular n-gon is a line joining two vertices and lying inside the figure.

    a. How many diagonals are there?

    b. How many of these diagonals pass through the centre?

18. Eleven stars are arranged in a row as shown.

    * * * * * * * * * * *

    The object of the game is to divide the row into three groups—a left group, a middle group, and a right group. Each group must have at least one star. We can create a group by placing two vertical bars in the spaces between the stars. For example, in the diagram below,

    * * | * * * * * | * * * *

    the left group has two stars, the middle has five stars, and the right group has four stars.

    a. How many ways can the three groups of stars be formed?

    b. Consider the equation $x + y + z = 11$ where x, y, z are positive integers. How many different solutions are possible?

19. Be sure to do Question **18** before you attempt this one.
    Suppose that x, y, z, w are positive integers.

    a. How many solutions are there to the equation $x + y + z + w = 21$?

    b. How many of these solutions have $x \leq 3$?

    c. How many of these solutions have $x = y$?

20. Suppose you want to count the solutions to $x + y + z = 11$, where we allow x, y, z to be non-negative integers.

    a. Show that any solution to $x + y + z = 11$ with $x \geq 0$, $y \geq 0$, $z \geq 0$ is the same as a corresponding solution to $x_1 + y_1 + z_1 = 14$ with $x_1 > 0$, $y_1 > 0$, $z_1 > 0$.

    b. Find the number of solutions to $x + y + z = 11$ with $x \geq 0$, $y \geq 0$, $z \geq 0$.

21. Find the number of solutions to the equation $x + y + z + w = 21$ where x, y, z, w are integers with $x \geq -2$, $y \geq -1$, $z \geq 0$, $w \geq 1$.

Section 11.4 — Counting Sequences With Repeated Elements

In earlier sections, we looked at counting two types of sequences. We considered questions such as this one:

How many four-letter words can be formed from the seven letters *a*, *b*, *c*, *d*, *e*, *f*, *g* if
 i) no letter may be repeated?
ii) a letter can be repeated as often as we like?

Using the product rule, the answers are $P(7, 4)$ and 7^4, respectively. You might wonder what happens if some of the letters can be used two or three times. We can create many new questions by varying how often the letters can be used in the sequence.

In this section, we count sequences in which some of the elements are the same. In Section 11.3, we counted the number of possible subsets by counting sequences in two different ways. This time we use the methods that we have developed for counting subsets to count sequences with repeated terms.

EXAMPLE 1

A bit string is a sequence in which each term is 0 or 1. For example, 0001110100 is a bit string of length 10 with six 0s and four 1s. How many bit strings of length 10 can we make using six 0s and four 1s?

Solution

Since we are counting sequences, a good strategy is to look at how many ways we can build the sequence. Consider filling the following ten boxes.

Let's look at the example 0001110100 in more detail.

0	0	0	1	1	1	0	1	0	0

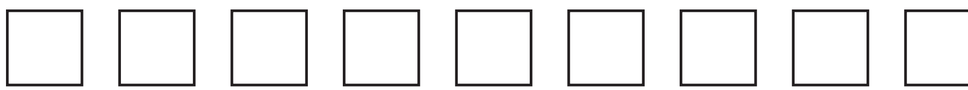
Note that the four 1s are found in positions $\{4, 5, 6, 8\}$. Once we specify the positions for the 1s, the sequence is completely determined. Hence building the sequence corresponds to selecting the four boxes from ten that will contain 1. The order of selection does not matter. There are $\binom{10}{4}$ such selections and, hence, there are $\binom{10}{4} = 210$ bit strings of length 10 with six 0s and four 1s. We could have specified the positions for the 0s; our answer would then be $\binom{10}{6}$, the same answer. In doing this example we showed that each bit string of length 10 with four 0s corresponds to one subset of size 4 selected from the set of boxes $\{1, 2, \ldots, 10\}$. Conversely, each of these subsets corresponds to exactly one bit string. Formally, there is a one-to-one correspondence between the set of bit strings of length 10 having four 0s and the set of subsets of size 4 selected from $\{1, 2, \ldots, 10\}$. It then follows that these two sets have the same number of elements.

In Example 1, we used only two symbols to form the sequence. In Example 2, we look at a more general question.

EXAMPLE 2 How many sequences of length 9 can be formed using four *a*s, three *b*s, and two *c*s?

Solution

We count these sequences by looking at how many ways we can construct them.



We can select the four positions for the *a*s in $\binom{9}{4}$ ways. For each of these ways, we can then select the three positions for the *b*s from the remaining five positions in $\binom{5}{3}$ ways. Finally, we can select the two positions for the *c*s from the remaining two positions in $\binom{2}{2}$ ways. Hence, there are $\binom{9}{4}\binom{5}{3}\binom{2}{2} = 1260$ sequences of length 9 that can be formed.

We can rewrite this answer in a more memorable form

$$\binom{9}{4}\binom{5}{3}\binom{2}{2} = \frac{9!}{5!4!}\,\frac{5!}{2!3!}\,\frac{2!}{0!2!} = \frac{9!}{4!3!2!}$$

The two examples can be generalized as follows.

The number of binary sequences of length n with r 1s and $(n - r)$ 0s is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

The number of sequences of length n that can be formed using k symbols with n_1 of the first type, n_2 of the second type, ..., and n_k of the kth type, where

$$n_1 + n_2 + \ldots + n_k = n \text{ is}$$

$$\binom{n}{n_1} \times \binom{n - n_1}{n_2} \times \ldots \times \binom{n_k}{n_k} = \frac{n!}{n_1! \times n_2! \times \ldots \times n_k!}$$

Each of these formulas can be developed using the methods shown in Examples 1 and 2. Now we look at some applications.

EXAMPLE 3

How many sequences of length 9 can be made from the letters of the name *Descartes*? Of these sequences, how many have the two *e*s side by side?

Solution

In the nine letters, there are two *e*s, two *s*s, and five other distinct letters. Thus, the number of sequences is $\frac{9!}{2!2!1!1!1!1!1!} = 90\,720$. To count the sequences in which the two *es* appear side by side, we put the two *e*s together as a single symbol $E = ee$. Now there are eight symbols to arrange, two *s*s and six other distinct ones including E. Thus, the number of such sequences is $\frac{8!}{2!}$. Note that we can leave out all the 1! factors for convenience.

EXAMPLE 4

Suppose 12 different coloured dice are rolled. What is the probability of getting two 1s, two 2s, …, and two 6s?

Solution

Any possible outcome can be represented by a sequence of length 12, where each term represents the outcome on a die of a particular colour, and with elements selected from {1, 2, 3, 4, 5, 6}. For example,

112233445566

corresponds to the first two dice coming up 1, the third and fourth dice coming up 2, and so on. The set of all outcomes U is the set of sequences of length 12 using any of the numbers from 1 to 6 as often as desired. Thus we have $n(U) = 6^{12}$.
The subset of sequences P with two 1s, two 2s, etc. is

$P = \{112233445566, 121233445566, \ldots\}$. Each element of P is a sequence of length 12 with two 1s, two 2s, and so on. There are $n(P) = \frac{12!}{2!2!2!2!2!2!}$ such

sequences. The probability that the required sequence will occur is $\frac{n(P)}{n(U)} = \frac{12!/(2!)^6}{6^{12}} \approx 0.0034$.

EXAMPLE 5

How many sequences of length 4 can be made using the letters *a, a, b, b, c, c*?

Solution

Let the set of all such sequences be $U = \{aabb, aabc, \ldots\}$.
We know how to count the sequences once we know which letters are to be used. For example, if there are two *a*s, one *b*, and one *c*, then there are $\frac{4!}{2!} = 12$ different sequences. Consider these two cases:

Case 1: Three different letters, two of one type and two others distinct (e.g., *aabc*)
Case 2: Two different letters, two of each type (e.g., *aabb*)

These cases correspond to disjoint subsets A and B with the property that $A \cup B = U$ so that $n(U) = n(A) + n(B)$.

For Case 1, there are $\binom{3}{1} = 3$ ways to select the letter which appears twice. For each of these choices there are 12 different sequences as shown above. Hence in Case 1, there are 36 different sequences.

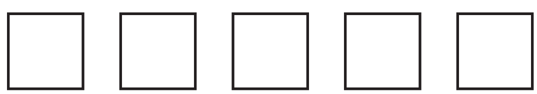
For Case 2, there are $\binom{3}{2} = 3$ ways to select the two letters, each of which appears twice. For each of these choices, the letters can be arranged in $\frac{4!}{2!2!} = 6$ ways. Hence, Case 2 contains 18 sequences. Combining the two results, we have $36 + 18 = 54$ sequences in U.

Exercise 11.4

Part A

Knowledge/Understanding

1. A binary sequence of length 5 is formed using 0s and 1s by filling the five boxes shown below.



a. List three such sequences.

b. How many such sequences can be made?

2. A sequence of length 5 is formed from the digits 1, 1, 2, 2, 3. How many sequences can be formed?

Part B

Knowledge/Understanding

3. The letters of the name *Mississauga* are rearranged. Such an arrangement is called an **anagram.**

a. How many possible anagrams can be formed?

b. How many of these anagrams start with the letter *s*?

c. How many of the anagrams start and end with *s*?

d. How many of the anagrams have the two *i*s side by side?

e. How many of the anagrams start with a vowel?

4. A string of lights has 12 sockets. How many arrangements can be made if there are two bulbs of six different colours available?

5. What fraction of sequences of eight flips of a coin, each giving *H* or *T*, results in exactly four *H*s?

6. How many binary sequences of length 5 have two or more 1s?

7. Ten trees—four pines, four cedars, and two spruce—are planted in two parallel rows of five trees. How many arrangements are possible if each row must have the same composition of trees, not necessarily in the same order.

8. Count all of the arrangements of the letters of the word *Descartes* that end with *s*.

9. Each of seven switches has two positions—off (*O*) and on (*I*).
   a. How many different ways can the switches be configured?
   b. Of these, how many configurations have exactly four switches turned on?
   c. How many of the configurations have at least two switches turned on?
   d. How many of the configurations have exactly four switches turned on including switch 1?

10. A bit string of length 10 is formed using two symbols, 0 and 1. Of all such strings with exactly six 1s, what fraction have exactly three 1s in the first five terms?

11. Suppose we want to create subsets of the ten digits {0, 1, 2, …, 9}.
   a. How many subsets can be created, including the empty set?
   b. How many of the subsets contain only digits less than 7?
   c. How many of the subsets contain 0 or 9?

Application

12. In a statistically designed experiment, there are four treatments labelled *A*, *B*, *C*, *D*, which are applied to 16 subjects in a random order. Each treatment is used four times, so that every subject receives exactly one treatment.
   a. How many different ways can the treatments be ordered? This is called a completely randomized design.
   b. Alternately, the subjects can be divided into groups of four people on a specific characteristic. Within each group, the four treatments appear once in a random order. For example, group 1 is the four heaviest members of the group, group 2 is the next set by weight, and so on. Given the four groups, how many different ways can the treatments be assigned? (This is called a randomized block design.)

Thinking/Inquiry/ Problem Solving

13. **a.** How many of the anagrams of the word *Mississauga* are palindromes?

b. How many are there for the word *Mississippi*?

14. Six different dice are rolled. What fraction of all the possible outcomes have

a. at least one repeated value?

b. two 2s, two 4s, and two 6s?

c. three odd values and three even values? (*Hint:* Count how many ways such a sequence can be constructed.)

Communication

15. In a binary sequence of length 10 made from 0s and 1s,

a. how many sequences are possible?

b. how many sequences have exactly r 1s where $0 \leq r \leq 10$?

c. Explain why $\binom{10}{0} + \binom{10}{1} + \ldots + \binom{10}{10} = 2^{10}$.

d. Verify this result by direct calculation.

16. Generalize the result in Question **10** to sequences of length n. Be sure to explain your reasoning.

17. a. Explain what type of sequence is represented by $\frac{10!}{2!3!5!}$.

b. Without evaluating the expression, explain why $\frac{50!}{20!15!15!}$ must be a positive integer.

c. Is $\frac{(3n)!}{(n-1)!n!(n+1)!}$ always a positive integer for $n \geq 1$?

18. Consider a bit string of length 8 having five 0s and three 1s. How many of these strings have at least two consecutive 1s?

19. How many bit strings of length 8 have at least one pair of consecutive 1s?

20. Consider arrangements of the symbols a, a, a, a, b, b, b, c, c, c. What fraction of these sequences contain a pair of consecutive cs? (*Hint:* Start building the sequence by arranging the as and bs.)

Part C

21. A random walk on the xy-plane starts at the point (0, 0) and moves at each step one unit to the right or one unit upwards. How many random walks end at the point (5, 3)?

22. A random walk starts at the point (0, 0) and at each step moves to the right one unit or upwards one unit.

 a. Show an equivalence between random paths and binary sequences, using E to represent a unit move to the right and V to represent a unit move vertically.

 b. How many paths are possible that end at the point (20, 12)?

 c. Of these paths, how many pass through the point (10, 10)?

 d. How many of the paths in part **b** pass through both the points (8, 4) and (12, 8)?

23. How many arrangements of six As, four Bs, and three Cs can be formed if

 a. all the As come before the first B?

 b. at least one A comes before the first B?

 c. at least one C occurs before the first A, and at least one A occurs before the first B?

24. A point on the plane starts at (0, 0). At each step, it moves one unit to the right and either one unit up or one unit down. That is, after one step, the possible positions are (1, 1) or (1, -1) and after two steps (2, 2) or (2, 0) or (2, -2). Note that there are two different paths to reach (2, 0). Consider the situation after 12 steps.

 a. What are the possible end positions?

 b. Show that each possible path can be represented as a binary sequence.

 c. How many paths end at the point (12, 0)?

Section 11.5 — A Strategy for Counting Problems

You now have the tools and strategies to solve a wide variety of counting problems. In this section, we review the tools and give you some ideas on how to approach a counting problem when you try to solve it.

The Tools

1. **The product rule:** If we can perform a first task in m ways and a second task in n ways, then we can perform the two tasks together in $m \times n$ ways.
2. **The sum rule:** If A and B are two subsets of a set U, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. This becomes $n(A \cup B) = n(A) + n(B)$ if A and B are disjoint.
3. **The rule of the complement:** $n(A) = n(U) - n(A)$. We cannot use this rule without defining the universal set U.
4. The number of ways of arranging n distinct symbols in a sequence of length is
   a. $P(n, r) = n(n-1)\dots(n-r+1) = \frac{n!}{(n-r)!}$ if each symbol can be used at most once
   b. n^r if each symbol can be used as often as we like
5. The number of ways to select a subset with r elements from a set of n elements is
$$\binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!}.$$
6. The number of ways to arrange n symbols, n_1 alike of the first type, n_2 alike of the second, …, n_k alike of the k^{th} type, where $n_1 + n_2 + \dots + n_k = n$, is $\frac{n!}{n_1!n_2!\dots n_k!}$.

The Strategy

You have done enough of the exercises and problems in Chapters 10 and 11 to realize that there is no single approach that we can use to solve every counting problem. Counting problems have several peculiarities:

- They all involve words, which means that we have to translate the problem into a mathematical notation before we can get started.
- They are easy to generalize, often in several different directions. We can exploit this property in reverse by looking at particular simplifications of a problem to make sure that we understand what is being asked and also to get an idea of how to carry out the counting.
- With a small change, we can turn a relatively straightforward problem into one that is much more difficult. This means that it is difficult to judge whether a problem is hard or easy when you first read it. Without a good strategy, you can be completely fooled.

To demonstrate the strategy, we look at two examples.

EXAMPLE 1 The letters of the alphabet are used to form a word of length k. No letter may be used more than once. How many of the words contain the letter z?

EXAMPLE 2 Suppose we have n symbols to arrange in a row. We are given that r of the symbols are identical (here called **special**) and that the remaining $n - r$ symbols are all distinct. How many arrangements can we make that start or end with the special symbol?

These two problems have been chosen because it is difficult to write down their answers without some careful thought. We attack such problems using the following steps. Not every step applies to every problem, but the steps give a useful guideline.

Step 1: Understand the problem

- Invent a notation to describe the objects that you want to count.
- Write down some specific examples.
- Decide if the objects you are dealing with are sequences or subsets. Does the order of the elements matter?

Step 2: Decide on an approach

- If the problem is general (i.e., if it involves ns and rs), try a simpler version with specific small values and list (and then directly count) all possible objects. Then try to find a formal counting approach that agrees with your direct count.
- Use a constructive approach—the number of objects is equal to the number of ways that we can build them.
- Start the construction by satisfying the restrictions first.
- If it is not obvious how to count the number of ways of constructing the objects directly, consider cases or the complement.

Step 3: Implement your approach

- Use the tools we have developed.
- Watch out for double counting or incomplete counting (especially with cases).
- Write an explanation that will be clear to someone else. Don't rely on formulae alone.
- If the problem is general, check the answer by looking at particular small cases where the count can be done directly.
- If your approach does not seem to be working, go back to Step 2.

Different approaches can lead to different expressions. If these expressions lead to numeric values, it is easy for you to see whether your answer agrees with that of a classmate. For answers involving symbols, you can compare special cases.

Now consider using this strategy for the examples. These examples are difficult problems chosen to show the power of the strategy.

EXAMPLE 1 The letters of the alphabet are used to form a word of length k. No letter may be used more than once. How many of the words contain the letter z?

Solution

We start with a special case, say $k = 3$, to make sure that we understand the problem. We want to count three-letter words such as *zab* or *abz*, all of which have distinct letters and include the letter z. As the examples show, order matters here—we want to count *zab* and *abz* as different words. Looking at the examples, we notice that once we have the three letters selected, we can arrange them to form $3! = 6$ different words. We want to count the number of ways that we can select the three letters. Since z must be included, we can choose the other two letters in $\binom{25}{2}$ ways. Hence, using the product rule, for $k = 3$, we obtain $3! \times \binom{25}{2}$. We apply the same method to the general case. Since z is included, we can select the other $k - 1$ letters for the word in $\binom{25}{k-1}$ ways. For each of these selections, we can arrange the k letters into $k!$ words. Hence there are $k! \times \binom{25}{k-1}$ words of length k that contain z.

Another approach to Example 1 is to use the rule of the complement to count the total number of k letter words, $P(26, k)$, and the number of such words that do not include z, $P(25, k)$. This approach gives an answer, $P(26, k) - P(25, k)$, that appears quite different from the first answer. With some effort, we can show that the two answers are algebraically the same.

EXAMPLE 2 Suppose we have n symbols to arrange in a row. We are given that r of the symbols are identical (here called special) and that the remaining $n - r$ symbols are all distinct. How many arrangements can we make that start or end with the special symbol?

Solution

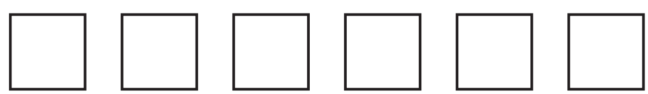
Step 1: Let's start with a specific problem, say $n = 6$, $r = 3$. We do not want to choose so small a problem that we lose the essential difficulties. Now we can choose the symbols A, A, A, B, C, D where A is designated to be special. We want to count how many arrangements we can make using these six symbols with A at the beginning or end. Some typical examples are *AABCAD*, *BCADAA*, *ABCDAA*. The third example raises an important aspect of the problem — are we counting arrangements that can start and end with the special symbol? The answer depends on how we interpret the word "or." Does it mean to include those arrangements that both begin and end with A or not? There is no correct answer here—the language of the question is too vague. Let's agree that we will include arrangements that begin and end with A.

Step 2: Based on Step 1, we might consider two cases:
Case 1: arrangements that start with A
Case 2: arrangements that end with A

One problem is that the cases are not disjoint. Some arrangements in Case 1 are also in case 2. In set notation, if we let S be all the arrangements in Case 1 and T all the arrangements in Case 2, then we want to find $n(S \cup T)$, which means that we would have to find $n(S)$, $n(T)$, and $n(S \cap T)$.

An alternate approach is to consider the complement. That is, the set of all arrangements that do not begin or end with A. To use this approach, we also need to count the universal set of all arrangements of the symbols without restriction. Let's try this approach.

Step 3: We will start with the particular case $n = 6$, $r = 3$ to make sure that our approach is feasible. This is not necessary but is often helpful for building up courage to attack the general situation. The universal set U contains all sequences of length 6 made from A, A, A, B, C, D, so $n(U) = \frac{6!}{3!1!1!1!} = 120$.



To count the number of arrangements that start and end with a symbol other than A, we can fill the first box in three ways, and the last box in two ways (using only B, C, D). The middle four boxes are filled with an arrangement of four letters, three As and something different. There are $\frac{4!}{3!1!}$ different arrangements for the middle four boxes. The total number of arrangements is then $3 \times 2 \times \frac{4!}{3!1!} = 24$. Using the rule of the complement, the number of arrangements that begin or end with A is $120 - 24 = 96$.

Since our approach works well, we can now attack the general problem. If U is the set of all arrangements of n symbols with r that are alike and all the rest are neither, then $n(U) = \frac{n!}{r!}$. To count the number of arrangements that neither start nor end with the special symbol, the first position can be filled in $n - r$ ways, and the last in $n - r - 1$ ways. The middle $n - 2$ positions can then be filled with an arrangement of $n - 2$ symbols, of which r are the same in $\frac{(n-2)!}{r!}$ ways. Using the product rule, there are $(n - r) \times (n - r - 1) \times \frac{(n-2)!}{r!}$ possible arrangements. Finally, using the rule of the complement, the number of arrangements that begin or end with the special symbol is

$$\frac{n!}{r!} - (n - r) \times (n - r - 1) \times \frac{(n-2)!}{r!} =$$

$$\frac{n \times (n-1) \times (n-2)!}{r!} - (n - r) \times (n - r - 1) \times \frac{(n-2)!}{r!}$$

$$= \frac{(n-2)!}{r!} \times [n(n-1) - (n-r)(n-r-1)]$$

$$= \frac{(n-2)!}{r!} \times r \times (2n - r - 1)$$

$$= \frac{(n-2)!}{(r-1)!} \times (2n - r - 1)$$

There is not much merit in this algebra since the final answer is not very illuminating. The unsimplified version is likely just as useful. To make sure that there are no errors, you should verify that we get 96 when $n = 6$, $r = 3$.

One final worry is whether we have covered all the possible situations. For example, if $r = 1$ so that there are no repeated symbols, does our formula give the correct answer? Substituting in the above expression, we get $\frac{(n-2)!}{0!} \times (2n - 2) = 2 \times (n-1)!$. This is the correct answer since we place the special symbol in two ways at the beginning or end and then the other symbols in $(n - 1)!$ ways. You can also check that we get the correct answer in the other extreme case, $r = n$.

We have solved two difficult problems with tools and strategies. The following problems are chosen to give you practice solving counting problems. Some of these are also difficult. Be brave and you will succeed!

Exercise 11.5

Part B

1. The letters of the alphabet are used to form a word of length k. How many of these words contain the letter z if we allow letters to be repeated as often as we like? See Example 1.

2. A subset of k integers is selected from the set 1, 2, …, 100. In how many of these subsets are the selected integers consecutive?

3. A subset of five numbers is selected from the integers 1, 2, …, n. How many of these subsets have largest element L where $1 \leq L \leq n$?

4. Consider the set of numbers $\{1, 2, 3, \ldots, 2n + 1\}$. How many subsets of size $o + e$ can be formed with o odd numbers and e even numbers?

5. An arrangement of length r is constructed from a set of n distinct symbols that include the letter A. How many of these arrangements contain A if

   a. no symbol may be used more than once?

   b. each symbol may be used up to r times?

6. Suppose you have n symbols of which r are identical (called special as in Example 2), and the remaining $n - r$ are distinct. How many arrangements of these symbols can you make if the arrangement must start with a special symbol?

7. A binary sequence of length n has exactly k 1s. How many such sequences can you make if the sequence begins or ends with 1?

Part C

8 How many sequences of fixed length n can be formed by arranging the integers 1, 2, …, n so that 1 and n are separated by exactly one term?

9. A sequence of length r is formed using the integers 1, 2, …, 1000, at most once. In how many of these sequences does 1 occur before 2?

10. A sequence of length $2n$ is formed using two 1s, two 2s, and so on to two ns. How many of these sequences have at least one 2 before a 1?

11. How many arrangements can be formed from a As, b Bs and c Cs if no two of the As are consecutive?

Key Concepts Review

Most counting problems are variations of standard questions or can be converted into such questions. We do not need basic concepts such as the product rule to solve every question. We can define some expressions for common situations.

The first thing to decide is whether order matters. If it does, you are counting sequences; if the elements are distinct, the terms *permutation* and *arrangement* are also used. If it does not, you are counting subsets or combinations.

Next you must decide what restrictions have been placed on the objects you are counting and whether you can deal with these restrictions and be left with a standard question. You may have to separate the objects into cases. This happens when the way you deal with the restriction affects the number of ways to complete the objects; it also occurs when counting subsets, if dealing with the restriction imposes order on the object.

Section 11.5 gave some hints on problem solving. As was pointed out, you need to remember the following formulas:

Counting Sequences

a. Arrangements of r elements from a set of n distinct elements ($P(n, r)$ or nPr)

$$P(n, r) = n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

If $r = n$, we get $P(n, n) = n!$
If $r > n$, $P(n, r) = 0$

b. Arrangements of r elements from a set of n elements, with unlimited repetition allowed, number of sequences $= n^r$

c. Arrangements of n symbols of k types, with n_1 symbols of type 1, n_2 of type 2, etc., so that $n_1 + n_2 + \ldots + n_k = n$, number of sequences is $\frac{n!}{n_1 n_2! \ldots n_k!}$

Counting Subsets

Subsets of r elements from a set of n distinct elements $\left[\binom{n}{r} \text{ or } nCr\right]$

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{P(n, r)}{r!}$$

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

Rich Learning Link wrap-up

investigate and apply

CHAPTER 11: VOTING SYSTEMS

Voting systems in Canada and around the world vary. Each has its own strengths and weaknesses. Canada's system gives voice to disparate regions but can result in governments that do not have a majority of the popular vote. Proportional representation gives seats in parliament in direct proportion to the popular vote, but if regions are to be heard, allocating the seats can be a problem. In 1951, mathematical economist Kenneth Arrow proved that there is no consistent method of making a fair choice among three or more alternatives. That is, no election procedure can always fairly decide the outcome of an election that involves three or more candidates.

Investigate and Apply

1. How many ways can 12 seats in a small parliament be assigned to three political parties if the votes are in the proportions 8 to 3 to 1?
2. How many ways can 120 seats in a parliamentary house be assigned to three political parties if the votes are in the proportions 80 to 30 to 10? (Most scientific calculators will not be able to answer this—try to find computer software that can.)
3. A 120-seat parliamentary house represents three regions: 20 seats for the east; 60 seats for the central region; and 40 seats for the west. Proportional representation is used in each of these regions. After an election, Party A wins 8 seats in the east, 47 in the centre, and 25 in the west. Party B wins 10 seats in the east, 8 seats in the centre, and 12 seats in the west. Party C wins 2 seats in the east, 5 seats in the centre, and 3 seats in the west. How many ways are there to assign these seats?

INDEPENDENT STUDY

Research a recent federal or provincial election and determine the number of seats each party would have if proportional representation seating had been used. How many ways could these seats have been allocated? How does the proportional representation seating compare to the actual number of seats awarded to each party?

Research a voting system, past or present, that used proportional representation. How did they deal with determining who gets what seat?

What did Kenneth Arrow mean by *fair choice*?

Review Exercise

1. A sequence of length 6 is formed from the digits {0, 1, 2, …, 9}. If no repetition is allowed, how many of these sequences can be formed if
   a. there are no restrictions?
   b. the sequence starts with 7?
   c. the sequence starts with a digit less than 7?
   d. the first two terms are both less than 3?
   e. the first two terms are both less than 7?
   f. the sum of the first two terms is 7?
   g. at least one of the first two terms is less than 7?

2. A password for a voice-mail system is a sequence of five digits selected from {0, 1, 2, …, 9} with unlimited repetition allowed. How many passwords can be formed if
   a. there are no restrictions?
   b. the first digit cannot be 0?
   c. the first and last digit are 9?
   d. the first or last digit is 9?
   e. the first and last symbol are the same?
   f. all the digits are distinct?
   g. at least two different digits must be used?
   h. the digit 9 must be included?

3. A cable contains 12 wires that are colour coded. There are three green, three red, three black, and three white wires. How many subsets of four wires can be selected if
   a. there are no restrictions?
   b. there is exactly one wire of each colour?
   c. only green and red wires are used?
   d. at least one green wire is used?
   e. at least two green wires are used?
   f. exactly two different colours are used?
   g. exactly three different colours are used?

4. A bit string of length 8 is a binary sequence made using 0s and 1s. How many such strings can be constructed if
   a. there are no restrictions?
   b. the string starts and ends with 0?
   c. the string starts or ends with 0?
   d. the string has exactly 2 0s?
   e. the string has exactly 2 0s and starts with 10?
   f. the string has exactly 2 0s, which occur consecutively?
   g. the string has exactly two 0s, one in the first half and one in the second half?
   h. the string has at least 2 0s?

5. A landscaper plans to plant a row of ten trees. There are five cedars, three pines, and two spruces available. How many different arrangements of the ten trees can be planted if
   a. there are no restrictions?
   b. all trees from each species must be planted side by side?
   c. a spruce is planted at each end of the row?
   d. the trees at the ends of the row are the same species?
   e. the two spruces cannot be planted side by side?
   f. the row starts or ends with a spruce tree?
   g. no two cedars can be side by side?

6. Explain, using words only, why $P(100, 10) = \binom{100}{10} \times 10!$.

7. A sequence of length 4 is formed using the letters $\{a, b, c, d, e, f\}$ without repetition. Explain what is wrong with the following arguments and provide a correct solution with a clear explanation.
   a. To count the number of sequences that start or end with a vowel, consider two cases
      Case 1: sequence starts with a vowel
      Case 2: sequence ends with a vowel
      In Case 1, there are 2 choices for the first letter, 5 choices for the second, 4 choices for the third, and 3 choices for the fourth, so there are $2 \times 5 \times 4 \times 3 = 120$ sequences that start with a vowel. In Case 2, start with 2 choices for the last letter and so on, so there are 120 sequences that end with a vowel. Hence there are 240 sequences that start or end with a vowel.
   b. To count the number of sequences that contain at least one vowel, select the vowel in 2 ways. The remaining three letters can be chosen in $\binom{5}{3}$

ways. Then the four selected letters can be arranged in 4! ways so there are $2 \times \binom{5}{3} \times 4! - 480$ sequences that contain at least one vowel.

8. The five letters $\{a, b, c, d, e\}$are arranged to form a word. If all possible words are arranged in a list in alphabetic order,
   a. how many words come before *ceadb*?
   b. how many words are between *adcbe* and *dacbe*?
   c. what is the 61^{st} word in the list?

9. A subset of size r is formed from a set with n elements that include A and B. How many of the possible subsets contain
   a. both A and B?
   b. A or B?
   c. neither A nor B?

10. A sequence of length r is formed from a set with n elements that include A, B, and C. How many of the sequences have
   a. A occurring before B?
   b. A occurring before B occurring before C?
   c. A and B occurring before C?

Chapter 11 Test

Achievement Category	Questions
Knowledge/Understanding	all
Thinking/Inquiry/Problem Solving	7-9
Communication	2-4
Application	5, 8

1. Evaluate $P(10, 3) - \binom{10}{3}$.

2. A five-digit number is formed using the digits from the set $\{1, 2, \ldots, 9\}$ with no repetition. How many of these numbers
   a. start with an even digit?
   b. start and end with an even digit?
   c. have exactly three even digits?

3. Given that $P(n, r)$ is the number of sequences of length r that can be formed using n symbols with no repetition allowed, prove that $\binom{n}{r} = \frac{P(n, r)}{r!}$

4. A committee of four people is to be formed from six men and six women. How many committees can be formed if
   a. there are no restrictions?
   b. both Bob and Mary must be on the committee?
   c. Bob and Mary will not serve on the committee together?

   You may leave your answer in unsimplified form. Be sure to explain your reasoning.

5. You are given a collection of 20 multiple choice questions, four each with possible answers A, B, C, D, or E. How many different sets of ten questions can be formed if the set has exactly two questions with each possible answer?

6. All the letters of the word *Toronto* are used to form words of length 7. How many of these words have the two Ts separated by at least one other letter?

7. Binary sequences of length n are formed using 0s and 1s. How many of these sequences start or end with 1?

8. A password with 6, 7, or 8 characters is formed using the 26 letters of the alphabet and the ten digits selected from the set $\{0, 1, 2, \ldots, 9\}$. Digits and letters may be repeated. How many passwords can be formed if a password must contain at least one letter and at least one digit?

9. The universal set U has N elements, including the letter A. Show that the fraction of all subsets of size r that contain A is $\frac{r}{N}$.

Extending and Investigating

ASSESSING ALGORITHMS

An algorithm is a recipe that a computer uses to solve a specific class of problems. For example, there are algorithms that sort lists or search strings of characters. A Web search engine uses a complex algorithm to quickly search a huge text string in order to identify sites that you are seeking. Computer scientists are interested in studying the properties of algorithms to see how efficient and fast they are. Here, we examine one way to assess algorithms.

You have learned the Gauss-Jordan algorithm for solving systems of three linear equations, written in matrix form, such as

$$\left[\begin{array}{ccc|c} 3 & 3 & -1 & 7 \\ 1 & 2 & 3 & 2 \\ 2 & -1 & -2 & 1 \end{array}\right]$$

A computer can use the same algorithm to solve ten linear equations in ten unknowns or, in general, n linear equations in n unknowns. To assess the algorithm, we count the number of additions, subtractions, multiplications, and divisions it takes to solve the equations. The time taken to solve the system is closely related to the total number of operations required. For the above example, let the number of operations be a_3, where the subscript indicates the number of equations we are solving. Since we only care about how many operations are needed and not the actual solution, we represent the entries in the matrix as *.

$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right]$$

We do not worry about special cases when, for example, the (1, 1) entry is 1 or a 0 appears that requires an interchange of rows.

The first step of the algorithm is to divide every entry in the first row by the first element. This requires four divisions. Next, we multiply each element in the first row by the (2, 1) entry and subtract from the second row. This requires another eight operations. We repeat this step for the third row, giving another eight operations. We have used a total of 20 operations to reduce the matrix to the form

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right]$$

The last two rows correspond to two equations in two unknowns, and we require a_2 operations to solve these equations. Finally, to find the first unknown, we substitute the solutions of these two equations into the equation corresponding to the first row and solve. This requires two multiplications and two subtractions, so we have $a_3 = a_2 + 24$. You can use the same argument to show that $a_2 = a_1 + 11$, and, obviously, $a_1 = 1$, so we have $a_2 = 12$, $a_3 = 36$.

A computer can use this algorithm to solve a system of n equations. You can use the same counting method to show that $a_n = (n + 1) + 2(n + 1)(n - 1) + 2(n - 1) + a_{n-1}$ or, simplifying, $a_n = 2n^2 + 3n - 3 + a_{n-1}$, $n \geq 2$. We can use this formula to find a_n recursively for any value of n; that is, we use the formula and a_3 to find a_4 and then repeat. The table gives the number of operations required for solving up to ten equations in ten unknowns.

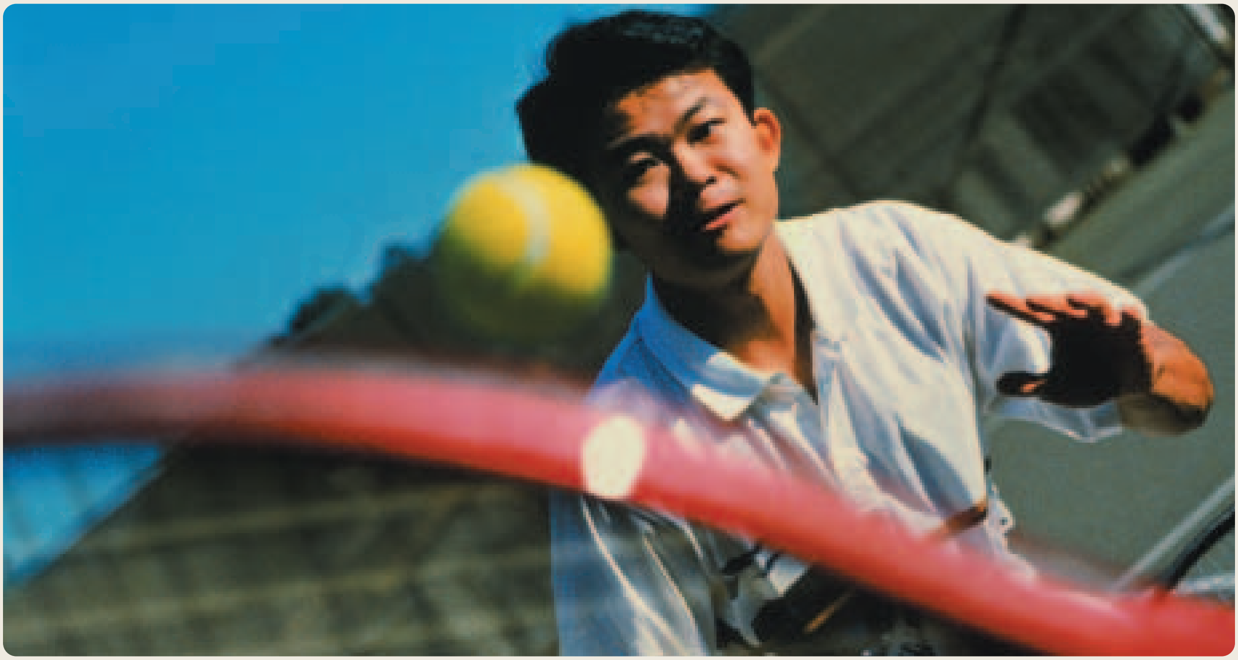
n	a_n
1	1
2	12
3	36
4	77
5	139
6	226
7	342
8	491
9	677
10	904

To assess the algorithm more generally, we can show that $a_n = \frac{4n^3 + 15n^2 - 7n - 6}{6}$. Note that a_n is approximately proportional to n^3 as n gets large; that is, $\frac{a_n}{n^3} \approx \frac{2}{3}$ for large values of n.

You might wonder if it is possible to find a better algorithm that requires substantially fewer operations; for example, one in which the number of operations required is proportional to n^2 as n gets large. The answer here is *no*. However, in certain cases, when n is large and many of the coefficients are 0 (this is called a sparse system of equations), a more efficient algorithm can be found.

Chapter 12

SEQUENCES



If you release a tennis ball from your hand, will it always drop directly to the ground? A scientist would answer this question by repeating the experiment many times in carefully controlled circumstances before drawing a conclusion about the consequence of a continuous or connected series of actions. In mathematics, a set of numbers can be arranged according to some rule or sequence. A closer look at sequences of numbers and sequences of functions, statements, and diagrams will lead to important and practical applications in finance, computer science, and medicine.

CHAPTER EXPECTATIONS In this chapter, you will

- use sigma notation, **Section 12.1**
- solve problems using counting principles, **Section 12.1, 12.2**
- solve problems involving permutations and combinations, **Section 12.2**
- prove formulas for the sums of series, **Section 12.3**
- understand mathematical induction, **Section 12.3**
- prove the binomial theorem, **Section 12.4**
- prove relationships between coefficients in Pascal's triangle, **Section 12.4**
- describe the connections between Pascal's triangle, values of $\binom{n}{r}$ and values for binomial coefficients, **Section 12.4**
- determine terms in the expansion of a binomial, **Section 12.4**

Review of Prerequisite Skills

This chapter deals with sequences. Here we review some ideas about arithmetic and geometric sequences.

ARITHMETIC SEQUENCES

In an arithmetic sequence, the difference between consecutive terms is a constant. In an arithmetic sequence having first term $a_1 = a$ and constant difference d, successive terms are $a_2 = a + d$, $a_3 = a + 2d$, $a_4 = a + 3d$, and $a_n = a + (n - 1)d$. If $a = 4$ and $d = 3$, we obtain the arithmetic sequence 4, 7, 10, …, $4 + (n - 1)3$, …. In this sequence, $t_n = 4 + (n - 1)3 = 3n + 1$. Every linear function defines an arithmetic sequence if the variable has the set of positive integers as its domain. For $f(n) = 3n - 7$, we obtain the sequence $-4, -1, 2, 5, \ldots, 3n - 7, \ldots$.

The sum of the first n terms of an arithmetic sequence is

$$S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d) = \frac{n}{2}[2a + (n - 1)d]$$

For the arithmetic sequence 2, 7, 12, …, $5n - 3$, …, , the value of a is 2, d is 5, and the sum of the first 15 terms is

$$\begin{aligned} S_{15} = 2 + 7 + 12 + \ldots + 72 &= \frac{15}{2}[4 + 14 \times 5] \\ &= \frac{15}{2}[74] \\ &= 555 \end{aligned}$$

GEOMETRIC SEQUENCES

In a geometric sequence, the ratio of consecutive terms is constant. In a geometric sequence having first term $g_1 = a$ and constant ratio r, successive terms are $g_2 = ar$, $g_3 = ar^2$, $g_4 = ar^3$, …, $g_n = ar^{n-1}$. If $a = 2$ and $r = 3$, we obtain the geometric sequence 2, 6, 18, 54, …, $2 \cdot 3^{n-1}$, …..

An exponential function of the form $f(n) = a \bullet b^{n-1}$ defines a geometric sequence if the variable has the set of positive integers as its domain. For $f(n) = 7 \cdot 2^{n-1}$, we obtain the sequence 7, 14, 28, 56, …, $7 \cdot 2^{n-1}$, ….

The sum of the first n terms of a geometric sequence is

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} = a\frac{r^n - 1}{r - 1}$$

For the geometric sequence with $a = 7$ and $r = 3$, the sum of the first 12 terms is

$$S_{12} = 7\frac{3^{12} - 1}{3 - 1}$$
$$= \frac{7}{2}(3^{12} - 1)$$

Exercise

1. For the following sequences, identify them as arithmetic, geometric, or something else. In all cases, $n > 1$.
   a. $t_n = n$
   b. $t_n = (-2)^n$
   c. $t_n = n + (-2)^n$
   d. $t_n = n(-2)^n$
   e. $t_n = 7n - 5$
   f. $t_n = 3 \cdot 2^{n-1}$

2. In an arithmetic sequence, the second term is 7 and the fifth term is 16. Find the tenth term and the sum of the first ten terms.

3. If the sum of the first five terms of an arithmetic sequence is 30 and the sum of the first ten terms is 10, find the sum of the first twenty terms.

4. If $a_1, a_2, \dots, a_n, \dots$ is an arithmetic sequence with first term a and common difference d and $b_n = a_{2n-1}$, $n \geq 1$, find an expression for the general term b_n and show that $b_1, b_2, \dots, b_n, \dots$ also form an arithmetic sequence.

5. In a geometric sequence, the first term is 2 and the fifth term is 32. Find the sum of the first ten terms.

6. Can a sequence be both arithmetic and geometric? Explain.

7. Consider the sequence $t_n = \binom{n+2}{n}$, $n \geq 1$.
   a. Evaluate the first five terms of the sequence. Is it arithmetic?
   b. Consider the new sequence defined by $a_n = t_{n+1} - t_n$, $n \geq 1$. Is this sequence arithmetic?

8. Suppose a geometric sequence with general term $g_n = ar^{n-1}$, $n \geq 1$ has both a and r positive. What can you say about the sequence with terms $a_n = \log(g_n)$, $n \geq 1$?

Rich Learning Link investigate

CHAPTER 12: RECURSIVE SEQUENCES

All living things receive their DNA from the generation that came before them. Changes to the DNA as it is passed along are fundamental to the evolution of species. There are parallel ideas in sociology (cultural evolution) and in philosophy (Hegel's dialectic method). There are also parallels in mathematics. Fractals are widely used mathematical objects that can be constructed by repeating a process in which each step adds greater complexity to the results from the previous step. Recursively defined sequences are numerical sequences in which each term is found from the previous term (or terms) through some specific process. Among other things, recursively defined sequences are used to solve complicated mathematical equations indirectly when no direct method exists.

Investigate

Possibly the most famous recursively defined sequence in mathematics is the Fibonacci sequence, named in honour of Leonardo da Pisa, who lived around the beginning of the 13th century and whose nickname was Fibonacci. He introduced the sequence in his book *Liber Abaci*, published in 1202, as the solution to the following problem: How many pairs of rabbits will be produced in a year if every month each pair of adult rabbits gives birth to a new pair that becomes mature in the following month? The answer is the twelfth term in the sequence defined by $t_1 = 1$, $t_2 = 2$, $t_n = t_{n-1} + t_{n-2}$, $n = 3, 4, 5, \ldots,$. The first few terms of this sequence are 1, 2, 3, 5, 8, 13, 21, 34, etc. The numbers themselves are called Fibonacci numbers.

Fibonacci numbers arise in several other contexts. They answer questions like "How many sequences of 0s and 1s of length $n-1$ have no 0,0 sub-sequence?" and "How many subsets of the integers from 1 to $n-1$ contain no consecutive integers?"

The ratios of the Fibonacci numbers approach the Golden Ratio $\frac{1 + \sqrt{5}}{2}$.

This ratio occurs in nature and is used in art and architecture because it is considered to be aesthetically pleasing. It can also be approximated using the recursive sequence $t_1 = 1$. Calculate the first ten terms of this sequence $t_n = 1 + \frac{1}{t_{n-1}}$, $n = 2, 3, \ldots$.

DISCUSSION QUESTIONS

1. When is a direct formula for the *n*th term in a sequence preferable to a recursive formula?
2. What are fractals and where are they used?

Section 12.1 — Sequences

We define a sequence of numbers by its terms, one term for each positive integer. That is, a sequence has the form $t_1, t_2, \ldots, t_n, \ldots$ where t_1 is the first term, t_2 is the second term, and t_n is the n^{th} or general term.

Sequences are of practical importance. Financial calculations, such as monthly mortgage payments, are based on sequences. In computer science, many algorithms, such as those used to sort and merge lists, are defined as a sequence of operations. In medicine, sequences are used to model the growth and decline of epidemics of infectious diseases in a population.

In this chapter, we look at sequences of numbers and also sequences of functions, statements, and diagrams. The terms of a sequence may be functions, such as $t_1(x) = 1 + x$, $t_2(x) = (1 + x)^2, \ldots, t_n(x) = (1 + x)^n, \ldots$ or statements, such as $S_1, S_2, \ldots, S_n, \ldots$ where

S_1: the sum of the first natural number is 1

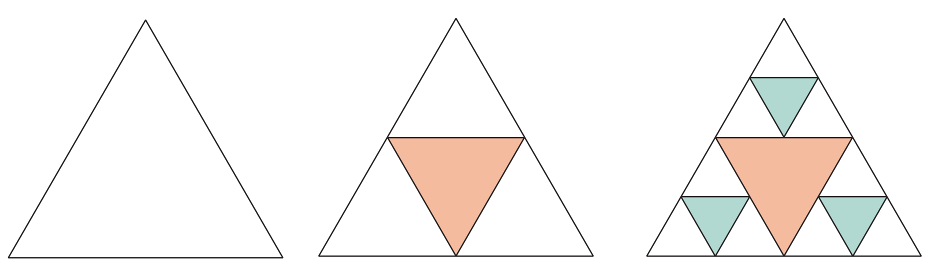
S_2: the sum of the first two natural numbers is 3

S_n: the sum of the first n natural numbers is $\frac{n(n + 1)}{2}$

We know that every term in this sequence of statements is true. In Section 3 of this chapter, we use a method of proof called mathematical induction to investigate other sequences of statements to see if they are true.

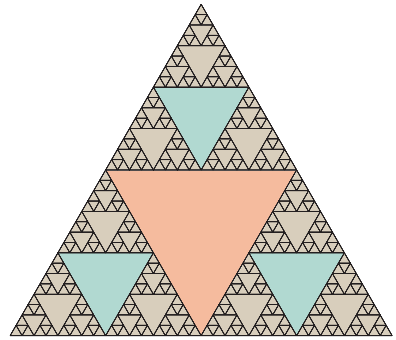
technology

We can also construct sequences of diagrams. One use of these sequences is to construct **fractals,** amazing computer-constructed diagrams, which are the subject of modern mathematical research. Enter *fractal* in a search engine and you will soon locate some beautiful examples. For example, the following sequence of diagrams, D_1, D_2, D_3, leads to **Sierpinski's triangle,** a simple fractal.



D_1 D_2 D_3

To produce this sequence of diagrams, we start with the first term, an equilateral triangle. To get the second term, join the midpoints of the three sides of the initial triangle and remove the equilateral triangle in the centre. This leaves three smaller equilateral triangles as shown. To produce the third term, repeat the removal process on all the remaining triangles in the second diagram. In general, we produce the n^{th} diagram by using the removal process on all the equilateral triangles in the $(n - 1)^{st}$ diagram. Here is what happens after $n = 6$.



D_6

As n gets large, the sequence of diagrams converges to Sierpinski's triangle. There are many Web sites that will show you dynamically how this sequence evolves. The above sequence of diagrams is defined recursively. In **recursive definitions** of a sequence, we do not give a formula for finding the nth term directly. Instead, we specify the first term and then define each term by a process applied to the preceding terms. **Recursion** as a way to define sequences is very important in computer science and other areas of application.

EXAMPLE 1 Provide a recursive definition for

a. the general arithmetic sequence
b. the general geometric sequence
c. the sequence of functions $t_1(x), t_2(x), \ldots, t_n(x), \ldots$ where $t_n(x) = (1 + x)^n$

Solution

a. The first term of the general arithmetic sequence is $t_1 = a$ and the difference between two consecutive terms is d. That is, $t_n - t_{n-1} = d$ for all $n \geq 2$. Rearranging this equation gives the recursive definition of the arithmetic sequence $t_1 = a$, $t_n = t_{n-1} + d$, $n \geq 2$.

b. The first term of the geometric sequence is $t_1 = a$ and the ratio of two consecutive terms is r. That is, $\frac{t_n}{t_{n-1}} = r$ for all $n \geq 2$. Again, we rearrange the equation to get the recursive definition of the geometric sequence $t_1 = a$, $t_n = r \times t_{n-1}$, $n \geq 2$.

c. For any fixed x, the sequence is a geometric sequence. Applying the result from above we have $t_1(x) = 1 + x$, $t_n(x) = (1 + x)t_{n-1}(x)$, $n \geq 2$.
Using the above examples, we can ask many mathematical questions about sequences. For example, what happens to the terms of a geometric sequence as n gets large? If you have taken calculus, then you know this is the limit of t_n as

$n \to \infty$. For the sequence of diagrams, we could ask about the area and the number of triangles in the n^{th} diagram, especially as n gets large.

We can also generate interesting questions by creating new sequences from a given sequence. There are many ways to do this.

Exercise 12.1

Part A

Knowledge/ Understanding

1. The following sequences are defined recursively. Evaluate the first five terms.
   a. $t_1 = 1, t_n = 2 - 3t_{n-1}, n \geq 2$
   b. $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}, n \geq 3$
   c. $g_1(x) = 1, g_n(x) = xg_{n-1}(x) + 1, n \geq 2$
   d. $h_1(x) = x, h_n(x) = 1 + 2h_{n-1}(x), n \geq 2$

2. In an arithmetic sequence, the third term is 12 and the 15th term is -48.
   a. Find the formula for the general term of the sequence.
   b. How many terms in the sequence are positive?

Part B

Application

3. A family is saving to pay for their child's university education. On the child's tenth birthday, the family puts \$2000 into a savings account that pays 5% annual interest. On each subsequent birthday up to and including the 18th, another \$2000 dollars is added to the account. Let $p_{10}, p_{11}, \ldots, p_{18}$ be the amount in the account after the payment is made on the 10th, 11th,..., 18th birthday.
   a. Show that the sequence p_n, $n = 10, 11, \ldots, 18$ satisfies the recursion $p_{10} = 2000, p_n = 1.05p_{n-1} + 2000, 11 \leq n \leq 18$.
   b. Using a calculator, evaluate p_{18}.

4. For any arithmetic sequence, show that the sum of any 20 consecutive terms is 20 times the average of the first and last term in the sum.

5. A geometric and arithmetic sequence have a common first term, 1. Show that if the second and third terms are also equal, then all terms are equal.

Knowledge/Understanding

6. Consider the geometric sequence with general term $t_n = 3 \times 2^{n-1}$, $n \geq 1$. Create a new sequence according to the instructions given and determine whether each new sequence is geometric.

a. The n^{th} term of the new sequence is the square of the n^{th} term of the given sequence.

b. The terms of the new sequence are the odd-numbered terms of the given sequence.

c. The first term of the new sequence is 3. The n^{th} term of the new sequence is $t_n \times t_{n-1}$, $n \geq 2$.

7. A sequence has general term $g_n = t_n \times t_{n-1}$, $n \geq 2$, where $t_n = 3 \times 2^{n-1}$, $n \geq 1$. In that this is a geometric sequence, what must be the value of g_1?

8. Suppose that the sequence $v_1, v_2, \dots, v_n, \dots$ is defined recursively so that $v_1 = 2$ and $v_n = 3 \times v_{n-1}$, $n \geq 2$. Verify that $v_n = 2 \times 3^{n-1}$ satisfies this recursive definition for all $n \geq 1$.

Communication

9. At the start of the n^{th} month, the remaining debt owed on a student loan of \$5000 is p_n, $n \geq 1$. Note that $p_1 = 5000$. The monthly interest rate is 0.75%. At the end of each month, the student makes a payment of \$100.

a. Show that $p_2 = 4937.50$

b. Explain why $p_n = 1.0075p_{n-1} - 100$ for $n \geq 2$.

c. Use this recursive definition and a spreadsheet program or calculator to determine how many months it takes before the loan is paid.

Thinking/Inquiry/Problem Solving

10. The numbers 1, 2, 3, … are written in a spreadsheet with ten columns. The first row is 1 to 10, the second 11 to 20, and so on. Let r_k be the number of terms of the arithmetic sequence $a_1 = 4$, $a_n = a_{n-1} + 3$, $n \geq 2$ that appear in the k^{th} row of the array. Find an expression for r_k.

11. An anti-arithmetic sequence $a_1, a_2, \dots, a_n, \dots$ is defined with first term a and the sum of any two consecutive terms a constant s. Find the general term of the sequence.

Communication

12. A sequence of points in the plane $P_1, P_2, \dots, P_n, \dots$ is defined recursively with P_1 given by (1, 0) and $P_{n+1} = (1, 2) + P_n$, $n \geq 2$.

a. Sketch the position of the first four points in the sequence.

b. Prove that every point in the sequence lies on a straight line.

Part C

13. Consider the function $f(x) = 2x(1 - x)$ and define a sequence $x_1, x_2, \ldots, x_n, \ldots$ recursively, with x_1 between 0 and 1, and $x_n = f(x_{n-1})$, $n \geq 2$.

    a. If $x_1 = 0.5$, evaluate x_n for all $n \geq 2$

    technology

    b. Suppose $x_1 = 0.3$. Use a calculator or spreadsheet to evaluate $x_2, x_3, \ldots, x_{15}$. Describe the behaviour of the sequence as n increases.

    c. Repeat part **b** for a variety of values for x_1, $0 \leq x_1 \leq 1$. What can you conclude from this investigation?

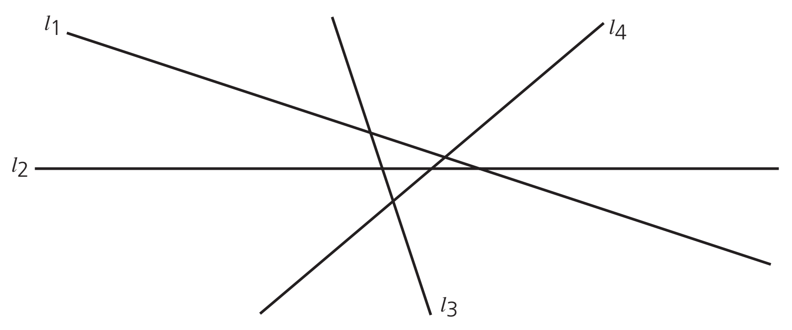
    d. Repeat parts **b** and **c** if $f(x) = 3x(1 - x)$.

14. A sequence of lines $l_1, l_2, \ldots, l_n, \ldots$ is drawn in the plane so that no two are parallel and no three intersect in a common point. The first four lines are shown on the diagram. Let $s_1, s_2, \ldots, s_n, \ldots$ be a sequence of numbers corresponding to the number of distinct regions that are created by the lines.

    a. Evaluate s_1, s_2, s_3 and s_4.

    b. Explain why $s_n = s_{n-1} + n$, $n \geq 2$.

    c. Show that $s_n = \frac{n^2 + n + 2}{2}$, $n \geq 2$ satisfies the recursive definition in **b**.



15. A sequence of functions with general term $f_n(x)$ is defined recursively. The first term $f_1(x) = g(x) = \frac{x}{1 + x}$ and $f_n(x) = g(f_{n-1}(x))$, $n \geq 2$.

    a. Find an expression for $f_n(x)$ and show that it satisfies the recursion.

    b. Sketch a graph of $y = g(x)$ and the line $y = x$ on the same set of axes.

    c. Find all values of x_0 for which the sequence $f_1(x_0), f_2(x_0), \ldots, f_n(x_0), \ldots$ is constant.

    d. Are there any values x_1 so that the sequence $f_1(x_1), f_2(x_1), \ldots, f_n(x_1), \ldots$ alternates in value?

16. In the sequence of Sierpinski triangles, let a_n be the number of equilateral triangles in the n^{th} diagram.

    a. Evaluate a_1, a_2, a_3, and a_4.

b. Explain why the sequence $a_1, a_2, \ldots, a_n, \ldots$ satisfies the recursion $a_n = 3a_{n-1}, n \geq 2$.

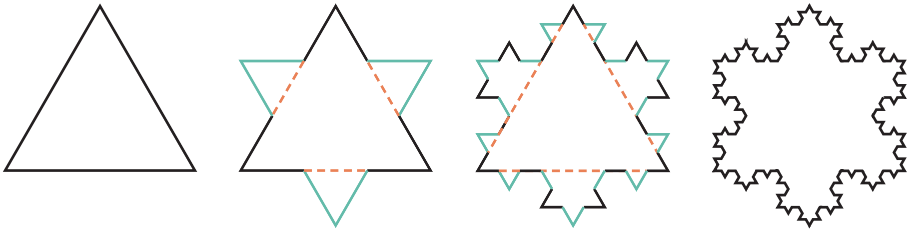
c. Verify that $a_n = 3^{n-1}, n \geq 2$ satisfies this recursive definition.

d. If the length of the side in the original triangle is 1, develop a recursive definition for the length of the side b_n of the small equilateral triangles in the n^{th} diagram.

e. Find a formula for the general term b_n.

f. Find a formula for A_n, the fraction of the area of the original triangle remaining in the nth diagram. What happens as n gets large?

17. The Koch snowflake is produced by recursively operating on a sequence of diagrams. The first term in the sequence D_1 is an equilateral triangle with side length 1. The second term D_2 is formed by replacing the middle third of each side by two other line segments of the same length as shown. D_n is produced by replacing the middle third of all lines in D_{n-1} by two line segments of equal length. D_1, D_2, D_3, and D_4 are shown below. Determine what happens to the area and perimeter of D_n as n gets large.



Section 12. 2 — Partial Sums and Sigma Notation

In the introduction to this chapter, we reviewed the formulae for the sum of the first n terms of both arithmetic and geometric sequences. For instance, if $t_n = a + (n - 1)d$, $n = 1, 2, \ldots$ is an arithmetic sequence, then the sum is

$$S_n = t_1 + t_2 + \ldots + t_n = \frac{n}{a}[2a + (n - 1)d]$$

Creating a new sequence in which each term is the sum of the terms of a given sequence is so common that we use a special language and notation to describe it. For any sequence of numbers $t_1, t_2, \ldots, t_n, \ldots$, we determine the nth term S_n of the new sequence to be the sum of the first n terms of the given sequence. That is,

$$S_1 = t_1, S_2 = t_1 + t_2, \ldots, S_n = t_1 + t_2 + \ldots + t_n$$

The terms S_n of the new sequence are called the **partial sums** of the terms of the original sequence.

We use a special notation called **sigma notation** to express S_n compactly. We write

$$S_n = \sum_{i=1}^{n} t_i$$

Note that $\sum$ is the upper case Greek letter sigma (the equivalent of our letter s), chosen to remind us that we are constructing a sum. The i is called the **index** and the values below and above $\sum$ give the **range of the index** in the summation. The notation tells us to add the terms t_i as the index i ranges from 1 to n in steps of 1. That is, S_n is the sum of the terms t_i for $i = 1, 2, \ldots, n$. There is nothing magical about sigma notation—it is simply a convenient way to express the sum of terms of a sequence.

EXAMPLE 1

For a sequence with general term t_n, write the following sums in $\sum$ notation.

a. The sum of the first 50 terms.
b. $t_4 + t_5 + \ldots + t_{25}$
c. The sum of the first 25 odd-numbered terms.

Solution

a. The sum is $t_1 + t_2 + \ldots + t_{50} = \sum_{i=1}^{50} t_i$

b. The sum is $t_4 + t_5 + \ldots + t_{25} = \sum_{j=4}^{25} t_j$

c. The sum is $t_1 + t_3 + t_5 + \ldots t_{49} = \sum_{k=1}^{25} t_{2k-1}$

The index in sigma notation can be any letter. Its purpose is to tell us what terms to include in the sum. By custom, we use lower case letters such as i, j, and k for the index.

EXAMPLE 2 Expand each of the following.

a. $\sum_{i=1}^{8} i^2$ b. $\sum_{j=50}^{56} ar^j$ c. $\sum_{k=0}^{5} \binom{10}{k}$ d. $\sum_{i=2}^{7} 3$

Solution

a. $\sum_{i=1}^{8} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2$

b. $\sum_{j=50}^{56} ar^j = ar^{50} + ar^{51} + ar^{52} + ar^{53} + ar^{54} + ar^{55} + ar^{56}$

c. $\sum_{k=0}^{5} \binom{10}{k} = \binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} + \binom{10}{5}$

d. $\sum_{i=2}^{7} 3 = 3 + 3 + 3 + 3 + 3 + 3$

EXAMPLE 3 Evaluate each of the following.

a. $\sum_{k=1}^{30} k$ b. $\sum_{i=2}^{6} i^2$ c. $\sum_{j=2}^{7} 3$

Solution

a. $\sum_{k=1}^{30} k = 1 + 2 + 3 + 4 + \ldots. + 30$

$= \frac{30 \cdot 31}{2}$

$= 465$

b. $\sum_{i=2}^{6} i^2 = 2^2 + 3^2 + 4^2 + 5^2 + 6^2$

$= 4 + 9 + 16 + 25 + 36$

$= 90$

c. $\sum_{j=2}^{7} 3 = 3 + 3 + 3 + 3 + 3 + 3$

$= 18$

EXAMPLE 4 Evaluate each of the following.

a. $\sum_{i=1}^{7} 3i$ b. $3\sum_{i=1}^{7} i$

Solution

a. $\sum_{i=1}^{7} 3i = 3 + 6 + 9 + 12 + 15 + 18 + 21$ (an arithmetic sequence)

$= \frac{7}{2}[6 + (7 - 1)3]$

$= \frac{7}{2}[24]$

$= 84$

$$\text{b. } 3\sum_{i=1}^{7} i = 3[1 + 2 + 3 + 4 + 5 + 6 + 7]$$
$$= 3 \times \frac{7 \cdot 8}{2}$$
$$= 84$$

The rules of arithmetic apply to summation notation. If $u_1, u_2, \ldots, u_n, \ldots$ and $v_1, v_2, \ldots, v_n, \ldots$ are two sequences and a is a constant, then

$$\sum_{i=1}^{n} au_i = a\sum_{i=1}^{n} u_i \qquad \sum_{i=1}^{n}(u_i + v_i) = \sum_{i=1}^{n} u_i + \sum_{i=1}^{n} v_i$$

The expression on the left uses the fact that a is a common factor for each term in the sum. The expression on the right reflects the fact that when we add, we can switch the order of the terms without changing the total.

EXAMPLE 5 For the arithmetic sequence defined by $f(n) = 4n - 3$, $n \geq 1$, determine the sum of the first 40 terms.

Solution

$$S_{40} = \sum_{i=1}^{40} (4i - 3)$$
$$= \sum_{i=1}^{40} 4i - \sum_{i=1}^{40} 3$$
$$= 4\sum_{i=1}^{40} i - \sum_{i=1}^{40} 3$$
$$= 4[1 + 2 + 3 + \ldots + 40\} - [3 + 3 + 3 + \ldots + 3]$$
$$= 4\frac{40 \cdot 41}{2} - 40 \times 3$$
$$= 3280 - 120$$
$$= 3160$$

EXAMPLE 6 If $a_1, a_2, \ldots, a_n, \ldots$ are the terms of an arithmetic sequence with first term 7 and common difference 2, express $s_n = \sum_{i=1}^{n} a_i$ in terms of $\sum_{i=1}^{n} i$, the sum of the natural numbers from 1 to n.

Solution

The ith term of the sequence is $a_i = 7 + (i - 1)2 = 5 + 2i$. The partial sum s_n is

$$S_n = \sum_{i=1}^{n} a_i$$
$$= \sum_{i=1}^{n} (5 + 2i)$$
$$= \sum_{i=1}^{n} 5 + \sum_{i=1}^{n} 2i$$

$$= 5n + 2\sum_{i=1}^{n} i$$

For any arithmetic sequence with general term $a_i = a + (i - 1)d = (a - d) + id$, we can see that $\sum_{i=1}^{n} a_i = n(a - d) + d\sum_{i=1}^{n} i$. The advantage of this derivation is that you can work out the sum of the terms of an arithmetic sequence by remembering only the sum of the natural numbers.

$$\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$$

EXAMPLE 7

For any sequence $t_1, t_2, \ldots, t_n, \ldots,$, define a new sequence with general term $d_n = \mathrm{t}_{n+1} - t_n$, the difference between consecutive terms of the original sequence. Evaluate $\sum_{j=1}^{n} d_j$ in terms of the original sequence.

Solution

The sum of the differences is

$$\sum_{j=1}^{n} d_j = \sum_{j=1}^{n} (t_{j+1} - t_j) = \sum_{j=1}^{n} t_{j+1} - \sum_{j=1}^{n} t_j$$

The two sums on the right side are almost the same. The first includes all terms from the second to the $(n + 1)^{\text{st}}$. The second includes all terms from the first to the n^{th}. Hence, taking the difference, the terms $t_2, \ldots, t_n$ disappear so we have

$$\sum_{j=1}^{n} d_j = t_{n+1} - t_1$$

This result is useful, and there are numerous other interesting sequence results that can be determined by using similar techniques.

EXAMPLE 8

By using the result of Example 7 with the sequence defined by $t_n = n^3$, determine an expression for $\sum_{i=1}^{n} i^2$.

Solution

Using the result from Example 7, we have $\sum_{j=1}^{n} d_j = (n + 1)^3 - 1^3 = n^3 + 3n^2 + 3n$.

$d_j = (j + 1)^3 - j^3 = 3j^2 + 3j + 1$ Substituting,

$$\begin{aligned} n^3 + 3n^2 + 3n &= \sum_{j=1}^{n} d_j \\ &= \sum_{j=1}^{n} (3j^2 + 3j + 1) \\ &= 3\sum_{j=1}^{n} j^2 + 3\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 \end{aligned}$$

Since $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ and $\sum_{j=1}^{n} 1 = n$, we can substitute to get an expression for $\sum_{j=1}^{n} j^2$.

$$3\sum_{j=1}^{n} j^2 = n^3 + 3n^2 + 3n - 3\frac{n(n+1)}{2} - n$$

$$= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2}$$

$$= \frac{2n^3 + 3n^2 + n}{2}$$

Then $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

We can evaluate many other partial sums using the same method.

Do not be fooled by summation notation. In any situation, if you have doubts about the notation, write the sum out explicitly. The exercises will give you lots of practice using sigma notation.

Exercise 12.2

Part A

Knowledge/Understanding

1. Express the following sums in sigma notation.

a. $1 + 2 + 3 + \ldots + 25$

b. the sum of the first fifteen terms of the sequence with general term $t_n = 3 \times 2^{n-1}, n \geq 1$

c. the sum of the squares of the positive integers from 50 to 100

d. the sum of the first 30 odd-numbered terms of the sequence with general term $t_n, n \geq 1$

2. Expand the following sums expressed in sigma notation.

a. $\sum_{i=1}^{10} i$ b. $\sum_{j=3}^{7} (j+1)^2$ c. $\sum_{i=1}^{n} \frac{1}{i}$

Communication

3. On a statistics test, several students simplified the expression by cancelling an x_i from the numerator and denominator on the left side, as shown

$$\frac{\sum_{i=1}^{5} x_i^2}{\sum_{i=1}^{5} x_i} = \sum_{i=1}^{5} x_i$$

where $x_1, \ldots, x_5$ were a sequence of numbers. Explain why the students all lost marks for this simplification. (*Hint:* Pick any five numbers for the x_i.)

Part B

4. Consider three sequences $u_1, u_2, \ldots, u_n, \ldots$; $v_1, v_2, \ldots, v_n, \ldots$; and $w_1, w_2, \ldots, w_n, \ldots$. Construct three such sequences of numbers with $n = 3$ to see if the following is true.

$$\frac{\sum_{i=1}^{n} u_i v_i}{\sum_{i=1}^{n} u_i w_i} = \frac{\sum_{i=1}^{n} v_i}{\sum_{i=1}^{n} w_i}$$

Can we cancel the u_i from the expression on the left?

5. Suppose that $a_1, a_2, \ldots, a_{100}$ is a sequence of 100 numbers with average value A.
   a. Write an expression for A using sigma notation.
   b. Consider the new sequence with terms defined by $b_n = a_n - A$ for $1 \le n \le 100$.
   c. Show that $\sum_{n=1}^{100} b_n = 0$.

6. For the geometric sequence with general term $g_n = 3^{-(n-1)}$, $n \ge 1$, determine
   a. the sum of the first ten terms
   b. the sum of the first n terms for all $n \ge 1$
   c. $\sum_{j=1}^{8} g_{2j-1}$

7. Consider the geometric sequence with general term $g_n = ar^{n-1}$.

   Evaluate $\dfrac{\sum_{i=1}^{60} g_i}{\sum_{i=61}^{120} g_i}$.

Thinking/Inquiry/ Problem Solving

8. Find the sum of the first n terms of the sequence 9, 99, 999,..... Use the result to find the sum of the first n terms of the sequences 1, 11, 111, ... and *k, kk, kkk,* … where k represents a digit from 2 to 8.

9. Use the fact that $\binom{n}{k}$ is the number of binary sequences of length n with exactly k 1s to evaluate $\sum_{k=0}^{n} \binom{n}{k}$.

Knowledge/Understanding

10. Suppose that $t_n = a + (n-1)d,\ n \ge 1$ and $s_n = ar^{n-1},\ n \ge 1$ are an arithmetic and a geometric sequence. Evaluate $\sum_{i=1}^{n} (-1)^i t_i$ and $\sum_{i=1}^{n} (-1)^i s_i$.

Thinking/Inquiry/Problem Solving

11. Consider the arithmetic sequence with general term $a_n = 1 + 3(n-1),\ n \ge 1$ and the geometric sequence with general term $g_n = 2^n,\ n \ge 1$.

a. Evaluate $\sum_{i=1}^{10} g_i$ and $\sum_{i=1}^{10} a_i$.

b. Define a new sequence t_n by selecting the $g_n{}^{th}$ term from the arithmetic sequence. That is, $t_1 = a_2,\ t_2 = a_4,\ t_3 = a_8$, and so on. Find $\sum_{i=1}^{10} t_i$.

12. Consider the sequence $t_n = n^2,\ n \ge 1$ and the sequence of differences $d_n = t_{n+1} - t_n,\ n \ge 1$. Use the method of Example 8 to find a simple expression for the sum of the first n natural numbers $\sum_{i=1}^{n} i$.

13. Consider the sequence $t_n = \frac{1}{n},\ n \ge 1$ and the sequence of differences $d_n = t_{n+1} - t_n,\ n \ge 1$. Use the method of Example 8 to find a simple expression for $\sum_{i=1}^{n} \frac{1}{i(i+1)}$.

Part C

14. Find a simple expression for $\sum_{i=1}^{n} a_i^2$, where $a_i = 2i - 3$.

15. Suppose that $t_1, .., t_n$ are the first n terms of a sequence and we construct a new sequence with $s_j = \sum_{j=1}^{n} t_j,\ j = 1, \ldots, n$. Prove that $\sum_{j=1}^{n} jt_j = \sum_{j=1}^{n} s_j$.

16. Use the result of question **15** to evaluate $\sum_{j=1}^{n} jg_j$, where g_j is the general term of a geometric sequence.

17. Use the method of Example 8 to show that the sum of the cubes of the first n natural numbers is $\left(\sum_{j=1}^{n} i\right)^2$, the square of the sum of the first n natural numbers.

Section 12.3 — Mathematical Induction

In this section, we introduce a new method of proof with the peculiar name **mathematical induction.** You will recall that we compared induction and deduction in the first chapter. Curiously, mathematical induction is a deductive method of proof.

You can see the motivation for mathematical induction in the following activity.

Suppose a sequence is defined by the recursion $t_n = 2t_{n-1} + 1$, $n \geq 2$, with $t_1 = 1$. Find an expression for the general term and prove that the expression is correct.

We start by evaluating the first few terms of the sequence. We have $t_1 = 1$, $t_2 = 2(1) + 1 = 3$, $t_3 + 2(3) + 1 = 7$, $t_4 = 2(7) + 1 = 15$, $t_5 = 2(15) + 1 = 31$.

Looking at the pattern, the formula $t_n = 2^n - 1$ gives the correct answers for $n \leq 5$. However, we have no idea if this is the correct formula for larger values of n. For instance, we can easily check that the less obvious formula

$$t_n = \frac{12 - 18n + 23n^2 - 6n^3 + n^4}{12}$$

also gives the correct answers for the first five terms. We use mathematical induction to verify that $t_n = 2^n - 1$ is the correct formula.

Suppose that we want to show that every term in a sequence of statements S_1, S_2, …, S_n, …is true. There are three steps in the proof using mathematical induction.

Step 1: Show that the first statement in the sequence S_1 is true.
Step 2: Given that any one term in the sequence is true, prove that the next term is also true.
Step 3: Combine the results of the first two steps to conclude that every statement in the sequence is true.

Why do these steps actually prove that all the statements are true? Step 1 verifies that the first statement is true. Once we have finished Step 2, we use it recursively in Step 3. Since we have shown that S_1 is true, then Step 2 tells us that S_2 is also true. Now, given that S_2 is true, and using the result from Step 2 again, S_3 is also true, and so on.

Now we use mathematical induction to prove that we have found the correct formula in the above recursion.

EXAMPLE 1 If $t_n = 2t_{n-1} + 1$, $n \geq 2$ with $t_1 = 1$, prove that $t_n = 2^n - 1$ for all $n \geq 1$.

Proof

Step 1: Substituting $n = 1$, we get $2^1 - 1 = 1$, so the formula is correct for $n = 1$.

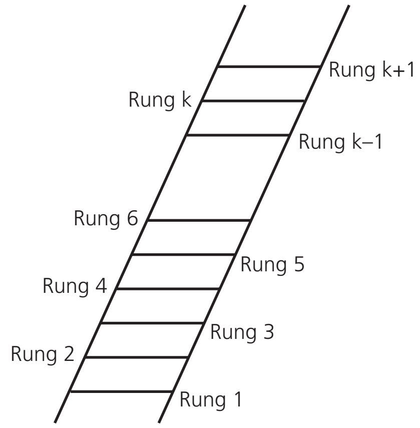
Step 2: Now suppose it is given that the formula is correct for some value of n, say $n = k$. That is, we are given that $t_k = 2^k - 1$. To complete Step 2, we need to show that the formula is correct for the next value, $n = k + 1$. Using the recursion, we have

$$\begin{aligned} t_{k+1} &= 2t_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

which is correct. This completes Step 2.

Step 3: Since the formula is correct for $n = 1$ (Step 1), we use Step 2 recursively to conclude that the formula is correct for $n = 2$, then $n = 3$, and so on for all n. The proof is complete.

One analogy for proof by mathematical induction is the task of devising a method to reach any rung on an infinite ladder.



Step 1 gets us on the first rung. Step 2 gives us a way to go from any rung to the next. That is, if we have already reached rung 4, we can use Step 2 to get to rung 5. Step 3 puts Steps 1 and 2 together to give us a way to reach any rung on the ladder, one rung at a time.

A common use of mathematical induction is to prove the correctness of a formula for the partial sums of the terms of a sequence. That is, if we are given a sequence $a_1, a_2, \ldots, a_n, \ldots$ with partial sum $s_n = a_1 + \ldots + a_n$, $n \geq 1$ and a postulated formula for s_n, we use induction to prove that the formula is correct.

EXAMPLE 2 Prove that $1 \times 1! + 2 \times 2! + \ldots + n \times n! = (n + 1)! - 1$ for all $n \geq 1$.

Solution

Note that the left side of the expression is the partial sum s_n of the terms of the sequence with general term $a_n = n \times n!$ Our goal is to prove that $s_n = (n + 1)! - 1$ for all $n \geq 1$.

Step 1: If $n = 1$, then $s_1 = a_1 = 1$ and $(1 + 1)! - 1 = 1$, so the formula is correct for $n = 1$.

Step 2: Suppose the formula is correct for some value of n, say $n = k - 1$. Then

$$\begin{aligned} s_k &= s_{k-1} + a_k \\ &= (k! - 1) + k \times k! \\ &= k!(1 + k) - 1 \\ &= (k + 1)! - 1 \end{aligned}$$

as required. The formula is correct for k if it is correct for $k - 1$.

Step 3: Given that the formula is correct for $n = 1$, we apply Step 2 recursively to conclude that the formula is correct for all values of $n \geq 1$.

Some Observations about Proof by Induction

1. In any proof using mathematical induction, we can do Steps 1 and 2 in either order. Step 1 is usually easier, so we do it first. Note that if Step 1 fails, then S_1 is false, and we can stop. It is not true that every term in the sequence of statements is true. You might wonder if it is possible to carry out Step 2 if Step 1 fails. See Exercise 10 for an example of this.
2. To many students, a proof by mathematical induction feels wrong because, in Step 2, it seems that we assume the truth of what we are trying to prove. The ladder analogy is helpful. What we are trying to prove is that we can climb to any rung. What we assume in Step 2 is that we have reached a particular rung. Then, we prove that we can climb to the next rung. Step 1 puts us on the first rung. Combining the two steps and using recursion (Step 3) gives us a way to climb to every rung.
3. Step 3 is the concluding step and is the same in every application of mathematical induction. Steps 1 and 2 depend on the particular result we are trying to prove. We always write Step 3 to demonstrate to the reader the logic of the proof.
4. A key component of Step 2 is to find a connection between consecutive terms in the sequence of statements. If the problem is to verify a formula for a partial sum, that is, to prove for all n that

$$t_1 + t_2 + \ldots + t_n = f(n)$$

where $f(n)$ is a postulated function, then we can connect consecutive statements, since

$$t_1 + t_2 + \ldots + t_n = (t_1 + t_2 + \ldots + t_{n-1}) + t_n$$

For other problems (see Example 3), it is more difficult to make the connection.

5. For Step 2, we show that if any one term in the sequence of statements is true, then so is the next term. We can take the given term as the $(k-1)^{st}$ term and show that the k^{th} term is true. If it is more convenient, we can take the k^{th} term as given and then show that the $(k+1)^{st}$ term is true. For partial sums, it is easier to start from the $(k-1)^{st}$ term.
6. We only use mathematical induction to establish the truth of a sequence of statements. This means that the statements can be ordered. There is a first statement, second statement, and so on. Sometimes students are tempted to try to use mathematical induction to prove theorems such as

   Prove that $x^2 + 2x - 3 \geq 0$ *for all real values of* $x \geq 1$.

   Mathematical induction cannot be used here because for a given value of x, there is no next term. We cannot express this theorem as a sequence of statements.
7. Mathematical induction can be used only when we know the answer. For example, if we are not given a formula for a partial sum, we cannot use mathematical induction. The reason for the name *mathematical induction* is that we often guess the answer based on an observed pattern, and use this method of proof to verify that our guess is correct.
8. Mathematical induction is only one method of proof. For a given sequence of statements, there are often simpler and more illustrative methods.

EXAMPLE 3 Use mathematical induction to prove that $a_n = \frac{n^3 - n}{3}$ is an integer for all $n \geq 1$.

Solution

For Step 1, when $n = 1$, $a_1 = \frac{1^3 - 1}{3} = 0$, which is an integer. The first statement in the sequence is true. For Step 2, we assume that the $(n-1)^{st}$ statement is true. That is, we are given that $a_{n-1} = \frac{(n-1)^3 - (n-1)}{3}$ is an integer. With this given information and a bit of algebra, we now prove that a_n is also an integer. Note that if we can show that $a_n - a_{n-1}$ is an integer, then we can conclude that $a_n = a_{n-1} + (a_n - a_{n-1})$ is an integer. We have

$$\begin{aligned} a_n - a_{n-1} &= \frac{n^3 - n}{3} - \frac{(n-1)^3 - (n-1)}{3} \\ &= \frac{n^3 - n - (n-1)^3 + n - 1}{3} \\ &= \frac{n^3 - n - n^3 + 3n^2 - 3n + 1 + n - 1}{3} \\ &= \frac{3n^2 - 3n}{3} \\ &= n(n-1) \end{aligned}$$

which is always an integer for n an integer. Hence, we have shown that if a_{n-1} is an integer, then so is a_n.

Applying Step 3, we know the first statement is true. Repeatedly applying Step 2, we conclude that $\frac{n^3 - n}{3}$ is an integer for any value of $n \geq 1$.

There is a much easier way to prove the result in Example 3.

EXAMPLE 3 (REVISITED)

Prove that $a_n = \frac{n^3 - n}{3}$ is an integer for all $n \geq 1$.

Solution

We can factor the numerator of a_n to get

$$a_n = \frac{n(n^2 - 1)}{3} = \frac{(n - 1)n(n + 1)}{3}$$

After factoring, we can see that the numerator is the product of three consecutive integers. One of these must be divisible by 3, so the given expression is always an integer. The proof is complete.

In summary, mathematical induction is a method of proof that can be used to show that a sequence of statements is true. We start by verifying the first term in the sequence directly (Step 1). For Step 2, we prove that any term in the sequence is true *if we are given that the previous statement is true.* Then in Step 3, we repeatedly use Step 2 to conclude that the second statement is true, since the first is, the third is true since the second is, and so on. We conclude that each statement is true.

Exercise 12.3

Part B

Communication

1. Consider the sequence $t_1, t_2, \ldots, t_n$ defined recursively by $t_1 = 1$ and $t_n = n \times t_{n-1}$. We want to prove that $t_n = n!$ for all $n \geq 1$ using mathematical induction. Here we break the proof down into its simplest pieces.
   a. What is the first step in the proof?
   b. Is it true that $t_1 = 1!$?
   c. Write clearly what is given for Step 2.
   d. Complete Step 2.
   e. Using the results from **b** and **d**, explain why $t_2 = 2!$.
   f. Explain how the results from **b** and **d** can be used to conclude that $t_n = n!$ for all $n \geq 1$.

2. Consider the arithmetic sequence with n^{th} term $t_n = a + (n - 1)d$, where a and d are given constants. Use mathematical induction to prove that the sum of the first n terms, $S_n = t_1 + t_2 + \ldots + t_n$, is given by the formula $S_n = \frac{n[2a + (n - 1)d]}{2}$ for all $n \geq 1$.

   a. For Step 1, show that the formula is correct if $n = 1$.

   b. For Step 2, establish a connection between S_{n-1} and S_n. Explain why $S_n = S_{n-1} + t_n$.

   c. For Step 2, we are given that the formula is correct for the sum of the first $n - 1$ terms. That is, we are given that $S_{n-1} = \frac{\{n-1\}[2a + (\{n-1\} - 1)d]}{2}$. Show that the formula is correct for the sum of the first n terms using this given information.

   d. Combine Steps 1 and 2 to conclude that the formula is correct for all values of $n \geq 1$.

Knowledge/Understanding

3. Using mathematical induction, prove that $1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$ for all $n \geq 1$.

4. Using mathematical induction, prove that the following statements are true for all $n \geq 1$.

   a. $1 + 3 + \ldots + (2n - 1) = n^2$

   b. $1 + 4 + \ldots + (3n - 2) = \frac{n(3n - 1)}{2}$

   c. $1^2 + 2^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$

   d. $(1)(2) + (2)(3) + \ldots + (n)(n + 1) = \frac{n(n + 1)(n + 2)}{3}$

   e. $\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \ldots + \frac{1}{(n)(n + 1)} = \frac{n}{n + 1}$

   f. $1 \times 1! + 2 \times 2! + \ldots + n \times n! = (n + 1)! - 1$

5. If $t_n = \frac{n^3 + 3n^2 + 2n}{6}$, $n \geq 1$, use mathematical induction to prove that t_n is an integer for all $n \geq 1$. (*Hint:* show that $t_{n+1} - t_n$ is an integer for all n.)

Knowledge/Understanding

6. The sequence a_n satisfies the recursion $a_n = a_{n-1} + (n - 1)^2$, $n \geq 2$ with $a_1 = 0$. Use mathematical induction to prove that $a_n = \frac{n(n - 1)(2n - 1)}{6}$ for all $n \geq 1$.

7. Consider the sequence defined by $e_n = n^2 - n$ for $n \geq 1$.

   a. Use mathematical induction to prove that every term in the sequence is even.

   b. Construct a simpler proof.

8. Prove that $n(n + 5)$ is even for all $n \geq 1$.

9. Prove that the sum of the cubes of three consecutive positive integers is always divisible by 9.

10. A sequence with general term t_n is defined recursively by $t_n = n \times t_{n-1}$ for $n > 1$ with $t_1 = 2$.

   a. Suppose we try to prove that $t_n = n!$ for all $n \geq 1$. Show that Step 2 of the induction proof works but that Step 1 fails. What do you conclude about the statement $t_n = n!$?

   b. Prove that $t_n = 2(n!)$ for all $n \geq 1$.

11. Suppose A is the matrix $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$. Prove that $A^n = \begin{pmatrix} 1 & 0 \\ na & 1 \end{pmatrix}$.

12. Consider expanding the product of n factors $(a_1 + b_1)(a_2 + b_2) \ldots (a_n + b_n)$ into a sum of terms. For example, $(a_1 + b_1)(a_2 + b_2) = a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2$.

   a. Prove that the number of terms in the expansion is 2^n for all $n \geq 1$.

   b. Prove that every term in the expansion contains one letter from each of the n factors.

13. Consider the sequence with general term defined by the recursion $v_n = 2v_{n-1} + n$ for all $n > 1$ with $v_1 = 2$. Prove that $v_n = 5 \times 2^{n-1} - 2 - n$ for all $n \geq 1$.

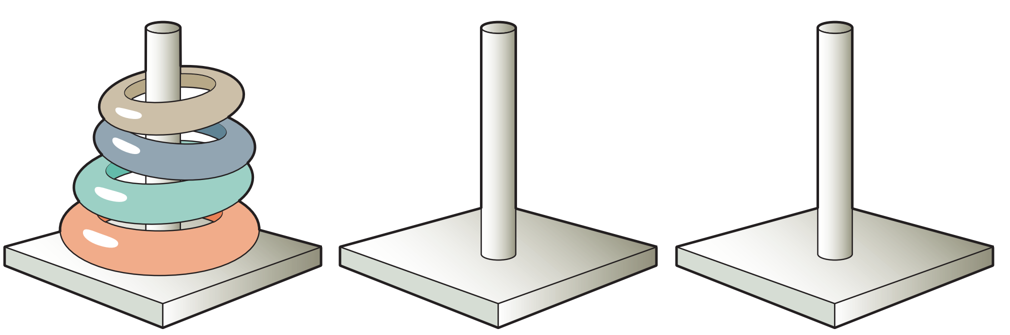
Communication

14. Mathematical induction can be used to provide a formal proof to statements that are obvious but difficult to prove. Here is an example. Suppose we have two sequences of numbers, $a_11, a_2, \ldots, a_n, \ldots$, and $b_1, b_2, \ldots, b_n, \ldots$. Use mathematical induction to prove that $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$ for all $n \geq 1$.

15. Suppose a is a positive number. Prove using mathematical induction that $(1 + a)^n \geq 1 + na$ for all $n \geq 1$.

Application

16. In a puzzle called the Tower of Hanoi, there are a number of disks, each with a hole in the centre, that can fit over one of three pegs. The disks are all of different radii and are initially placed on one peg in decreasing size from bottom to top. The initial position for the game with four disks is shown in the diagram.



The object of the game is to move all the disks to another peg. Disks are moved one at a time from one peg to another with the only restriction being that a larger disk can never be placed on a smaller one. Let M_n be the minimum number of moves (a move corresponds to moving one disk from one peg to another), where n is the number of disks.

a. Show that $M_1 = 1$ and $M_2 = 3$.

b. Use mathematical induction to verify that $M_n = 2^n - 1$.

Part C

technology

17. Drawing conclusions from observed patterns can be dangerous. Therefore, we need the formality of mathematical induction. Consider the sequence defined by $t_n = 41 + n + n^2$ for $n \geq 1$.

    a. Using a spreadsheet or other program, show that $t_1, t_2, \ldots, t_{39}$ are prime numbers.

    b. Can you conclude that t_n is prime for all $n \geq 1$?

    c. Verify that t_{40} is not prime.

18. Suppose n straight lines are drawn in the plane so that no two are parallel and no three are concurrent (i.e., no three intersect in a common point). Let T_n be the number of distinct regions that are formed.

    a. Show that $T_1 = 2$, $T_2 = 4$, $T_3 = 7$.

    b. Show that $T_n = T_{n-1} + n$.

    c. Use mathematical induction to show that $T_n = \frac{n^2 + n + 2}{2}$ for all $n \geq 1$.

19. Construct two proofs to verify that $2n^3 + 3n^2 + n$ is divisible by 6 for all $n \geq 1$.

20. Consider the sequence $f(n) = \frac{1}{n}$ for $n \geq 1$ and the partial sum $s(n) = \sum_{i=1}^{n} f(i)$.

    a. How many terms are in the sequence $2^{k-1}, 2^{k-1} + 1, 2^{k-1} + 2, \ldots, 2^k - 1$?

    b. Verify for any positive integer k that $f(2^{k-1}) + f(2^{k-1} + 1) + \ldots + f(2^k - 1) > \frac{1}{2}$ (you do not need mathematical induction to show this).

    c. Prove that $s(2^k) > \frac{k}{2}$ for $k \geq 1$.

    d. Discuss the behaviour of $s(n)$ as n gets large.

21. Consider the sequence $g(n) = \frac{1}{n^2}$ for $n \geq 1$ and the partial sum $t(n) = \sum_{i=1}^{n} g(i)$.

    Prove that $t(n) \leq 2 - \frac{1}{n}$ for all $n \geq 1$. (Remarkably, $t(n)$ approaches $\frac{\pi^2}{6}$ as n gets large.)

Section 12.4 — The Binomial Theorem

In this section, we look at the **Binomial Theorem,** one of the most famous results in mathematics. The Binomial Theorem was known by the Islamic mathematician al-Karaji in the 10th century and was rediscovered by the British scientist Sir Isaac Newton in the 17th century. The idea behind the theorem is very simple. The algebraic expression $a + b$ is called a binomial. You will remember or can quickly deduce that

$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

The binomial theorem gives the expansion in terms of powers of a and b for the expression $(a + b)^n$ for any positive integer n.

At first glance, expanding $(a + b)^n$ looks like a difficult algebra problem. It is remarkable that we can turn this algebra problem into a simple counting problem. The trick is to look carefully at how the multiplication of binomial factors works. Consider, for example, expanding the product of three binomials $(a_1 + b_1)(a_2 + b_2)(a_3 + b_3)$.

Here we include subscripts in the binomials so that we can see what happens in the expansion. A bit of work gives

$$(a_1 + b_1)(a_2 + b_2)(a_3 + b_3) = (a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2)(a_3 + b_3)$$
$$= a_1a_2a_3 + a_1a_2b_3 + a_1b_2a_3 + a_1b_2b_3 + b_1a_2a_3 + b_1a_2b_3 + b_1b_2a_3 + b_1b_2b_3$$

The key observation is that each of the terms on the right contains exactly one letter from each of the three binomials. For example, $a_1b_2a_3$ has a from the first and third binomials and b from the second. The first line of the expansion shows that the same result is also true for the product of two binomials. To prove the Binomial Theorem, we first need to show that this observation is true for the product of any number of binomials.

THEOREM

In the expansion of the n binomial factors $(a_1 + b_1)(a_2 + b_2)\ldots(a_n + b_n)$, each term contains exactly one symbol from each factor.

Proof

We use mathematical induction. For Step 1, the statement is true for $n = 1$ and as shown above, also true for $n = 2$ and $n = 3$. Now suppose we are given that in the product of $n - 1$ binomials, each term contains exactly one symbol from each factor. We write $(a_1 + b_1)(a_2 + b_2) \ldots (a_{n-1} + b_{n-1}) = t_1 + t_2 + t_3 + \ldots + t_m$, where we are given that each term t_i has exactly one symbol from each factor. Then

$$(a_1 + b_1)(a_2 + b_2) \dots (a_n + b_n) = [(a_1 + b_1)(a_2 + b_2)\dots(a_{n-1} + b_{n-1})](a_n + b_n)$$
$$= [t_1 + t_2 + \dots + t_m](a_n + b_n)$$
$$= t_1a_n + t_1b_n + \dots + t_ma_n + t_mb_n$$

Each term in this expansion has exactly one symbol from each factor. This completes Step 2. Combining the two steps, we conclude that the theorem is true for all $n \geq 1$.

Now we look at expanding $(a + b)^n$. We start with the special case $n = 3$ to see how the proof works. We have $(a + b)^3 = (a + b)(a + b)(a + b)$. Taking an a or b from each factor, we get terms of the forms a^3, a^2b, ab^2, and b^3. The question is how many? We get a^2b by selecting b from one of the factors and a from the other two. Since there are three factors, we can select one b in $\binom{3}{1}$ ways. Having selected the b, there is only one way to choose as and that is from every factor b is not taken from. Each of these selections gives a term a^2b and, combining like terms, the coefficient of a^2b in the expansion is $\binom{3}{1}$. Similarly, the coefficients of a^3, ab^2, and b^3 are $\binom{3}{0}$, $\binom{3}{2}$, and $\binom{3}{3}$, respectively. Thus, we have $(a + b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$. We use the same process to expand $(a + b)^n$.

THE BINOMIAL THEOREM

The expansion of $(a + b)^n$ is

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \dots + \binom{n}{k}a^{n-k}b^k + \dots + \binom{n}{n}b^n$$

Proof

In the expansion of the Binomial Theorem $(a + b)^n = (a + b)(a + b) \dots (a + b)$. The terms in the expansion are formed by picking either a or b from each of the n factors. A term with k bs and $(n - k)a$s can be simplified to $a^{n-k}b^k$. There are $\binom{n}{k}$ ways of selecting k bs from the n factors. For each of these, there is exactly one way to select the $(n - k)a$s. Hence, there are $\binom{n}{k}$ terms in the expansion that simplify to $a^{n-k}b^k$. Combining these terms we get $\binom{n}{k}a^{n-k}b^k$. Thus the expansion becomes

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \dots + \binom{n}{k}a^{n-k}b^k + \dots + \binom{n}{n}b^n$$

The general term is $\binom{n}{k}a^{n-k}b^k$.

The first term in the expansion corresponds to selecting no bs and, therefore, a from each of the n factors. The last term corresponds to choosing all bs from every factor and, therefore, no as.

Because the symbols for n choose k appear so prominently in the expansion, they are often called the **binomial coefficients.** Here are some examples of the use of the Binomial Theorem.

EXAMPLE 1 Find the coefficients of x^2, x^8, and x^k in the expansion of $(1 + x)^{20}$.

Solution

From the binomial theorem we have

$$(1 + x)^{20} = \binom{20}{0} + \binom{20}{1}x + \binom{20}{2}x^2 + \ldots + \binom{20}{k}x^k + \ldots + \binom{20}{20}x^{20}$$

The coefficients of x^2, x^8, and x^k are $\binom{20}{2}$, $\binom{20}{8}$, and $\binom{20}{k}$, respectively.

EXAMPLE 2 Determine the coefficient of x^4 in the expansion of $(2 - 3x)^7$.

Solution

The general term in the expansion is $\binom{7}{k}(2)^{7-k}(-3x)^k$. For the coefficient of x^4, we set $k = 4$. The coefficient of x^4 is $\binom{7}{4}2^3(-3)^4 = 2^3 \cdot 3^4\binom{7}{4}$.

Expressions such as this can always be simplified to a numerical value. Unless a numerical value is specifically requested, however, it is usual to leave the answer as given here.

EXAMPLE 3 If the coefficient of x^3 in the expansion of $(1 + 2x)^n$ is 160, find the value of n.

Solution

The general term in the expansion is $\binom{n}{k}(2x)^k = \binom{n}{k}2^k x^k$.

For the coefficient of x^3, $k = 3$, so $\binom{n}{3}2^3 = 160$ or $\binom{n}{3} = 20$.

Then $\frac{n(n-1)(n-2)}{3!} = 20$

$$n(n-1)(n-2) = 120$$

Solving, by trial and error, we get $n = 6$.

EXAMPLE 4 Prove that the sum of the binomial coefficients is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$.

Solution

This identity is easy to prove using the Binomial Theorem. We have

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n$$

Substitute $x = 1$ to get $(1 + 1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + .. + \binom{n}{n}$, as required.

Exercise 28 at the end of this section gives another way to prove this result.

The last example gives one of many identities involving the binomial coefficients. Note that in general, the sum of the coefficients in the expansion of $(a + bx)^n$ is found by setting $x = 1$ to give $(a + b)^n$.

If we write down the coefficients in the expansion of

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n$$

in a series of rows, one for each value of n, we get an array of numbers called Pascal's triangle. It is named after its discoverer, the great French mathematician and philosopher Blaise Pascal (1623–1662). Here are the first five rows.

$$\begin{array}{c} \binom{0}{0} \\ \binom{1}{0} \quad \binom{1}{1} \\ \binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\ \binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\ \binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \end{array}$$

We set the first row to $\binom{0}{0} = 1$ to preserve the symmetry. This means that the fifth row, for example, corresponds to the binomial coefficients with $n = 4$. If we replace the binomial coefficients by their values, we get

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \end{array}$$

Here we have listed the first seven rows of Pascal's triangle. Several patterns are apparent. Two sides of the triangle have only 1s, reflecting the fact that $\binom{n}{0} = \binom{n}{n} = 1$ for all values of $n \geq 0$. One number in from the edge, we see the sequence of natural numbers, since $\binom{n}{1} = n$. The sum of the $(n + 1)^{st}$ row is 2^n, which restates the result of Example 4. In the interior of the triangle, we see that every number is the sum of the two closest numbers in the row above.

For example, $\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$. We show below that this observation is generally true. We provide two proofs. In Exercise 25, you are asked to consider another.

EXAMPLE 5

Pascal's Identity Theorem

If $n \geq 1$, $1 \leq k \leq n$, then $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof 1

Suppose we have $n + 1$ distinct letters, one of which is A. There are $\binom{n+1}{k}$ subsets of size k that can be constructed from these letters. Now we count these subsets in a second way. Every subset of size k does not contain A (Case 1) or does contain A (Case 2). The two cases are disjoint. There are $\binom{n}{k}$ subsets of size k that can be formed from the n letters other than A. For case 2, A is in the subset. The remaining $k - 1$ letters can be selected in $\binom{n}{k-1}$ ways. Hence the number of subsets of size k that can be constructed from $n + 1$ distinct letters is $\binom{n}{k} + \binom{n}{k-1}$. Equating the two counts gives the identity.

Proof 2

Consider $(1 + x)^{n+1} = (1 + x)(1 + x)^n$

$$= (1 + x)\left[\binom{n}{0} + \ldots + \binom{n}{k-1}x^{k-1} + \binom{n}{k}x^k + \ldots + \binom{n}{k}x^n\right]$$

From the binomial theorem, the coefficient of x^k in the expansion of $(1 + x)^{n+1}$ is $\binom{n+1}{k}$. On the right side, the coefficient of x^k is $1 \times \binom{n}{k} + 1 \times \binom{n}{k-1}$.

Since the two expressions are equal, the coefficients of every power of x are equal. It follows that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

In the following exercises and problems, we explore more consequences of the Binomial Theorem.

Exercise 12.4

Part A

Knowledge/ Understanding

1. Write out the complete expansion of the following binomial expressions.

a. $(a - b)^5$ b. $(1 + x)^6$ c. $(x - 2y)^4$

d. $(1 - s^2)^5$ e. $\left(x + \frac{1}{x}\right)^7$ f. $(z^3 - b^2)^5$

2. a. How many terms are there in the expansion of $(1 + x)^{25}$?

b. In unsimplified form, write the coefficients of x^4 and x^{23}.

c. Determine the coefficient of x^{17}.

d. Determine the coefficient of x^3 and x^{10}.

e. Determine the largest coefficient.

f. Determine the sum of the coefficients.

3. Determine the general term for each of the following binomial expansions.

a. $(1 - 3x)^{15}$ b. $(5 - 10x)^{20}$

c. $(a + x^2)^{13}$ d. $\left(a^3 + \frac{1}{a^2}\right)^{10}$

4. In the expansion of $(2 - x)^{12}$, find

a. the term containing x^k

b. the coefficient of x^{10}

5. Determine the coefficient of x^4 in the expansion of $\left(2x - \frac{3}{x}\right)^8$.

6. Determine the sum of the coefficients in the expansion of

a. $(1 - 3x)^4$ b. $(1 - 5x + 2x^2)^5$

Part B

Communication

7. Explain why the sum of the coefficients in the expansion $(1 - x)(1 + x)^n$ is 0 for all values of n.

Knowledge/ Understanding

8. Evaluate the following.

$$\binom{6}{0}\left(\frac{1}{3}\right)^6 + \binom{6}{1}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \binom{6}{2}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + \binom{6}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + \binom{6}{4}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 +$$
$$\binom{6}{5}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^5 + \binom{6}{6}\left(\frac{2}{3}\right)^6$$

9. Determine the first four terms in the expansion of $(1 + x + x^2)(1 - x^2)^8$ in ascending powers of x.

10. Determine the coefficient of x^3 in $f(x) = (3 - 2x)(1 + x)^{12}$.

11. In the expansion of $(1 + x)^8 - (1 - 2x)^7$, determine the coefficient of x^5.

12. Determine the coefficient of x^4 in the expansion of

 a. $(2 - 3x)(1 + 3x)^6$ b. $(1 + x - x^2)(1 - 2x)^7$

13. Which term in the expansion of $\left(\frac{1}{a} - 3a^2\right)^{24}$ contains

 a. a^{-15} b. a^{10}

14. In the expansion of $(1 - x)^n$, the coefficient of x^2 is 15. Determine the value of n.

15. a. Determine the general term in the expansion of $(z^2 - 2z^5)^{10}$.

 b. Determine the middle term.

 c. Determine the coefficient of z^{41}.

 d. In which term does z^{36} occur?

 e. Does z^{30} occur in this expansion?

16. In the expansion of $(1 - x^2)(1 + x)^{2n}$, the third term is $189x^2$. Determine n.

17. In the expansion of $(a + x)^8$, the coefficient of x^7 is 24. Determine a.

Communication

18. Suppose $x \geq 0$. Explain why $(1 + x)^n \geq 1 + nx$ for all $n \geq 1$.

19. The first three terms of a binomial expansion are $1 + 21x + 189\,x^2$. Determine the function that gives this expansion.

Thinking/Inquiry/
Problem Solving

20. By making use of the fact that $(1 + x)^3(1 + x)^n = (1 + x)^{n+3}$, prove that
$$\binom{n}{r} + 3\binom{n}{r-1} + 3\binom{n}{r-2} + \binom{n}{r-3} = \binom{n+3}{r}.$$

21. Show that the coefficient of x^k in the expansion of $S(x) = 1 + (1 + x) + (1 + x)^2 + \ldots + (1 + x)^{n-1}$ is $\binom{n}{k+1}$. Use the fact that $S(x)$ is the sum of terms in a geometric sequence.

Thinking/Inquiry/
Problem Solving

22. Suppose r and s are roots of the quadratic equation $x^2 + 2x - 1 = 0$. Prove that $r^n + s^n$ is an integer for all values of $n \geq 1$.

23. Use the fact that $(1 + x)^n = (x + 1)^n$ to show that $\binom{n}{n-r} = \binom{n}{r}$ for $0 \le r \le n$.

24. Suppose you have a set of n distinct objects that you want to divide into two disjoint subsets of size r and $n - r$. Use this problem to explain why $\binom{n}{n-r} = \binom{n}{r}$ without any calculation.

25. If $n \ge 1$, $1 \le k \le n$, prove that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ algebraically by expanding and simplifying the binomial coefficients.

26. Use the expansion of $(1 - x)^{10}$ to prove that
$1 - \binom{10}{1} + \binom{10}{2} - \binom{10}{3} + \ldots - \binom{10}{9} + \binom{10}{10} = 0$

27. Use the expansion of $(1 + x)^{12} + (1 - x)^{12}$ to verify that
$\binom{12}{2} + \binom{12}{4} + \ldots + \binom{12}{12} = 2^{11} - 1$

28. In Example 4, we used the binomial expansion of $(1 + x)^n$ to prove that $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$. To see another way to prove this identity, consider the following counting problems.
    a. How many binary sequences (a sequence with each term 0 or 1) of length n can you construct?
    b. How many binary sequences of length n with exactly k 1s can you construct?
    c. Combine the results of **a** and **b** to verify the identity.

29. One diagonal (a line parallel to the edge) of Pascal's triangle begins 1, 3, 6, 10, …. Find the general term of this sequence.

Application

30. Show that the sum of all the entries in Pascal's triangle down to and including the n^{th} row is $2^n - 1$.

Part C

31. If the coefficient of x^3 in the expansion of $\left(2x + \frac{1}{x}\right)^n$ is 672, find the value of n.

32. Consider Vandermonde's identity
$\binom{m+n}{k} = \binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \ldots + \binom{m}{k}\binom{n}{0}$, where $k \le m$, $k \le n$.

a. Use the fact that $(1 + x)^{m+n} = (1 + x)^m(1 + x)^n$ to establish the identity.

b. Verify the identity by counting the number of subsets of size k that can be selected from a set of n red objects and m blue objects.

33. Suppose that you have $n + 1$ 0s and r 1s.

a. How many binary sequences can you construct that end in exactly k 1s, $0 \leq k \leq r$?

b. Prove the identity $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \ldots + \binom{n+r}{r} = \binom{n+r+1}{r}$.

34. Suppose that r_1 and r_2 are the roots of the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are rational numbers. Show that $r_1^n + r_2^n$ is rational for all $n \geq 1, n \in N$.

The Binomial Theorem is true for a negative integer exponent. The following questions will help you to see what happens if the exponent is negative.

35. The expression $\binom{-n}{k} = \frac{(-n)!}{(-n-k)!k!}$ has no meaning, since the definition of factorial requires that we use positive integers. However, recall that $\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$. Use this to prove that $\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}$.

36. Since $(1 - x)^{-1} = \frac{1}{1-x}$, we can see what the expansion of $(1 - x)^{-1}$ looks like by dividing 1 by $1 - x$. Do the division for five terms, and note that the series obtained appears to have an infinite number of terms. Repeat this division for $(1 + x)^{-1}$, $(1 - x)^{-2}$, $(1 + x)^{-2}$.

37. Let $S_1(x) = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$. Determine $S_1(x) - xS_1(x)$. From this, show that $S_1(x) = (1 - x)^{-1}$.

38. Let $S_2(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots + (n + 1)x^n + \ldots$. Determine $S_2(x) - xS_2(x)$. Use the result from the previous Question to show that $S_2(x) = (1 - x)^{-2}$.

39. From the previous questions, we can guess that the Binomial Theorem for negative exponent must give

$$(1-x)^{-n} = 1 + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + \dots + \binom{-n}{k}(-x)^k + \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k}(-x)^k$$

where $n \geq 1$ and the series has an infinite number of terms. Use $n = 2$ in this expression for five terms and compare the expansion with the one you obtained in Question 38.

40. Determine the first four terms and the general term for each of the following.

a. $(1-x)^{-1}$ b. $(1+x)^{-2}$ c. $(1-x)^{-3}$

d. $(1+x)^{-4}$ e. $(1-2x)^{-4}$ f. $(1+3x)^{-5}$

41. Question 39 establishes the first step required for an inductive proof that the Binomial Theorem is true for a negative integer exponent. For Step 2, by assuming that $S_n(x) = (1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$, use the induction method to prove that the theorem holds for all $n \geq 1$.

Key Concepts Review

You should be able to

1. interpret recursive definitions of sequences and identify a sequence by writing out a number of terms

2. express sequences recursively

3. interpret arithmetic and geometric sequences

4. simplify expressions given in $\sum$ notation

5. evaluate sequence sums expressed using $\sum$ notation

6. determine, using mathematical induction or other means, whether or not a given expression is true for all values of the variable

7. determine, using the Binomial Theorem, the value of the coefficient in any given term in a binomial expression

8. use the Binomial Theorem in establishing simple combinatorial identities

Rich Learning Link wrap-up

investigate and apply

CHAPTER 12: RECURSIVE SEQUENCES

Recursively defined sequences are used in many areas of applied mathematics where direct approaches to solving problems are not available. Sometimes they provide a natural first step toward a direct solution. For example, many people see the pattern in 1, 3, 7, 15, 31, ... as "each successive term is found by doubling and adding one." This is the sequence from the famous Tower of Hanoi puzzle. People are usually less likely to notice that the *n*th term can be found directly as $2^n - 1$.

Investigate and Apply

1. Prove, using mathematical induction, that the recursive formula $t_1 = 1$, $t_n = 2t_{n-1} + 1$, $n = 2, 3, \ldots$, is equivalent to $t_n = 2^n - 1$.
2. a. Calculate the first six terms of the sequence given by $t_1 = 1$,
   $$t_n = \frac{1}{2}\left(t_{n-1} + \frac{2}{t_{n-1}}\right),\ n = 2, 3, \ldots.$$
   b. Calculate the squares of each of the terms from part **a**.

   c. What are the terms in the sequence from part **a** doing?

   This question demonstrates an example of Newton's Method. It is a recursive method for finding successively more accurate approximations to solutions of an equation (in this case the equation is $x^2 = 2$).
3. a. Calculate the first 10 terms of the sequence $t_1 = 2\sqrt{2}$,
   $$t_n = 2^n\sqrt{2 - \sqrt{4 - \left(\frac{t_{n-1}}{2^{n-1}}\right)^2}},\ n = 2, 3, \ldots.$$
   b. Calculate the first 20 terms of the sequence $t_1 = 4$, $t_n = \left(1 - \frac{1}{(2_{n-1})^2}\right)t_{n-1}$, $n = 2, 3, \ldots.$

   c. Which sequence approaches pi faster?

INDEPENDENT STUDY

Investigate extensions to the Tower of Hanoi puzzle.

How many decimal places of pi have been computed? What method was used? What is the reason for calculating pi to so many decimal places?

(For students who have studied calculus: What is Newton's Method? Why does it work?)

What are recursive function calls in computer programming? How are they similar to recursively defined numerical sequences?

Review Exercise

1. An arithmetic sequence with general term a_n, $n \geq 1$ has the sum of the first two terms 5 and the sum of the third and fourth terms 17. Find $\sum_{i=1}^{10} a_i$.

2. Can a sequence be both arithmetic and geometric? Explain.

3. For the geometric sequence with general term $t_n = 3(-1)^n\left(\frac{1}{2}\right)^n$, $n \geq 1$, find
   a. the sum of the terms $t_1 + t_2 + \ldots + t_{99}$
   b. the sum of the odd numbered terms $t_1 + t_3 + t_5 \ldots + t_{99}$
   c. the sum of the reciprocal of the terms $\frac{1}{t_1} + \frac{1}{t_2} + \ldots + \frac{1}{t_{99}}$

4. Suppose $a_1, \ldots, a_n, \ldots$ is a sequence of positive numbers. The new sequences with general terms A_n and G_n are defined as the arithmetic average and geometric average of the first n terms $a_1, \ldots, a_n$. That is,
   $$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n}, \qquad G_n = (a_1 \times a_2 \times \ldots \times a_n)^{\frac{1}{n}}, n \geq 1$$
   a. Suppose that $a_1, a_2, \ldots, a_n, \ldots$ is an arithmetic sequence. Find expressions for A_n and G_n. Determine if the corresponding sequences are arithmetic, geometric, or neither.
   b. Repeat part **a** if $a_1, a_2, \ldots, a_n, \ldots$ is a geometric sequence.

5. A sequence of numbers x_n is defined by $x_1 = 1$ and $x_n = 2 + 3x_{n-1}$, $n \geq 2$.
   a. Evaluate the first six terms of the sequence.
   b. Using mathematical induction, prove that $x_n = 2.3^{n-1} - 1$, $n \geq 1$.

6. To model the population size of a country, let u_n be the size at the end of the nth year. The birth rate in any year is 1.2% (that is, the number of births is 0.012 times the population size at the start of the year) and the death rate is 0.8%. The emigration rate is 0.5% and the country admits 100 000 immigrants in any year. At the start of year 1, the population is 30 000 000.
   a. Establish a recursive definition for u_n.
   b. According to the model, will the population grow or shrink over time?
   c. What is the value for the number of immigrants needed to maintain a constant population?

7. A point P in the plane starts at the origin (0, 0). At any step n, the point moves by adding the vector $(2, -1)$ to the previous point. How many moves are required before the point escapes the circle $x^2 + y^2 = 400$?

8. A sequence of diagrams $D_1, D_2, \ldots$ is constructed recursively by starting with the unit square. A second square is constructed on the top $\frac{1}{3}$ of the first square as shown.

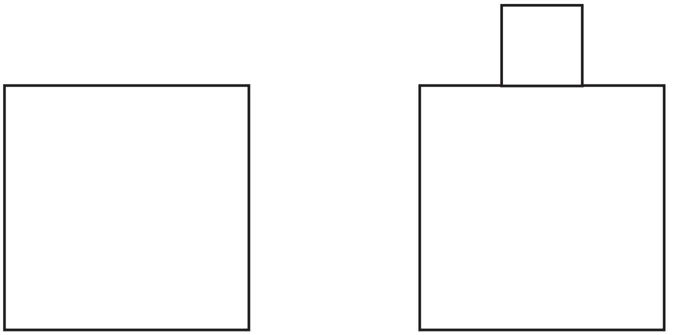


Diagram 1 Diagram 2

At each step, a new square is placed on top of the pile with one side the middle one third of the supporting square.

a. Establish recursions for the height and enclosed area of D_n.

b. What happens to the height and area as n gets large?

9. Consider the sequence with general term $t_i = (i + 1)i$, $i \geq 1$ and let the difference between two consecutive terms be $d_i = t_i - t_{i-1}$ for all $i \geq 2$.

a. Show that $d_i = 2i$.

b. Explain why $\sum_{i=2}^{n} d_i = t_n - t_1$.

c. Combine the results of **a** and **b** to prove that the sum of the first n natural numbers is $\sum_{i=2}^{n} i = \frac{(n+1)n}{2}$.

10. Using mathematical induction, prove that for all positive integers n that $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + .. + \frac{1}{(2n-1) \times (2n+1)} = \frac{n}{2n+1}$.

11. A sequence x_n is defined by $x_1 = 1$ and $x_n = 3x_{n-1} + 1$, $n \geq 2$. Use mathematical induction to prove

a. no term in the sequence is divisible by 3

b. the term x_{2k} is divisible by 4 for all $k \geq 1$

12. Using mathematical induction, show that $f(n) = 2^n - n(n+1)$ is positive for all integer values of $n \geq 5$.

13. The sequence of Fibonacci numbers with general term f_n satisfies the recursion $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$, $n \geq 3$. Use mathematical induction to prove that

$$f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$$

for all values $n \geq 1$.

14. Prove that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for all $n \geq 1$. (*Hint:* Simplify the right-hand side first.)

15. Using mathematical induction, prove that $p^n - 1$ can be factored as $(p-1)f(p)$ where $f(p)$ is a polynomial of degree $n - 1$.

16. In the expansion of $(x - y)^{12}$, find the coefficients of x^6y^6 and x^8y^8.

17. The coefficient of x^2 in the expansion of $(1 - 2x)^n$ is 24. What is n?

18. In the expansion of $\left(2x - \frac{1}{2y}\right)^{12}$, find the coefficient of $\left(\frac{x}{y}\right)^6$.

19. Use the Binomial Theorem to evaluate
$$1 - \binom{6}{1}2 + \binom{6}{2}2^2 - \binom{6}{3}2^3 + \binom{6}{4}2^4 - \binom{6}{5}2^5 + \binom{6}{6}2^6.$$

20. Use the expansion of $(1 + x)^{2n} + (1 - x)^{2n}$ to prove the identity
$$1 + \binom{2n}{2} + \binom{2n}{4} + \ldots + \binom{2n}{2n} = 2^{2n-1}.$$

Chapter 12 Test

Achievement Category	Questions
Knowledge/Understanding	all
Thinking/Inquiry/Problem Solving	5, 7, 8
Communication	1
Application	3, 6, 9

1. Explain the meaning of the expressionn $\sum_{i=1}^{99} i^2$.
2. In a geometric sequence with general term $g_n = ar^{n-1}$, $n \geq 1$, evaluate $\sum_{i=1}^{n} g_i^2$.
3. A student buys a car and receives a loan of $t_0 = \$4000$ with an interest rate of 1% per month. At the end of each month, the student pays \$200 on the loan. Let t_n, $n \geq 1$ be the amount remaining to be paid after the payment has been made at the end of the n^{th} month.
   a. Evaluate t_1.
   b. Develop a recursion for the sequence t_n.
4. If the sequence x_n, $n \geq 1$ satisfies the recursion $x_1 = 1$, $x_n = 2x_{n-1} + n - 2$, $n \geq 1$, use mathematical induction to prove that $x_n = 2^n - n$ for all $n \geq 1$.
5. Prove that $1^2 + 3^2 + \ldots + (2n - 1)^2 = \frac{4n^3 - n}{3}$ for all $n \geq 1$.
6. Consider the binomial expansion of $\left(x + \frac{1}{x}\right)^{12}$.
   a. Find the coefficient of x^4.
   b. What is the coefficient of the term that does not depend on x?
7. Evaluate $1 + 2\binom{n}{1} + 4\binom{n}{2} + \ldots + 2^n\binom{n}{n}$.
8. Suppose that a_n, $n \geq 1$ is an arithmetic sequence and $a_1 + a_3 + \ldots + a_{17} = 27$, $a_2 + a_4 + \ldots + a_{18} = 9$. Find $\sum_{i=1}^{36} a_i$.
9. Prove the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $1 \leq k \leq n - 1$.

Extending and Investigating

THE CAUCHY SCHWARZ INEQUALITY

You have learned about setting up and solving many different kinds of equations in the mathematics that you have studied in school. Mathematicians are also interested in studying inequalities. For example, you may have seen the famous geometric–arithmetic inequality for a set of positive numbers $a_1, a_2, ..., a_n$, which states that

$$\exp\left\{\frac{\sum_{i=1}^{n} \ln(a_i)}{n}\right\} \leq \frac{\sum_{i=1}^{n} a_i}{n}$$

In words, if you calculate an average by first taking the natural logarithm of each number, then finding the standard average, and, finally, exponentiating with base e, you will get an answer smaller than the standard average. This process could have interesting consequences if your mathematics teacher decided to use the geometric mean to report the class average on a test. Suddenly every student would look better against the average.

Another famous inequality is the Cauchy Schwarz inequality, actually discovered by the Russian mathematician Bunyakovskii in 1859, which applies to two sequences of numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$, with no restrictions on positivity. The inequality is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

This inequality guarantees, for example, that the correlation between two sets of numbers must fall between –1 and 1.

Surprisingly, we can prove this inequality with the simple properties of quadratic functions. Consider the function

$$f(x) = \sum_{i=1}^{n} (a_i + b_i x)^2$$

Since every term in the sum is a square, we know that $f(x) \geq 0$ for all values of x. If we expand each term in the sum and group like terms, we get

$$f(x) = \left(\sum_{i=1}^{n} a_i^2\right) + \left(2\sum_{i=1}^{n} a_i b_i\right)x + \left(\sum_{i=1}^{n} b_i^2\right)x^2$$

That is, $f(x)$ is a quadratic function in x, which is always greater than or equal to 0. Hence, the corresponding quadratic equation $f(x) = 0$ has at most one real root and the discriminant is less than or equal to 0. That is

$$\left(2\sum_{i=1}^{n} a_i b_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \leq 0$$

Simplifying and rearranging terms, we have the Cauchy Schwarz inequality.

Equality holds only if the equation $f(x) = 0$ has exactly one real root. Since $f(x) = \sum_{i=1}^{n} (a_i + b_i x)^2$ and every term in the sum is non-negative, we must have a_i the same multiple of b_i for each term if equality holds. For example, if $a_i = -2b_i$ for each i, $1 \leq i \leq n$ then $\left(\sum_{i=1}^{n} a_i b_i\right)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$.

The inequality and its properties have a geometric interpretation that you have already seen for $n = 3$. Suppose we have two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. Then we can write the Cauchy Schwarz inequality as

$$(\vec{a} \bullet \vec{b})^2 \leq |\vec{a}|^2 |\vec{b}|^2$$

or, rearranging the terms and taking the square root

$$-1 \leq \frac{\vec{a} \bullet \vec{b}}{|\vec{a}|\,|\vec{b}|} = \cos\theta \leq 1$$

In this case, the inequality is just a statement that the cosine of the angle between two vectors is always between –1 and 1. Equality is achieved when $\vec{a}$ is a multiple of $\vec{b}$ so that the angle between the vectors is 0.

This surprising connection between a purely algebraic expression and the geometric notions of angle can be extended to higher dimensions (for example, $n > 3$). Vectors are defined by adding additional coordinates, and the dot product and vector lengths are defined by making similar extensions. The cosine of the angle between two vectors is specified by exactly the same formula and is equivalent to the general version of the Cauchy Schwarz inequality.

Cumulative Review

CHAPTERS 10–12

1. A sequence of length 4 is used as a password. The elements of the sequence can be any of the digits selected from $\{0, 1, \ldots, 9\}$ and any of the 26 letters selected from $\{a, b, \ldots, z\}$. No letter or digit may be repeated. Let U be the set of all such passwords and

   A: the subset of passwords that start with a digit

   B: the subset of passwords that end with a digit

   a. Find $n(U)$.

   b. What passwords are in the subset $A \cup B$?

   c. Find $n(A \cup B)$.

2. In the set of numbers $U = \{1, 2, \ldots, 1000\}$, how many are not perfect squares?

3. A binary sequence of length 6 is formed using any number of 0s or 1s. How many such sequences are there if

   a. there are no restrictions?

   b. the sequence must have exactly three 1s?

   c. the sequence starts and ends with a different digit?

4. Find the coefficient of x^{10} in the binomial expansion of $\left(2x - \frac{1}{x}\right)^{20}$. Do not simplify your answer.

5. What is the product rule used in counting arguments?

6. In an arithmetic sequence $a_1, a_2, \ldots, a_n, \ldots, \sum_{i=1}^{10} a_i = 20$, and $\sum_{i=1}^{20} a_i = 60$. Find $\sum_{i=1}^{60} a_i$.

7. Using mathematical induction, prove that $1 + 3 + \ldots + (2n - 1) = n^2$ for all $n \geq 1$.

8. Prove Pascal's identity $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, $n \geq 1$, $1 \leq r \leq n$.

9. The sequence $x_1, x_2, \ldots$ is defined by the recursion $x_1 = 4$, $x_n = 3x_{n-1} - 2$, $n \geq 2$.

   a. Evaluate x_2, x_3 and x_4.

   b. Guess and prove a formula for x_n.

10. A mathematics club has five grade 12 students and six grade 11 students. A team of three students is to be picked for a contest. The team must have at least one grade 11 student. To count the number of possible teams, a club member uses the following argument.

    Select one of the six grade 11 students. Then pick the remaining two students from the other ten students in $\binom{10}{2} = 45$ ways, so there are $6 \times 45 = 270$ possible teams

    a. Explain why this argument is not correct.

    b. Calculate the correct answer.

11. A sequence of length 7 or 8 is formed from the nine letters of the word *Descartes*. How many of these sequences have the two letters *S* consecutively?

12. A sequence of length 5 is formed with terms selected from $\{1, 2, 3, \ldots, n\}$ where no two terms have the same value. How many different sequences are possible if the largest term must be less than or equal to r, $1 \le r \le n$?

Technology Appendix

OVERVIEW

This Technology Appendix includes a range of investigations that are referenced throughout the student text. A technology icon in the chapter margin refers you to a specific page in this appendix for investigative illustrations of concepts. This appendix uses investigative applications to support your understanding of mathematical concepts and relationships and to help you apply concepts in realistic problem situations through modelling.

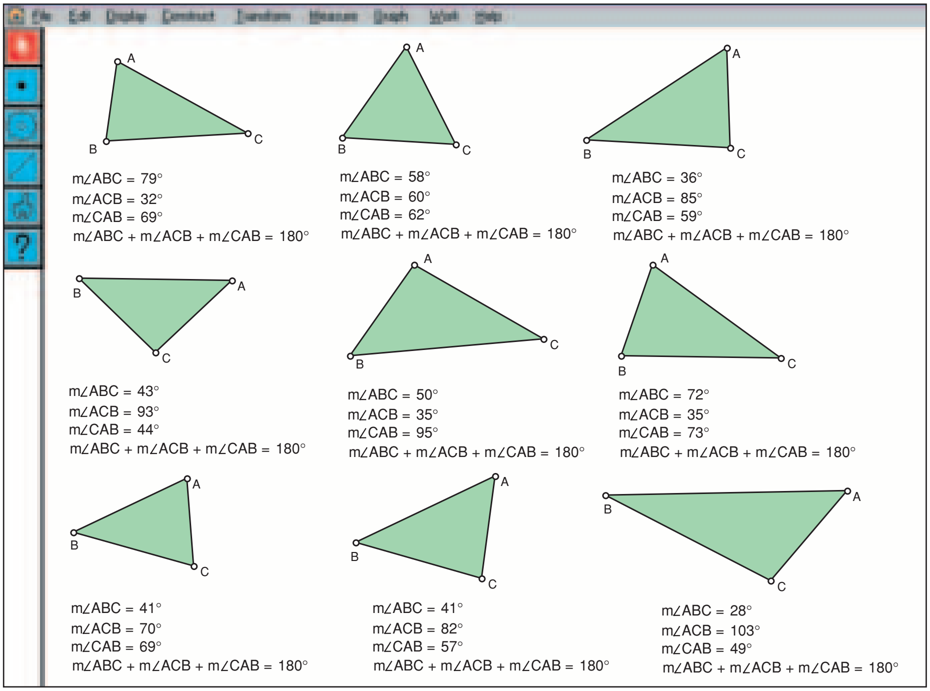
Contents

The dynamic geometry software used in this appendix is Geometer's Sketchpad by Key Curriculum Press.

SECTION 1.1 — Is Proof Enough?

Many great facts, theorems, and discoveries resulted from a person asking the question "I wonder if…." Unfortunately, the majority of the mathematics that you experience today involves proving theorems that have already been proven in the past. With the use of dynamic geometry software, however, you can prove theorems, as well as hypothesize. You can use your reasoning skills to construct an educated hypothesis, use dynamic geometry software to test your hypothesis (and watch it come to life — literally, in some cases), and finally, create a formal proof for the hypothesis.

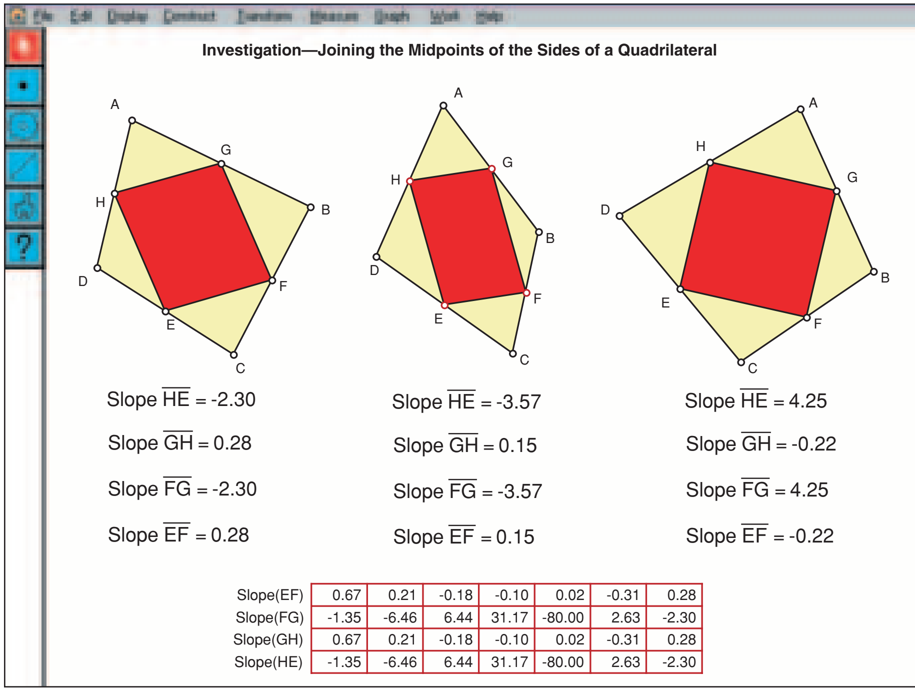
You do not create a proof by creating a sketch, manipulating it, or even animating it. A hypothesis cannot be considered a fact without a formal mathematical proof. Consider even the simplest example:



It certainly appears as though the sum of the angles in any triangle is 180º, and we may be completely confident that such a hypothesis is true, but even though we have nine cases (and we could make as many more as we wish), we cannot say we have *proved* the hypothesis. (Can you prove that the sum of the angles in any triangle is 180º? What other facts have you assumed in your proof?) Now consider the following example:

QUESTION: The midpoints of the sides of quadrilateral *ABCD* are joined to form a new quadrilateral *EFGH*. Hypothesize what type of quadrilateral *EFGH* is.

Solution: Consider the following sketches and tables created with dynamic geometry software:



Slope(EF)	0.67	0.21	-0.18	-0.10	0.02	-0.31	0.28
Slope(FG)	-1.35	-6.46	6.44	31.17	-80.00	2.63	-2.30
Slope(GH)	0.67	0.21	-0.18	-0.10	0.02	-0.31	0.28
Slope(HE)	-1.35	-6.46	6.44	31.17	-80.00	2.63	-2.30

It would be very tempting to conclude that *EFGH* is a parallelogram since all of the specific examples shown (the three sketches and the seven sets of data in the table) indicate that the opposite sides have equal slopes. But can we be sure that this is always the case? We cannot, although we can be extremely confident that our hypothesis is correct. Mathematics is built on proven facts, which lead to theorems, which are used to develop other, more powerful theorems. This is why we need to develop formal (or rigorous) proofs that are universally accepted as being true. Consider the following proof for the previous example:

$\overrightarrow{FG} = \overrightarrow{FB} + \overrightarrow{BG}$

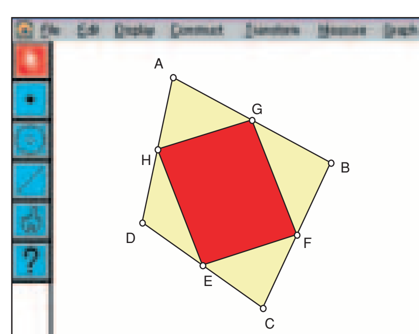
but $\overrightarrow{FB} = \overrightarrow{CF}$ and $\overrightarrow{BG} = \overrightarrow{GA}$ since G and F are midpoints

$\overrightarrow{FG} = \overrightarrow{CF} + \overrightarrow{GA}$

but $\overrightarrow{CA} = \overrightarrow{CF} + \overrightarrow{FG} + \overrightarrow{GA}$

or $\overrightarrow{CA} = \overrightarrow{FG} + \overrightarrow{FG}$

so $\overrightarrow{CA} = 2\overrightarrow{FG}$



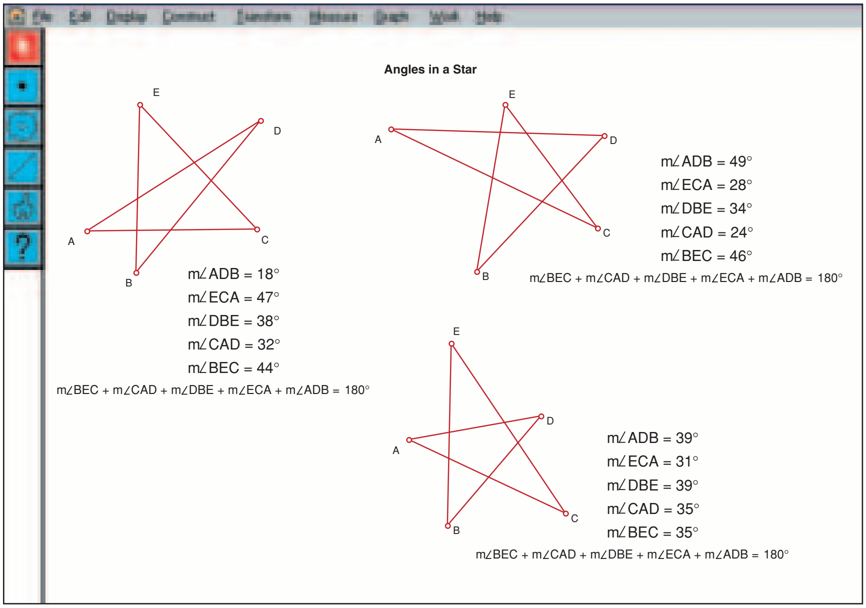
Using an identical argument, you can show that $\overrightarrow{CA} = 2\overrightarrow{EH}$, which implies that $\overrightarrow{EH} = \overrightarrow{FG}$, and by extension, *EFGH* is a parallelogram.

This proof does not depend on the position of the vertices of *ABCD*, so our diagram, although only one specific quadrilateral, actually represents *all* possible quadrilaterals when coupled with the formal proof.

It should be obvious that hypothesis and formal proof both have great value in mathematics, and that dynamic geometry software is a powerful tool for developing a viable hypothesis. By hypothesizing, you can take ownership of a theorem (rather than confirming what a mathematician discovered many years ago). By using dynamic geometry software, you can remember your ideas and findings in a dynamic and interesting form, and by proving an idea formally, you can confirm that your hypothesis is true.

SECTION 1.3 — EXAMPLE 4

Use Geometer's Sketchpad to investigate and explore star-crossed angles similar to those in Section 1.3, Example 4. There are many different ways to produce similar results using dynamic geometry software, especially with the constant development of shortcuts available to software users. Therefore, you are always encouraged to find alternative methods of creating these sketches.



1. Create a star freehand by drawing intersecting line segments in a continuous path.

2. By selecting three points at a time using the shift key, measure the angles at each vertex.

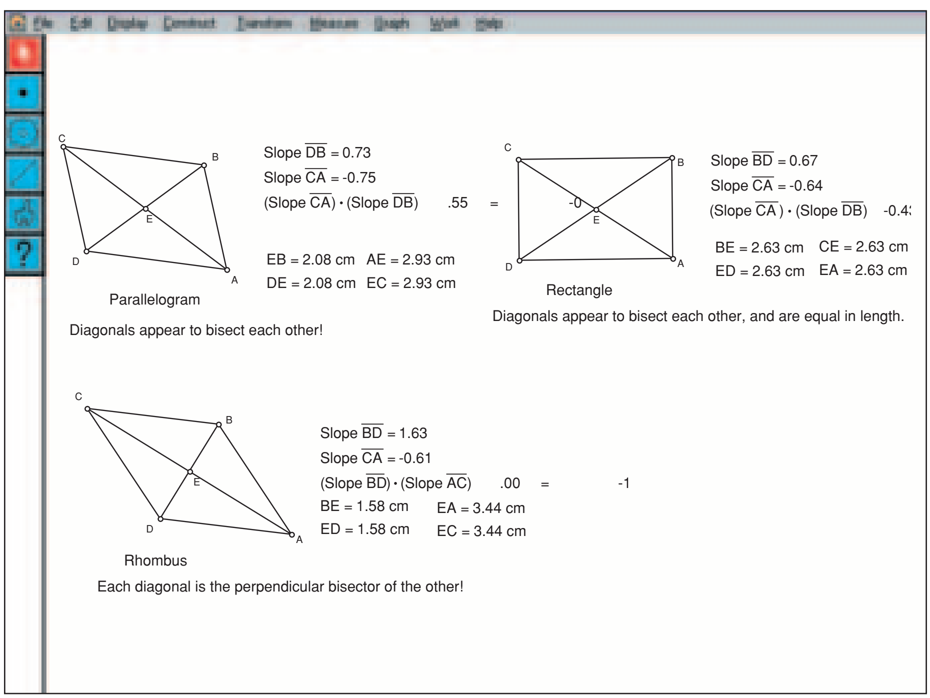
3. Under the Measure menu, choose **Calculate** to determine the sum of the angles.

4. Manipulate the vertices or make multiple copies of the sketch. What do you notice about the sum of the angles?

5. Develop a proof that this property is always true.

QUESTION 1.4.6

Use Geometer's Sketchpad to construct and explore diagonals of quadrilaterals similar to those in Exercise 1.4, Question 6.



Part 1 — Parallelogram

1. Construct a line segment and label it *AB*. Create another point, *C*. Then select point *B*, hold down Shift ▲, and select point *C*. Under the Transform menu, choose **Mark Vector** ($B \rightarrow C$), select line segment *AB* and point *A*, and then choose Transform, **Translate**. You will have to label the new point and change the label to *D* by double clicking on the default name (with the label tool). Join *BC* and *AD* to complete the parallelogram, and try moving the vertices to convince yourself that the shape remains a parallelogram when manipulated. *Note:* There are a number of ways of creating a parallelogram. Can you think of another way?

2. Construct segments *DB* and *AC*, and construct point *E*, the point at the intersection of the diagonals.

3. Select line segments *DB* and *CA* and choose Measure, **Slope**. Choose Measure, **Calculate**, and then click on "Slope *DB*" in the sketch, hit multiply (*), and then click on "Slope *CA*" in the sketch. What is the purpose of this step? What are we hoping to find?

4. Measure the side lengths (or distances between points) and make a hypothesis.

Part 2 — Rectangle

1. Construct line segment *AB*. Select point *B* and line segment *AB*, and then choose **Perpendicular Line** from the Construct menu. With the line selected, choose Construct, **Point on Object**, and then move the new point along the line to an appropriate position. Select the line and choose Display, **Hide Line**.

2. Select point *B* and then point *C*. Under Transform, choose **Mark Vector** ($B \rightarrow C$), select line segment *AB* and point *A*, and then choose Transform, **Translate**. Join *AD* to complete the rectangle, and try moving the vertices to convince yourself that the shape remains a parallelogram when manipulated. *Note:* There are a number of ways of creating a rectangle. Can you think of another way?

3. Construct segments *AC* and *BD* and the point at intersection, *E*.

4. Measure the lengths and slopes as in **Part 1** and make your hypotheses.

Part 3 — Rhombus

1. Construct line segment *AB*. Select point *A*, and then point *B*, and choose Construct, **Circle by Center + Point**. With the circle still selected, choose Construct, **Point On Object**. Label this new point *D* and join it to *A*. What is the purpose of this step? Select the circle and choose Display, **Hide Circle**.

2. Select point *A* and then point *B*. Under Transform, choose **Mark Vector** ($A \rightarrow B$), select line segment *DA* and point *D*, and then choose Transform, **Translate**. Label the new point *C* and join *BC* to complete the rhombus. Why does the figure remain a rhombus when the vertices are moved around? *Note:* There are many other ways of creating a rhombus. Can you think of any?

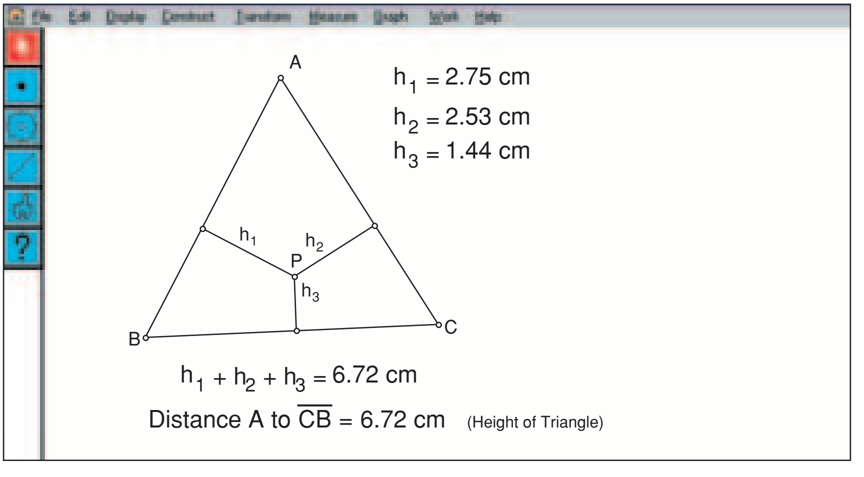
3. Construct segments *AC* and *BD* and the point at intersection, *E*.

4. Measure the lengths and slopes as in **Part 1** and make your hypotheses.

5. Is a rhombus the only quadrilateral that exhibits this property? By creating a random quadrilateral and moving the points, try to find another quadrilateral with diagonals that share some of the properties shown here.

CHAPTER 2 — REVIEW OF PREREQUISITE SKILLS, QUESTION 7

Use Geometer's Sketchpad to investigate and construct equilateral heights as an extension to Chapter 2, Review of Prerequisite Skills, Question 7.



1. Create an equilateral triangle by first creating two points, A and B. Select point A and choose Transform, **Mark Center** 'A'. Select point B and then Transform, **Rotate**, and enter 60º. Select the three points and Construct, **Segment** to complete the triangle.

2. Create a point P somewhere inside the triangle.

3. Select BC and point P, then Construct, **Perpendicular Line**. Construct the point of intersection of the new line and line segment BC, and join it to P. Select the line and use Display, **Hide Line** to simplify your sketch. Repeat this process on sides AB and AC.

4. Label the lengths from P as h_1, h_2, and h_3. Use Measure, **Calculate** to calculate the sum $h_1 + h_2 + h_3$.

5. Select point A and side BC and Measure, **Distance**.

6. Manipulate point P and the vertices of the triangle to verify that the property continues to hold.

7. Why does it work? Can you prove it?

SECTION 2.1 — CONGRUENCY IN TRIANGLES, INVESTIGATION 1

The following investigations will validate three different postulates we use when proving triangles congruent.

Open Geometer's Sketchpad and follow the instructions below.

SET UP

- Choose the Graph menu.
- Select the **Snap To Grid** option.
- In the Display menu, choose the **Preferences** option and follow these instructions:
  i) Under the **Measurements** category, select **Text Format**.
  ii) Under **Autoshow Labels for**, select **Points**.
  iii) Click OK.

INVESTIGATION 1 (SAS)

First you will construct two triangles.

1. Draw a line segment between any two points on your grid.

   How? Click on the Line Segment tool. Put the × on any point on the grid. Click and drag your mouse to another point on the grid to make a line segment.

2. Label the segment to have endpoints *A* and *B* (if they are not already).

   How? First click on the Arrow tool. Now, left-click on one endpoint of the line segment to highlight it. Go to the Display menu and click **Relabel Point**. Here, choose the letter you want to represent your point. Repeat the steps to label the other endpoint.

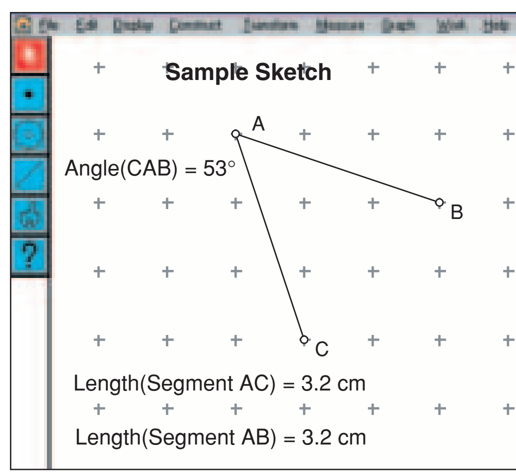
3. Measure the length of *AB*.

   How? First choose the Arrow tool. Now, left-click on your line segment. Go to the Measure menu and click **Length**. You should see **Length (Segment AB)** = (your length).

4. From point *A*, draw a second line segment to another point on the grid. Label this new endpoint *C*. Measure the length of *AC*.

5. Measure ∠*CAB*.

   How? First, click away from your diagram anywhere on the sketchpad (to deselect the line). While holding down **Shift ▲** click on point *C*, then *A*, and finally *B*. Release the shift key and go to the Measure menu and click **Angle**.



Now, let's create a second diagram identical to diagram *CAB*. Choose an area away from your first diagram and do the following:

6. Using the Line Segment tool, draw a line segment from one point to another on your grid, away from your first diagram.

7. Label the segment having endpoints *D* and *E*.

   How? First click on the Arrow tool. Now, left-click on one endpoint of the line segment to highlight it. Go to the Display menu and click **Relabel Point**. Repeat the steps to label the other endpoint.

8. Measure the length of *DE*.

   First click on the Arrow tool. Now, left-click on the line. Go to the Measure menu and click **Length**. You should see **Length (Segment DE)** = (your length).

9. Adjust the length of *DE* by clicking on the Arrow tool and grabbing one endpoint of your line segment (to do this, simply click on an endpoint and drag). Drag your endpoint along until its length is exactly the same as *AB* in your other diagram.

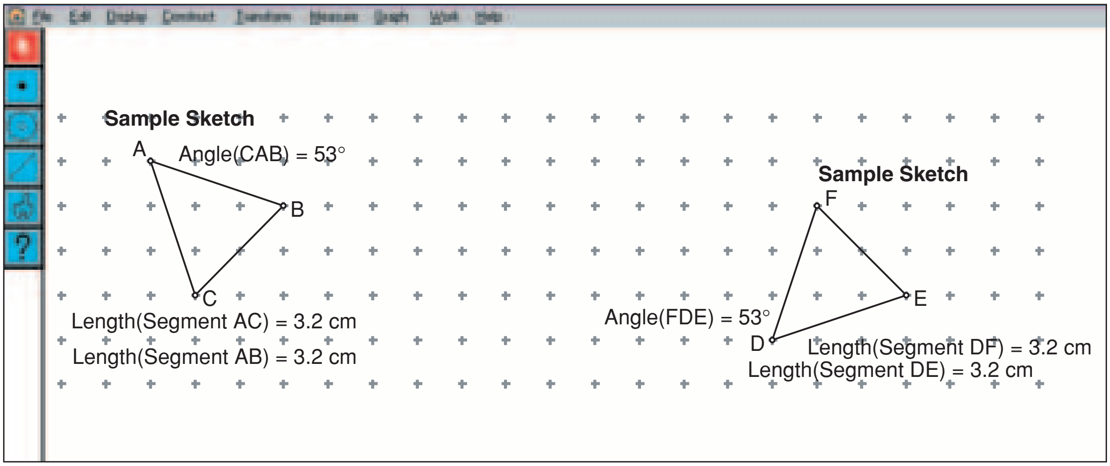
10. From point *D*, draw a second line segment. Label the endpoint *F*.

11. Now measure ∠*FDE* and line segment *DF*.

    How? To do this, first select the Arrow tool. While holding down the shift key, click on point *F*, then *D*, and finally *E*. Release the shift key and go to the Measure menu and select **Angle**. To measure *DF*, click on it and go to the Measure menu and select **Length**.

12. Deselect your points by clicking anywhere on the sketchpad. By dragging point *F* around, make the line segment *DF* the same length as *AC*, and at the same time make ∠*FDE* the same measurement as ∠*CAB*.

13. Complete the triangle in both diagrams by drawing in line segment *CB* in triangle *CAB*, and line segment *FE* in triangle *FDE*.



Answer the following questions:

1. How are the measurements of side *CB* and side *FE* related?

2. How does $\angle ACB$ relate to $\angle DEF$?

3. How does $\angle ABC$ relate to $\angle DEF$?

4. Calculate the area and perimeter of each triangle by following these instructions:

   Select all three points of your triangle. (Click on each while holding down the shift key).

   i) Go to the Construct menu and choose **Polygon Interior**.

   ii) Now go to the Measure menu and choose **Area**.

   iii) Go back to the Measure menu and choose **Perimeter**.

   iv) Do the same for your other triangle.

   a) Are these triangles congruent? Why or why not? Properly name the congruent triangles.

   b) What did you do early in your construction to guarantee your two triangles would be congruent?

   c) List all the properties that exist between two triangles that are congruent.

DID YOU KNOW?

You can also calculate the area of a triangle using Heron's formula. This formula is based on the side lengths of a triangle. The formula is given by

$A = \sqrt{s(s-a)(s-b)(s-c)}$

where a, b, c are the side lengths of your triangle and

$s = \frac{a+b+c}{2}$.

Show that Heron's formula works using your two triangles as test cases. You can use the built-in **Calculate** function in the Measure menu to assist you.

SECTION 2.1 — CONGRUENCY IN TRIANGLES, INVESTIGATION 2

INVESTIGATION 2 (SSS)

Open Geometer's Sketchpad, and then

- Choose the Graph menu.

- Select the **Snap To Grid** option.
- In the Display menu, choose the **Preferences** option and follow these instructions:
  i) Under the **Measurements** category, select **Text Format**.
  ii) Under **Autoshow Labels for**, select **Points**.
  iii) Click OK.

Construct two triangles using the following instructions:

1. Using the line segment tool, draw a line segment between any two points on your grid.

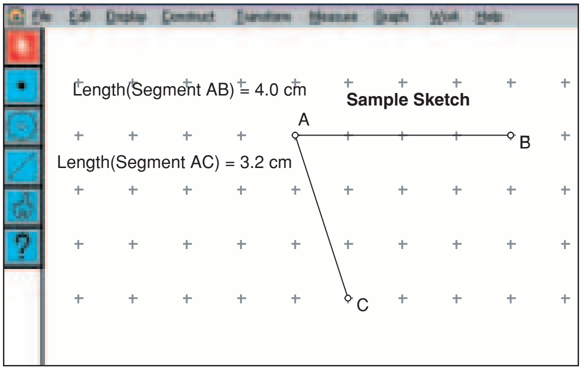
   How? Click on the Line Segment tool . Put the × on any point on the grid. Click and drag your mouse to another point on the grid to make a line segment.

2. Label the segment to have endpoints *A* and *B*.

   How? First click on the Arrow tool . Now, left-click on one endpoint of the line segment to highlight it. Go to the Display menu and click **Relabel Point**. Here, choose the letter you want to represent your point. Repeat the steps to label the other endpoint.

3. Measure the length of *AB*.

   How? First choose the Arrow tool. Now, left-click on your line segment. Go to the Measure menu and click **Length**. You should see **Length(Segment AB)** = (your length).



4. From point *A*, draw a second line segment to another point on the grid. Label this new endpoint *C*. Measure the length of *AC*.

5. Finally, connect *B* to *C* with a line segment. Measure side *BC*.

Now, let's create a second diagram identical to diagram *CAB*.

Choose an area away from your first diagram and follow these instructions:

6. Using the Line Segment tool, draw a line segment from one point to another on your grid.

7. Label the segment having endpoints *D* and *E*.

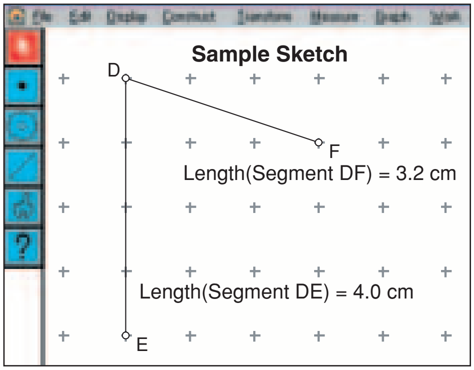
How? First click on the Arrow tool. Now, left-click on one endpoint of the line segment to highlight it. Go to the Display menu and click **Relabel Point**. Here, choose the letter you want to represent your point. Repeat the steps to label the other endpoint.

8. Measure the length of *DE*.

   How? First click on the Arrow tool. Now, left-click on the line. Go to the Measure menu and click **Length**. You should see **Length(Segment DE)** = (your length).

9. Adjust the length of *DE* by clicking on the Arrow tool and grabbing one endpoint of your line segment (to do this, simply click on an endpoint and drag). Drag your endpoint along until its length is exactly the same as *AB* in your other diagram.

10. From point *D*, draw a second line segment. Label the endpoint *F*. Measure *DF*.



11. Adjust the length of *DF* by clicking on the Arrow tool and grabbing one endpoint of your line segment (to do this, simply click on an endpoint and drag). Drag your endpoint along until its length is exactly the same as *AC* in your other diagram.

12. Finally, connect *E* to *F* with a line segment. Measure side *EF*.

13. Now, move point *F* so that *EF* equals *BC*.

14. Measure the angles $\angle BAC$ and $\angle EDF$. What do you notice about these angles? Predict what relationship might exist between

    a) $\angle ABC$ and $\angle DEF$

    b) $\angle BCA$ and $\angle EFD$

    Check your prediction by measuring these angles.

Calculate the area and perimeter of each triangle by following these instructions:
i) Select all three points of your triangle.
ii) Go to the Construct menu and choose **Polygon Interior**.
iii) Now go to the Measure menu and choose **Area**.
iv) Go back to the Measure menu and choose **Perimeter**.
v) Do the same for your other triangle.

15. Are these triangles congruent? Why or why not? Properly name the congruent triangles.

16. What did you do early in your construction to guarantee your two triangles would be congruent?

17. List all the properties that exist between two triangles that are congruent.

SOMETHING TO THINK ABOUT...

- What would be the relationship between two circles whose centre is the circumcentre of two congruent triangles?
- What would be the relationship between two triangles whose vertices are the midpoints of the sides of two congruent triangles?

SECTION 2.1 — CONGRUENCY IN TRIANGLES, INVESTIGATION 3

INVESTIGATION 3 (ASA)

Open Geometer's Sketchpad, and then
- Choose the Graph menu.
- Select the **Snap To Grid** option.
- In the Display menu, choose the **Preferences** option and follow these instructions:
  i) Under the **Measurements** category, select **Text Format**.
  ii) Under **Autoshow Labels for**, select **Points**.
  iii) Click OK.

Construct two triangles using the following instructions:

1. Using the Line Segment tool , draw a line segment between points $A(-10,1)$ and $B(-5,1)$ on your grid.

2. Label the segment to have endpoints A and B.

   How? Click on the Arrow tool . Now, left-click on one endpoint of the line segment to highlight it. Go to the Display menu, and click **Relabel Point**. Here, choose the letter you want to represent your point. Click OK. Repeat the steps to label the other endpoint.

3. Measure the length of AB.

How? Click on the Arrow tool. Now, left-click on your line segment. Go to the Measure menu and click **Length**. You should see **Length(Segment AB) =** (your length).

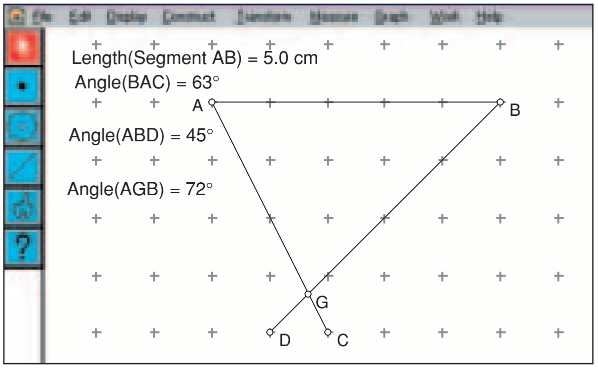
4. From point A, draw a second line segment to point $(-9, -1)$ on the grid. Label this new endpoint C. Measure $\angle BAC$.

5. From point B, draw a second line segment to point $(-7, -1)$ on the grid. Label this new endpoint D. Measure $\angle ABD$.

6. We have just created two fixed angles and a contained side. Let's complete the triangle by extending AC and BD until they intersect. *Note:* We must maintain angle BAC and angle ABD when we do this.

7. Click on the intersection of AC and BD with the Arrow tool, and label this point G.

8. Measure $\angle AGB$.



Now, let's create a second diagram that contains the same fixed angles and contained side.

9. Using the Line Segment tool, draw a line segment from $E(2, 4)$ to $F(7, 4)$ on your grid.

10. Label the segment having endpoints E and F.

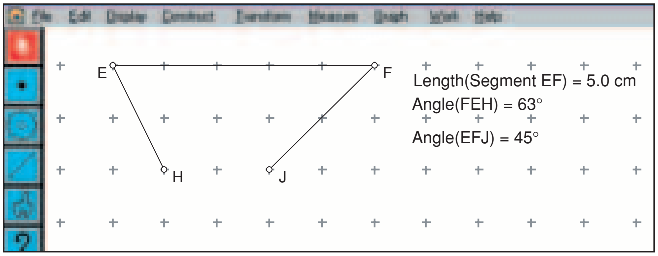
How? First click on the Arrow tool. Now, left-click on one endpoint of the line segment to highlight it. Go to the Display menu and click **Relabel Point**. Here, choose the letter you want to represent your point. Repeat the steps to label the other endpoint.

11. Measure the length of *EF*.

How? Click on the Arrow tool. Now, left-click on the line. Go to the Measure menu and click **Length**. You should see **Length(Segment EF)** = (your length).

12. Using the Line Segment tool, make an angle *FEH* that is equal to ∠*BAC*. Measure to check.

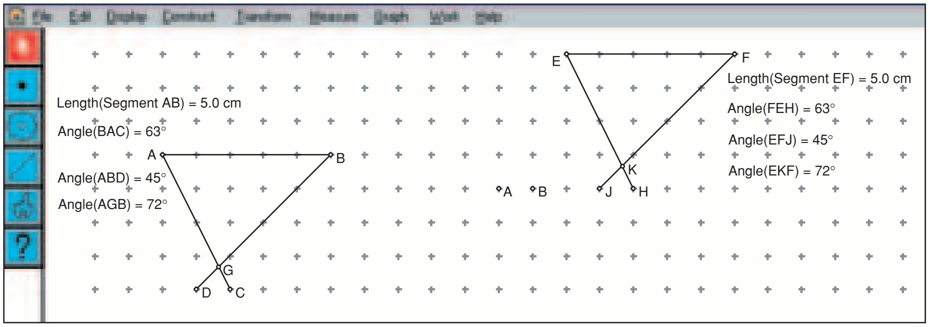
13. Make an angle *EFJ* that is equal to ∠*ABD* and going in the general direction of point *H*. Measure ∠*EFJ* to check.



14. Extend *FJ* and *EH* until they intersect. You must maintain the angles *FEH* and *EFJ* as they were.

15. Click on the intersection point of *FJ* and *EH* using the Arrow tool. Label this point *K*.

16. Measure ∠*EKF*. How does it compare with ∠*AGB*?



17. Are the two triangles you created congruent? Why or why not? Name the congruent triangles.

18. What did you do early in your construction to guarantee the triangles would be congruent?

19. Name all the corresponding congruent sides and angles in these triangles.

20. Show that the perimeter and area of each triangle is the same by constructing the **Polygon Interior** in Δ*ABG* and Δ*EFK* and then measuring the area and perimeter from the Measure menu.

21. List all the properties that exist between the two congruent triangles.

SECTION 2.1 — CONGRUENT QUADRILATERALS, INVESTIGATION

In this activity you will investigate the properties that exist between congruent quadrilaterals. At the end of this investigation you will be asked to
 i) list the properties of congruent quadrilaterals.
ii) explain how we prove that two quadrilaterals are congruent.

Open Geometer's Sketchpad, and then
- Choose the Graph menu.
- Select the **Snap To Grid** option.
- In the Display menu, choose the **Preferences** option and follow these instructions:
  i) Under the **Measurements** category, select **Text Format**.
  ii) Under **Autoshow Labels for**, select **Points**.
  iii) Click OK.

1. Draw a quadrilateral having the following coordinates as its vertices:

   $A(-10, 3)$, $B(-12, -1)$, $C(-9, -2)$, $D(-2, 0)$.

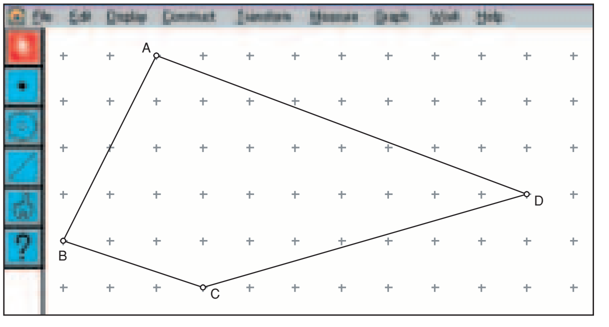
   How? Using the Line Segment tool, draw a line segment from $(-10, 3)$ to $(-12, -1)$. Draw another line segment from $(-12, -1)$ to $(-9, -2)$. Then draw a line segment from $(-9, -2)$ to $(-2, 0)$, and finally a line segment from $(-2, 0)$ to $(-10, 3)$.

2. Label the vertex at $(-10, 3)$ point A.

   Label the vertex at $(-12, -1)$ point B.

   Label the vertex at $(-9, -2)$ point C.

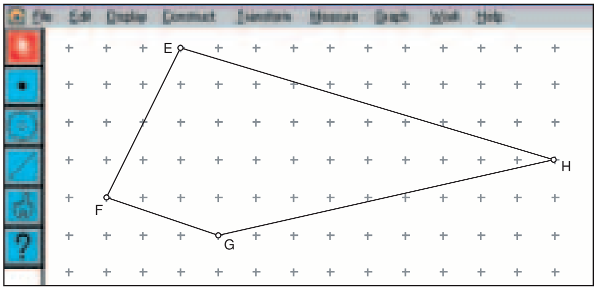
   Label the vertex at $(-2, 0)$ point D.



3. Create an identical quadrilateral on another part of the sketchpad using the coordinates

   $E(3, 3)$, $F(1, -1)$, $G(4, -2)$, $H(11, 0)$.

   Label the vertices of this quadrilateral $EFGH$.



4. Is quadrilateral $ABCD \cong$ quadrilateral $EFGH$? Why or why not?

5. Calculate the following measurements:

   a) $\angle ABC =$ ______________ and $\angle EFG =$ ______________
   b) $\angle BCD =$ ______________ and $\angle FGH =$ ______________
   c) $\angle DAB =$ ______________ and $\angle HEF =$ ______________
   d) $\angle ABC =$ ______________ and $\angle EFG =$ ______________
   e) $\overline{AB} =$ ______________ and $\overline{EF} =$ ______________
   f) $\overline{BC} =$ ______________ and $\overline{FG} =$ ______________
   g) $\overline{CD} =$ ______________ and $\overline{GH} =$ ______________
   h) $\overline{DA} =$ ______________ and $\overline{HE} =$ ______________

6. Calculate the area and perimeter of each quadrilateral using **Polygon Interior** in the Construct menu. What relationship exists between the area and perimeter of these corresponding quadrilaterals?

7. Write out the properties that exist between corresponding congruent quadrilaterals.

EXTENSION

We have just *shown* that the two quadrilaterals *ABCD* and *EFGH* are congruent. *Prove* that the quadrilaterals *ABCD* and *EFGH* are congruent.

SECTION 2.1 — EXAMPLE 1

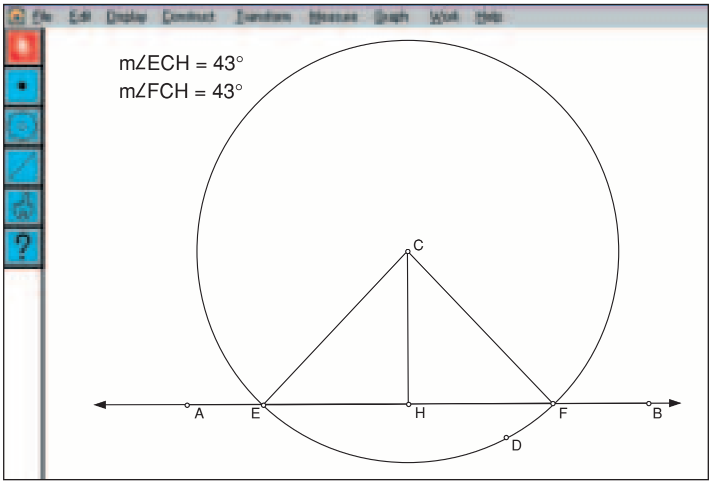
Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Section 2.1, Example 1.

1. Construct an isosceles triangle by first constructing any circle, plotting any two points on its circumference, and joining the points on the circumference to each other and then to the centre of the circle.

2. Construct a midpoint on the base side of the triangle and join the midpoint to the top vertex of the triangle, which is also the centre of the construction circle. This is the median.

3. Measure the angles that the median forms at the top vertex to show that they are equal.

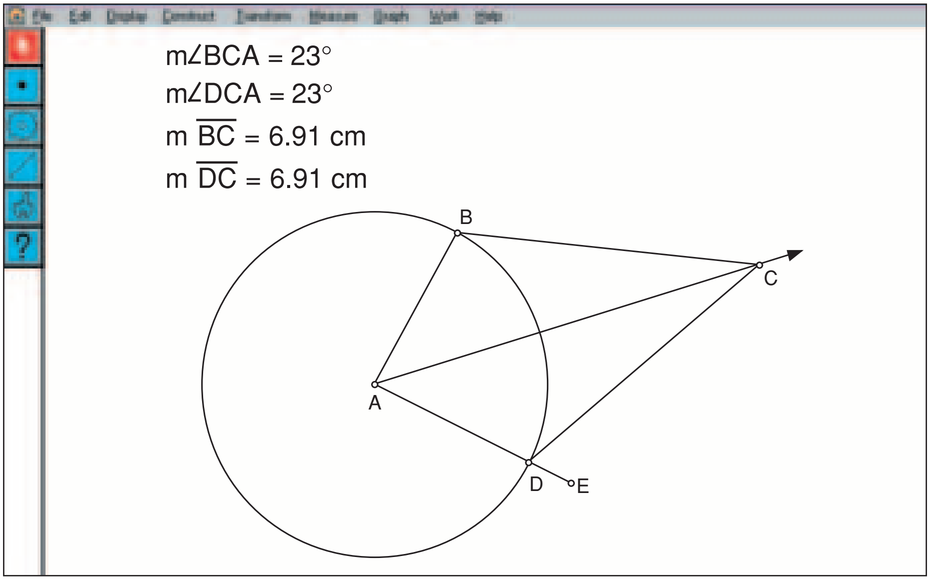
4. Drag the centre of the construction circle to show that this property remains true for many isosceles triangles.



QUESTION 2.1.7

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties required to be proven in Question 2.1.7:

1. Construct two equal sides of the quadrilateral *AB* and *AD* by constructing two radii of the same circle.
2. Construct an angle bisector of the angle between the two equal sides constructed in step **1**, and plot any point *C* on that angle bisector. Join that point to the other two points on the construction circle to form the required quadrilateral.
3. Measure the lengths of the two sides *BC* and *DC* just constructed to show that they are equal. Measure the two angles formed by the diagonal *AC* and vertex *C* to show that they are also equal.
4. Drag point *C* to show that these properties remain true for many such quadrilaterals.



QUESTION 2.1.11

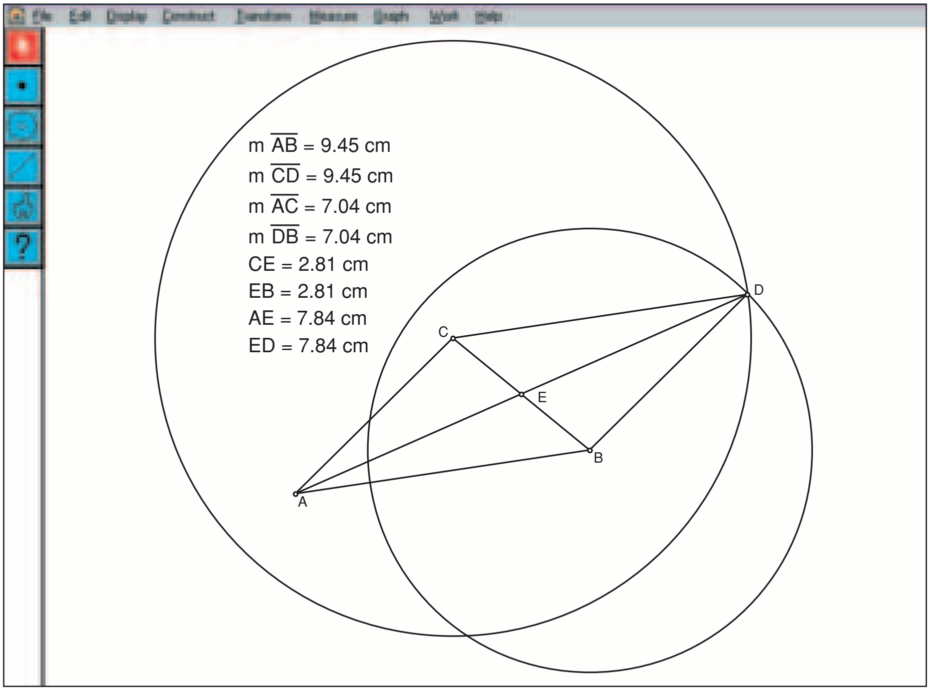
Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties required to be proven in Question 2.1.11.

1. Construct any two joined sides of a quadrilateral using **Segment** in the Construct menu.

2. Construct two circles with radii the lengths of the segments in step **1** and centred on the unattached endpoints of these segments. The point of intersection of these circles is the opposite vertex of the required quadrilateral. Plot this point, and join it to the other endpoints to form the quadrilateral with opposite sides equal.

3. Construct the diagonals of the quadrilateral, and plot the point of intersection of the diagonals. Measure the length of each part of the diagonals to show that the diagonals bisect each other.

4. Drag any vertex of the quadrilateral to show that the properties remain true for many such quadrilaterals.



SECTION 2.2 — PARALLELOGRAM AREA PROPERTY THEOREM

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Section 2.2, Parallelogram Area Property Theorem.

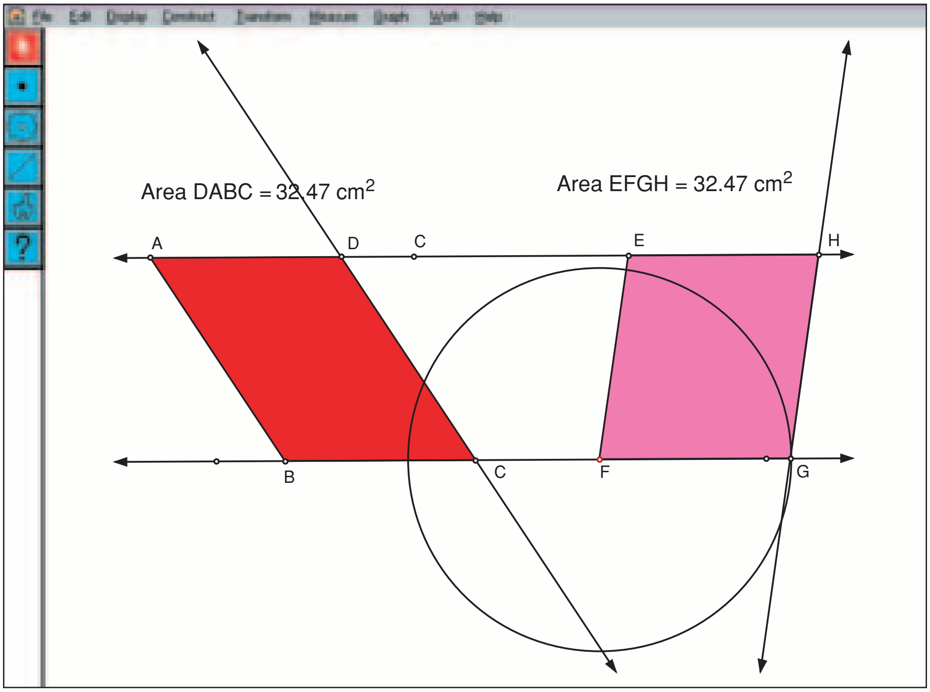
1. Construct any two parallel lines, one above the other, by first constructing a line, a point above the line, and a line parallel to the first line through the plotted point.

2. Construct any line segment *BC* on the lower line, and join the endpoint *B* of this segment to any point *A* on the upper parallel line to form side *AB*. Construct a line parallel to *AB* that passes through the other endpoint *C*. Plot the point of intersection of this line and the upper parallel line and label it *D*. This completes the construction of the first parallelogram *ABCD*.

3. To create the other parallelogram, plot another point *F* on the lower parallel line. Create base *FG* equal in length to *BC* using a construction circle with centre *F* and radius *BC*. Create the remaining sides of this parallelogram FGHE using the same method as in step **2**.

4. Construct the polygon interior of each parallelogram by selecting all of its vertices and using **Polygon Interior** in the Construct menu. The area will appear shaded. Select each area and measure it using **Area** in the Measure menu. Verify that the two areas are equal.

5. Check that this property remains true for many such parallelograms by dragging one of the vertices of the first parallelogram or any parallel line.



SECTION 2.2 — EXAMPLE 2

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Section 2.2, Example 2.

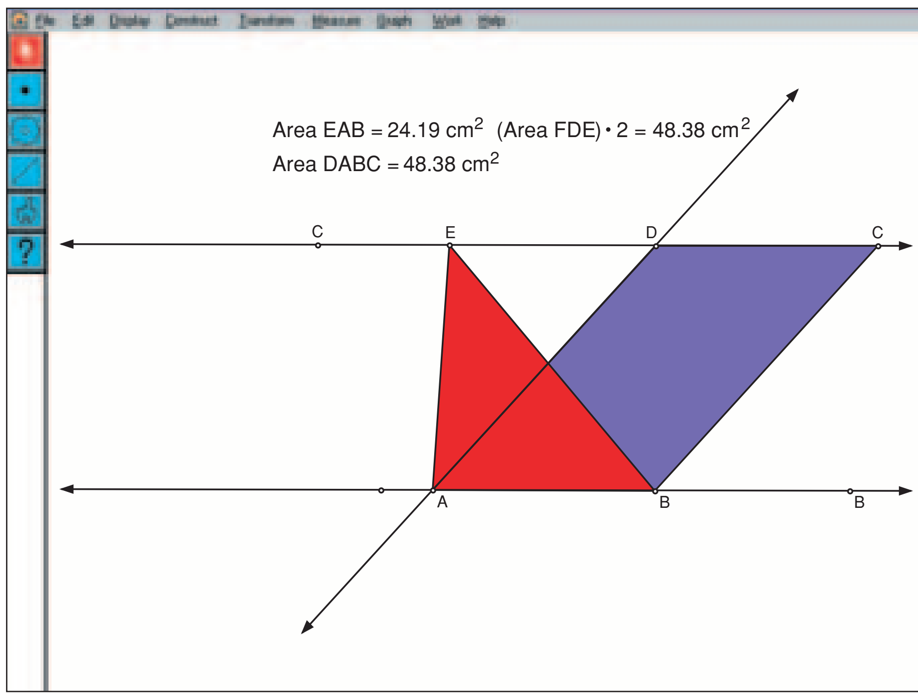
1. Construct any two parallel lines, one above the other, by first constructing a line, a point above the line, and a line parallel to the first line through the plotted point.

2. Construct any line segment *AB* on the lower line, and join the endpoint *A* of this segment to any point *D* on the upper parallel line to form side *AD*. Construct a line parallel to *AD* that passes through the other endpoint *B*. Plot the point of intersection of this line and the upper parallel line and label it *C*. This completes the construction of the parallelogram *ABCD*.

3. Construct any point *E* on the upper line, and join it to *A* and *B* to form the sides of the required triangle.

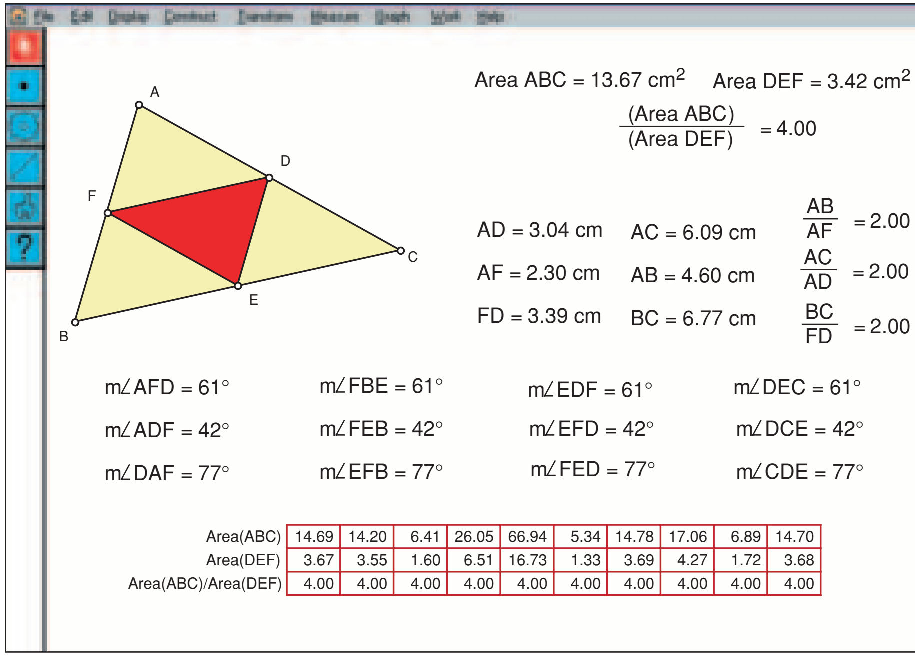
4. Construct the polygon interior of the parallelogram by selecting all of its vertices and using **Polygon Interior** in the Construct menu. The area will appear shaded. Construct the polygon interior of the triangle using the same method. Select each area and measure the area using the **Measure** tool. Verify that the area of the parallelogram is twice the area of the triangle by using **Calculate** in the Measure menu to find the product two times the triangle area.

5. Drag any vertex of the parallelogram or either parallel line to verify that the property holds true for many parallelograms and triangles.



QUESTION 2.2.2

Use Geometer's Sketchpad to investigate and construct midpoints in a triangle as an extension to Question 2.2.2. In this case, investigate the relationship between triangle areas rather than paralleloram areas.



1. Hold Shift ▲ while drawing three points with the Point tool.

2. Select Construct, **Point at Midpoint** and then Construct, **Segment**.

3. Use the Label/Text tool to label all of the points as shown.

4. Select points *A*, *B*, and *C* and Construct, **Polygon Interior**.

5. Choose View, **Display** and change the colour if you want.

6. Under the Measure menu, select **Area**.

7. Select *D*, *E*, and *F* and repeat steps **4** and **5**.

8. Under Measure, select **Calculate**, and then click on the measurement for "Area ABC" (on your sketch), press the division symbol (/) on the calculator, and click on the measurement for "Area DEF" on your sketch. This takes a little while to get used to. You may have to move the calculator window out of the way to click on your measurements. You should see the ratio of the areas appear on the screen.

9. Select each of the segments in the triangles and then select Measure, **Length**. *Note*: Since many of the segments such as *DC* do not exist as line segments on their own (just as part of the sides of the larger triangle), it may be easier to select the endpoints and then choose Measure, **Distance**.

10. Calculate the ratios of corresponding sides using Measure, **Calculate** (as in step **8**).

11. By selecting three points in order and choosing Measure, **Angle**, you can display all of the angle measures.

12. Select one of the vertices of the triangle and move it to various positions. Note which measurements change and which ones remain the same. What does this mean?

13. Hold the shift key and select **Area ABC**, **Area DEF** and the **Ratio** (Area ABC)/(Area DEF), and then choose Measure, **Tabulate**. A table will appear with one set of data in it. If you don't like the headings to the left of the table, you can double click on them to give them a more meaningful name.

14. Manipulate the original triangle by moving point *A*, *B*, or *C*.

15. Add more data to your table by double clicking on the data in the table.

16. Add as many sets of data as you wish to the table.

17. How many hypotheses can you make from this single diagram? How many could you prove?

QUESTION 2.2.6

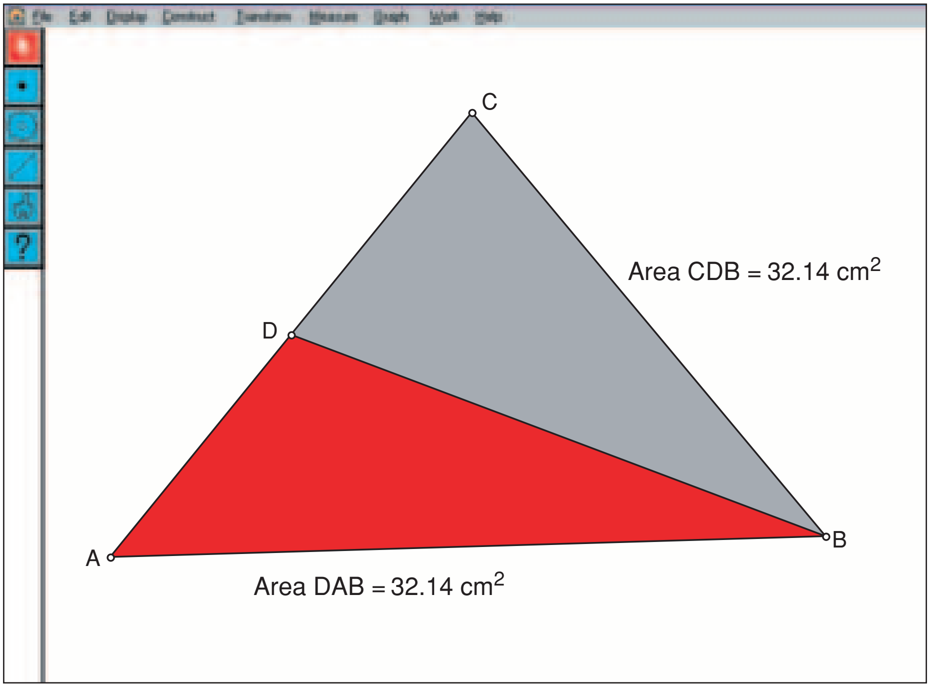
Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Question 2.2.6.

1. Construct any triangle and plot the midpoint of one side. Join the midpoint to the opposite vertex to form a median.

2. Construct the polygon interior of one of the smaller triangles by selecting all 3 vertices (one point being the midpoint) and using the Construct Interior tool. The constructed interior will be shaded. Repeat for the other, smaller triangle.

3. Measure the area of each smaller triangle by selecting the area and using Measure, **Area**. Verify that the areas are equal.

4. Drag any vertex of the large triangle to check that the property holds true for many triangles.



QUESTION 2.2.17

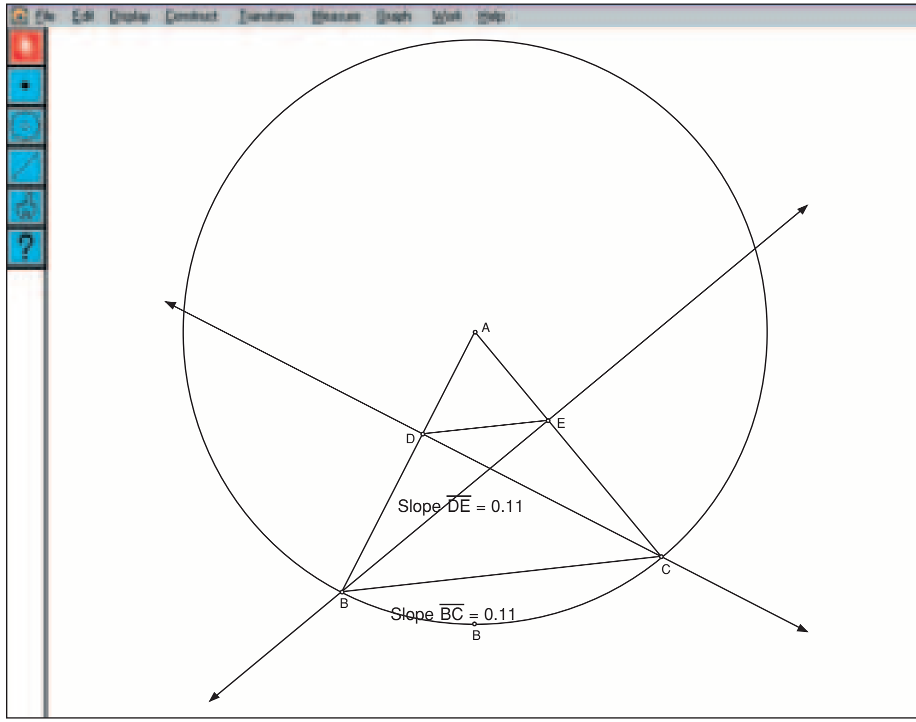
Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Question 2.2.17.

1. Construct an isosceles triangle using a construction circle with two radii drawn in it. Join the two points on the circumference to complete the triangle.

2. Construct two perpendiculars from each equal side of the triangle to the opposite vertices. Construct the points of intersection of these perpendiculars with the triangle sides.

3. Join these points of intersection to each other. Verify that this line is parallel to the base of the triangle by measuring the slope of each line segment.

4. Show that this property remains true for many isosceles triangles by dragging any vertex of the triangle.



SECTION 2.3 — THE RIGHT BISECTOR THEOREM

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Section 2.3 — The Right Bisector Theorem.

1. Construct any segment, the midpoint of the segment, and then the line perpendicular to the segment, through the midpoint. This is the right bisector line.

2. Plot a point anywhere on the right bisector and join it to the endpoints of the line segment.

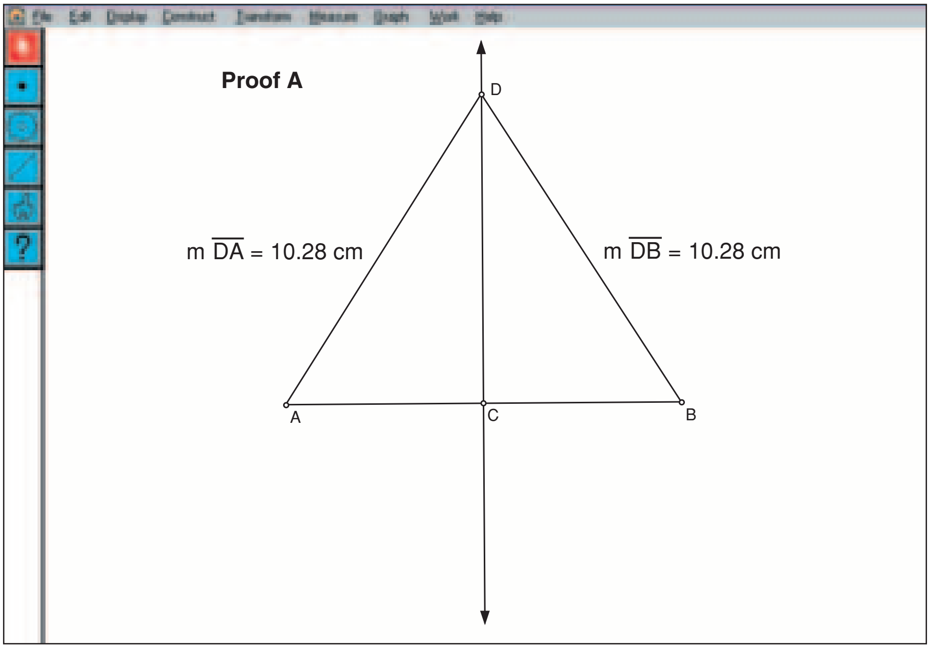
3. Measure the lengths of the lines joining the point on the right bisector to the endpoints of the segment to verify that they are equal in length.

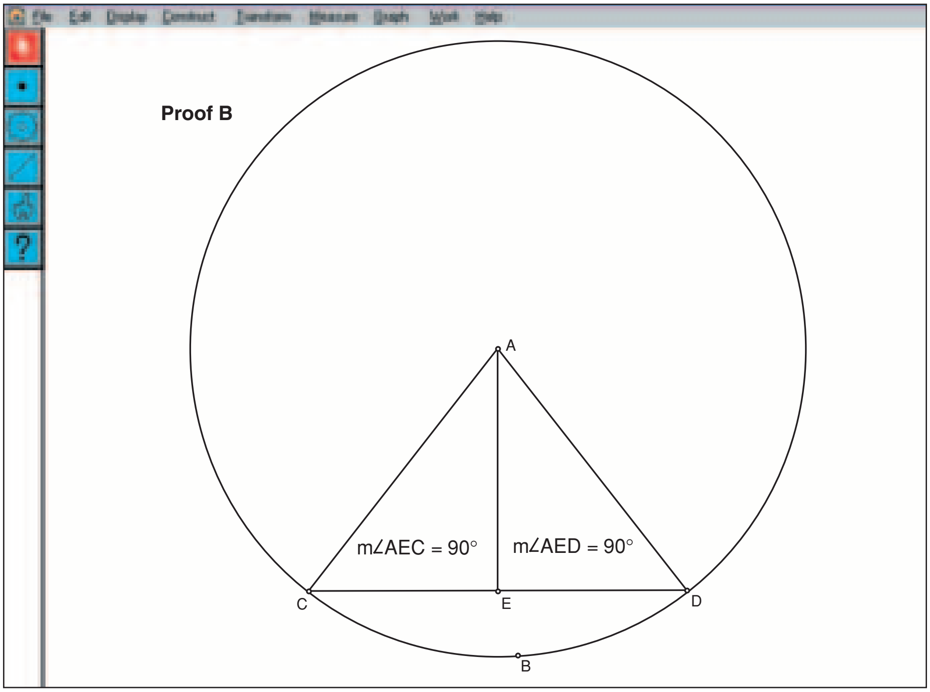
4. Show that this property holds true in general by dragging the point on the right bisector.

5. To demonstrate the converse, construct an isosceles triangle using a construction circle with two radii drawn in it. Join the two points on the circumference to complete the triangle. Plot the midpoint of the base of this isosceles triangle and join it to the opposite vertex.

6. To verify that this line is the perpendicular bisector of the base, measure the angles it makes with the base line.

7. To show that this property holds true in general, drag any vertex of the isosceles triangle.





SECTION 2.3 — ANGLE BISECTOR THEOREM

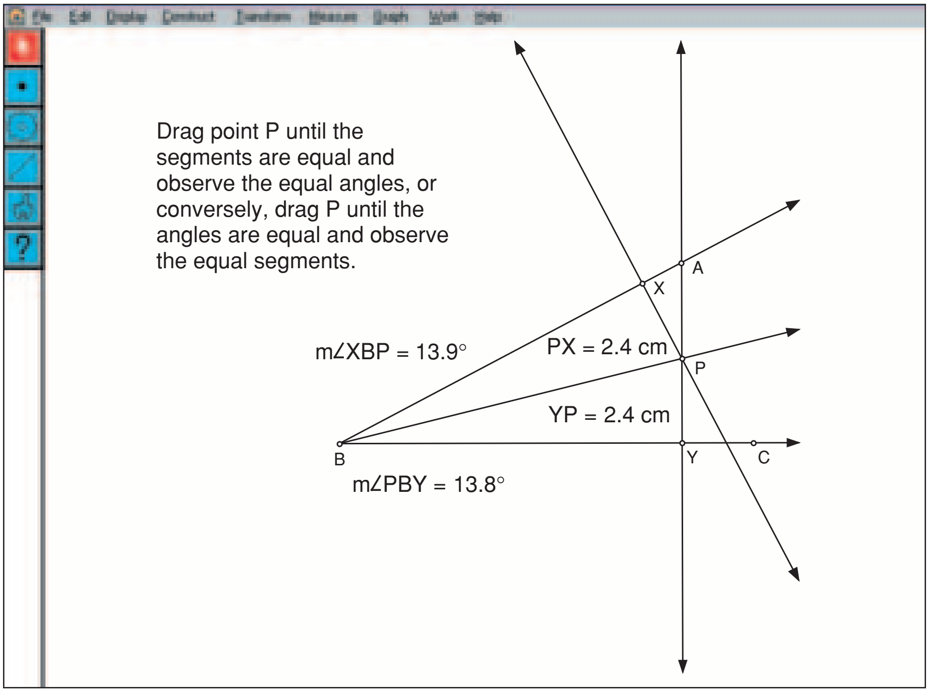
Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Section 2.3 — Angle Bisector Theorem.

1. Construct any angle *ABC* by constructing two rays with a common starting point *B*.

2. Construct any two points *X* and *Y* on the arms of this angle. Then, construct perpendiculars to the arms of the angle, through the points plotted on the arms of the angle.

3. Plot the point of intersection of these two perpendiculars and label it *P*.

4. Measure the lengths of segments *XP* and *YP* and angles *XBP* and *YBP*.

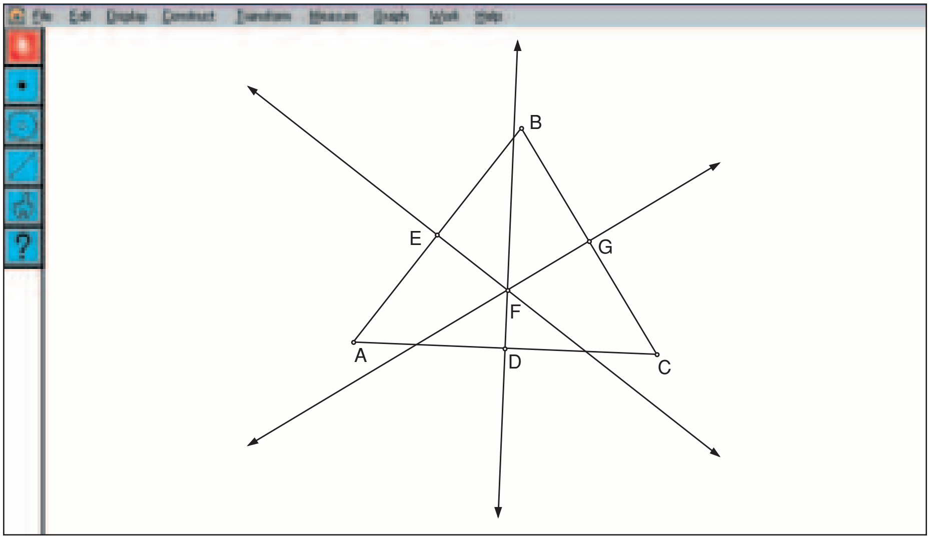


5. To demonstrate Part 1 of the theorem, drag point *P* until the segments are equal in length. Observe that the angles are equal. To demonstrate Part 2, drag point *P* until the angles are equal. Observe that the segments are equal.

QUESTION 2.3.11

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Question 2.3.11.

1. Construct any triangle and plot the midpoints of each side.
2. Construct perpendiculars to two of the sides through each midpoint. These are the perpendicular bisectors.
3. Plot the point of intersection of the perpendicular bisectors.
4. Construct the third perpendicular bisector, and verify that it passes through the point of intersection of the other bisectors.
5. Show that this property holds true for many triangles by dragging any vertex of the triangle.



SECTION 2.4 — THE PARALLEL LINES THEOREM

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Section 2.4 — The Parallel Lines Theorem.

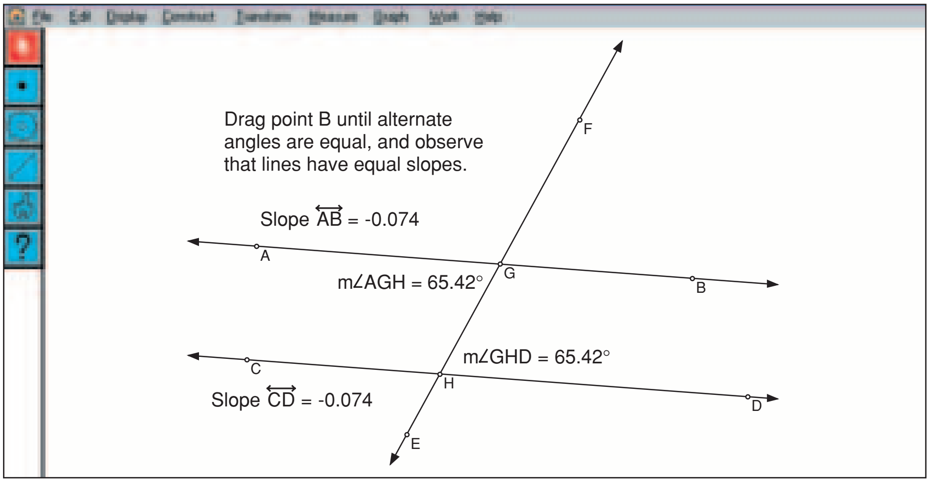
1. Construct any two non-intersecting line segments and a third line that crosses these two.

2. Plot the points of intersection of all the lines.

3. Measure any pair of alternate angles and corresponding angles.

4. Drag a point on either of the non-intersecting lines until the alternate angles and the corresponding angles are equal.

5. Show that the line segments are parallel by measuring the slopes of these line segments.



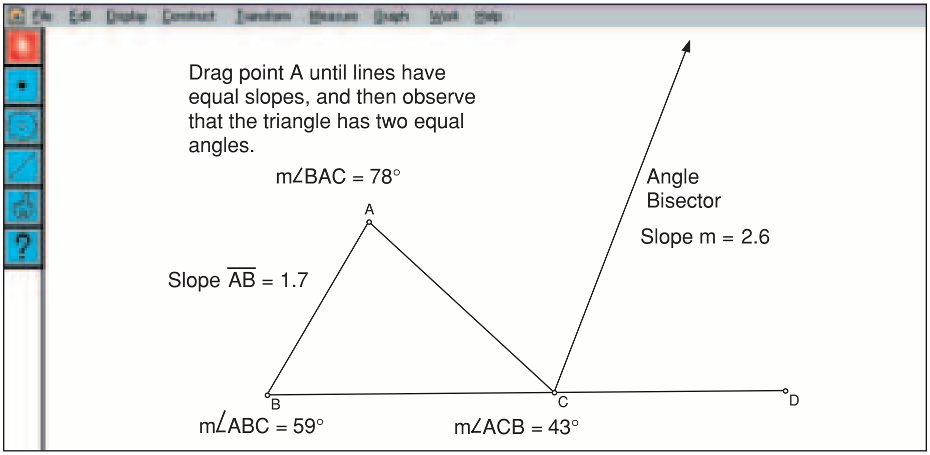
QUESTION 2.4.6

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Question 2.4.6.

1. To construct a triangle with one exterior angle, first construct a long line segment *BD*, plot a point *C* on it, and from that point, draw the other two sides of the triangle *AC* and *AB*.

2. Construct an angle bisector of the exterior angle *ACD*. Measure its slope and the slope of the nonadjacent side *AB*. Also measure the angles *ABC* and *BAC*.

3. Drag point *A* until the slope of the angle bisector is the same as side *AB*. Observe that the two measured angles are equal.



SECTION 2.5 — THE TRIANGLE AREA PROPERTY

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Section 2.5 — The Triangle Area Property.

1. Construct two parallel construction lines by first constructing a line, a point above the line, and another line through the point, parallel to the first line.

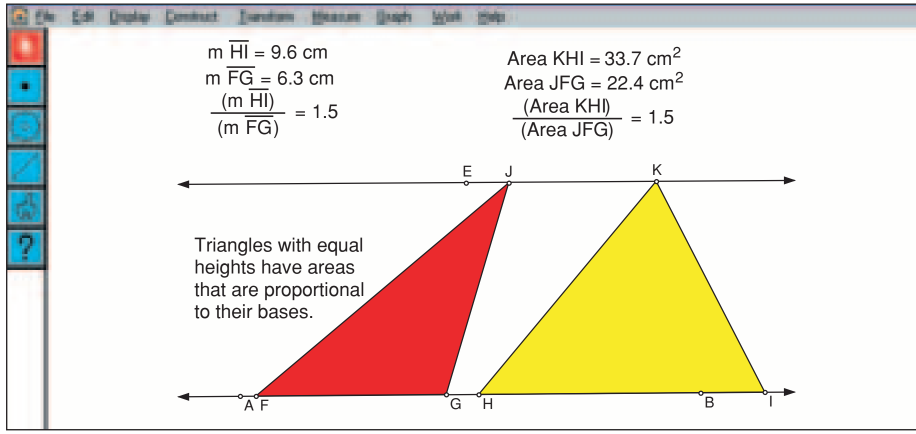
2. Construct the base of the first triangle by making a line segment on the lower parallel line and by joining the endpoints of this line segment to a point on the upper parallel line. Repeat this operation to construct another triangle beside the first that will have the same height.

3. Measure the lengths of the two triangle bases and calculate the ratio of the longer base to the shorter base using the Measure/Calculate menu.

4. Construct the polygon interior of each triangle by selecting the vertices of each, and measure the area of each triangle by selecting each interior and using the Measure/Area menu.

5. Calculate the ratio of the larger area to the smaller area using the Measure/Calculate menu, and observe that the ratios of the areas equal the ratio of the bases.

6. Show that this property remains true for many triangles by dragging the upper line up or down and/or dragging one of the endpoints of one of the triangle bases.



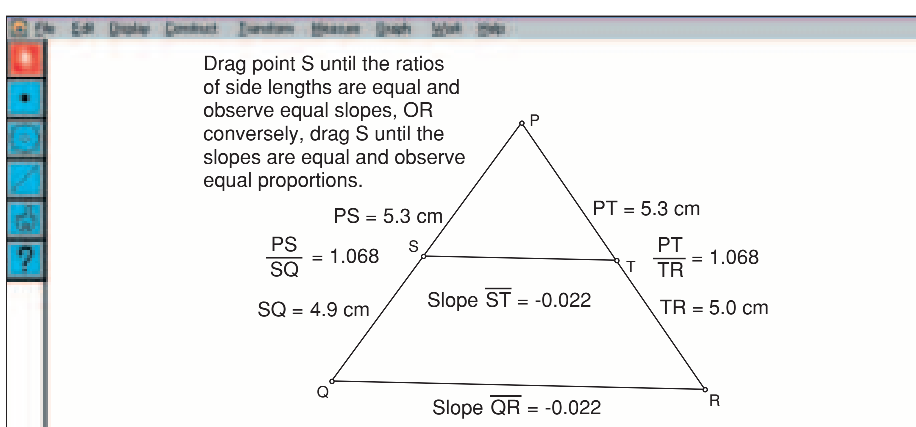
SECTION 2.5 — THE TRIANGLE PROPORTION PROPERTY THEOREM

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Section 2.5 — The Triangle Proportion Theorem.

1. Construct any triangle *PQR*, plot points *S* and *T* on each of two adjacent sides, and join these two points with segment *ST*. Measure the slopes of segments *ST* and *QR*.

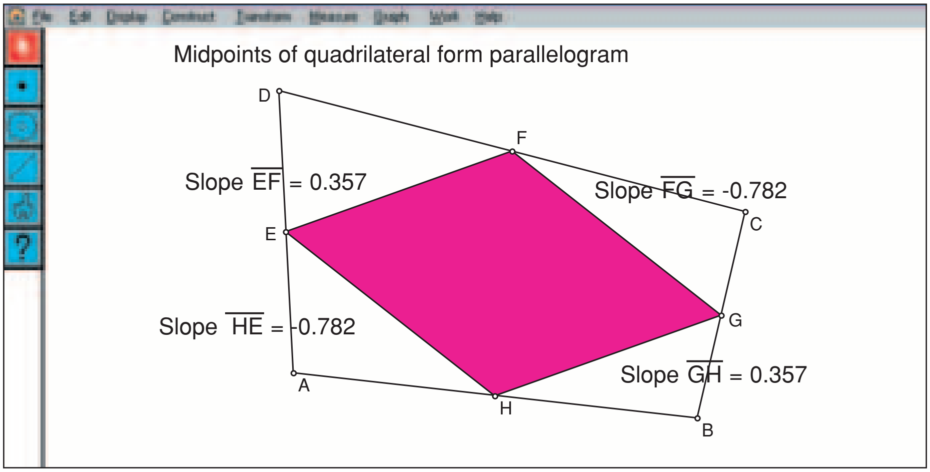
2. Measure the lengths of segments *PS* and *SQ* and calculate the ratio of their lengths. Repeat for segments *PT* and *TR*.

3. Drag point *S* until the slope of *ST* equals the slope of *QR*, indicating that these lines are parallel. Observe that the side ratios are also equal, or conversely, drag *S* until the side ratios are equal and observe that the line segment ratios are also equal.



SECTION 2.5 — EXAMPLE 4

Use Geometer's Sketchpad to construct and explore midpoint polygons similar to those in Section 2.5, Example 4.



1. While holding the shift key down, draw three points. Choose Construct, **Segment**, Construct, **Point at Midpoint**, and then Construct, **Segment** again. You have created a "Midpoint" triangle.

2. Label the vertices, and after selecting points *A*, *B* and *C*, choose Construct, **Polygon Interior** and Measure, **Area**. Do the same for triangle *DEF*, and choose appropriate display colours.

3. Using Measure, **Calculate**, determine the ratio of the area of the outer triangle to the area of the "Midpoint triangle." Manipulate the triangle. Does the ratio appear to be always true? Can you prove it?

4. Construct the midpoint quadrilateral, pentagon, and hexagon and calculate the area ratios. Manipulate the figures by moving the vertices. You may be surprised by what you find....

5. Can you make any general hypotheses regarding the ratio of areas of polygons formed this way?

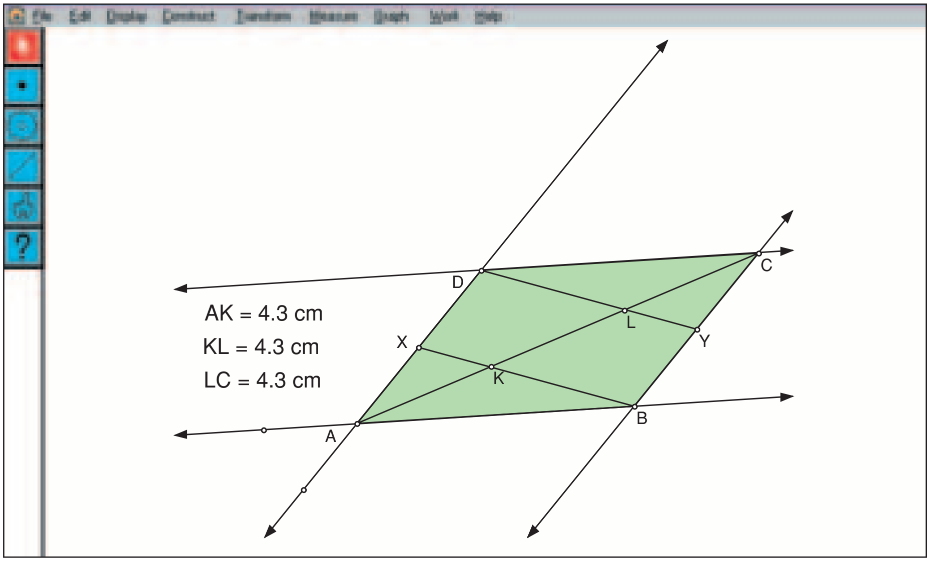
QUESTION 2.5.11

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Question 2.5.11.

1. Construct a parallelogram by using the method given in Section 2.5, Example 4, or as follows:
   i) Construct a construction line and construct a line segment *AB* on this line.
   ii) Plot a point *D* above the construction line and construct a line through points *AD*.
   iii) Construct a line parallel to *AB* through point *D* and, finally, a line parallel to *AD* through point *B*.
   iv) Plot the point of intersection of the last two lines constructed and label it *C*.
   v) Join points *A*, *D*, *C*, and *B*, in that order, to finish the construction of parallelogram *ABCD*.

2. Construct midpoints *X* and *Y* of sides *AD* and *BC,* respectively, and join points *BX*, *DY*, and *AC*. Plot the point of intersection of *XB* with *AC* and label it *K*, and also plot the point of intersection of *DY* and *AC* and label it *L*.

3. Measure the lengths of *AK*, *KL*, and *LC* to confirm that *BX* and *DY* trisect *AC*. Drag any vertex of the parallelogram to show that this property holds true for many parallelograms.



SECTION 2.6 — SIMILARITY IN TRIANGLES, INVESTIGATION

Use Geometer's Sketchpad to explore the following problem involving similar triangles.

Problem

a) If two triangles are constructed such that two pairs of corresponding angles are equal, are the triangles similar?
b) If two triangles are constructed such that two pairs of corresponding sides are proportional and the contained angles are equal, are the triangles similar?
c) If two triangles are constructed such that all three pairs of corresponding sides are proportional, are the triangles similar?

Write your predicted answer to each of the problems posed above, based on your knowledge of similar triangles.

Solution a): AA~

1. Construct any small triangle with a horizontal base. Measure the three angles of this triangle and move the measurements close to the angles they measure.

2. Construct a horizontal line segment that is longer than the base of your first triangle. Using the ray tool, construct any angles directed upwards, with vertices that are the endpoints of this horizontal line segment.

3. Measure these two angles, and adjust the size of the angles until they equal the corresponding angles in the first triangle. Plot the point of intersection of the rays to determine the vertex of the new triangle, and measure the angle at that vertex. Observe that this angle equals the corresponding angle in the first triangle.

4. Measure the sides of the triangles, and calculate the ratios of corresponding sides. Observe that these are equal because the triangles are similar.

Solution b): SAS~

1. Construct any small triangle with a horizontal base. Measure the three angles of this triangle and move the measurements close to the angles they measure. Also measure the length of the three sides of the triangle and move these measurements close to the sides they measure.

2. Construct a horizontal base for a second triangle that is longer than the base of the first triangle. Measure the length of this new base. Construct a second side for this triangle. Measure the length of this side and the angle it makes with the base. Adjust the size of this angle until it equals the corresponding angle in the first triangle.

3. Calculate the ratio of the length of the base of the second triangle to the length of the base of the first triangle. Also calculate the ratio of the length of the second side of the second triangle to the length of the corresponding side in the first triangle. Adjust the length of the second side of the second triangle until the two ratios of the side lengths are equal, making sure that the angle equality is maintained.

4. Complete the construction of the second triangle by joining the ends of the two sides previously constructed with a line segment. Measure the remaining angles of the second triangle and observe that the triangles are similar.

Solution c): SSS~

1. Construct any small triangle *ABC*. The following steps will use the dilation feature of Geometer's Sketchpad to construct a triangle with corresponding sides that are proportional. Plot any point *D* below the small triangle. Select point *D* and click on **Mark Centre** in the Transform menu to select this point as the centre of dilation.

2. Now select point *A* and create the image of *A* under a dilation by using the Transform menu, choosing **Dilate**, and entering a dilation factor of any value (Suggestion: use a a small factor like 2), and click on OK. This will plot image point A'. Repeat this process for the other two vertices and join them with line segments to create a triangle $A'B'C$.

3. Measure the lengths of all the sides of both triangles and calculate the ratios of each pair of corresponding sides. Observe that these ratios are all equal, showing that the sides of the triangles are proportional.

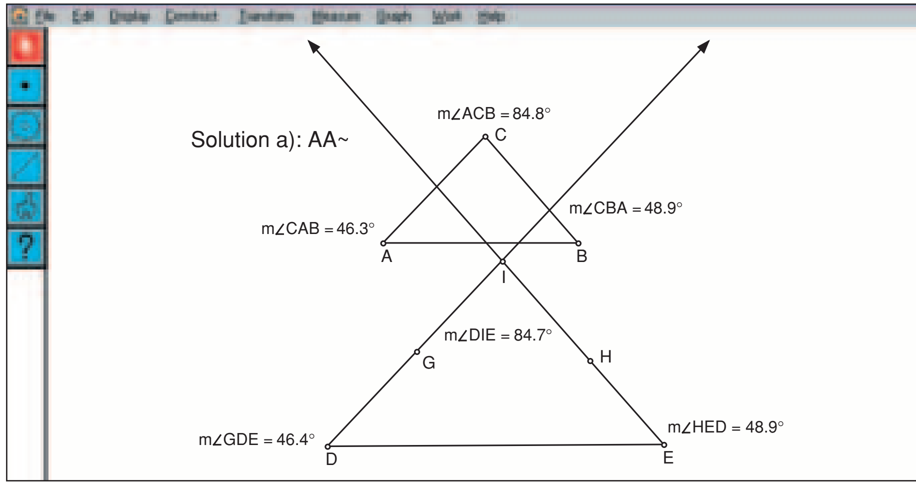
4. Measure all the angles in both triangles to verify that the triangles are similar.

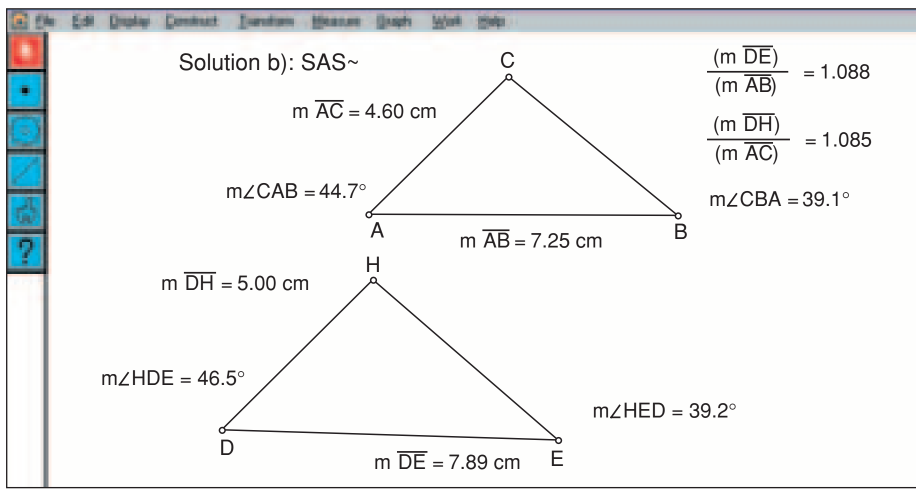
Observations

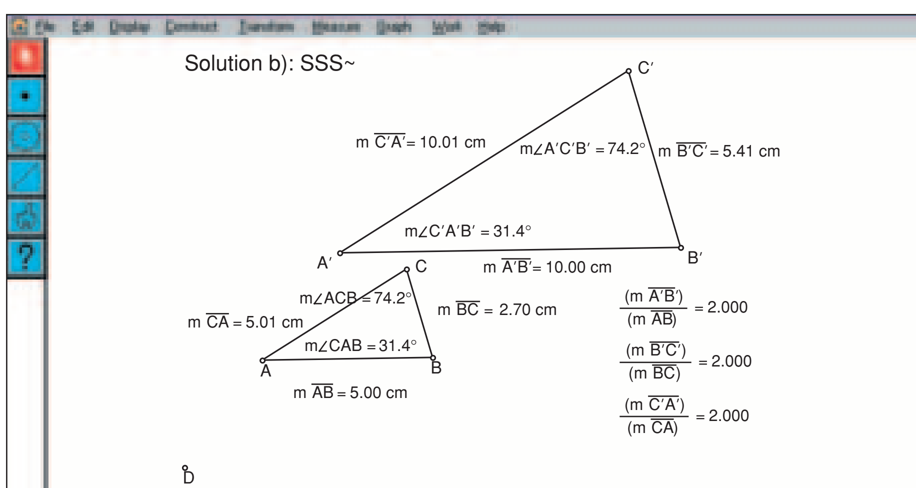
In your report, include printouts of the sketches you made in this investigation.

Conclusions

Write a summary of your findings in this investigation as they relate to the problem statement given.


Solution a): AA~
m∠ACB = 84.8°
C
m∠CBA = 48.9°
m∠CAB = 46.3°
A
B
I
m∠DIE = 84.7°
G
H
m∠HED = 48.9°
m∠GDE = 46.4°
D
E


Solution b): SAS~
C
(m DE) / (m AB) = 1.088
m AC = 4.60 cm
(m DH) / (m AC) = 1.085
m∠CAB = 44.7°
m∠CBA = 39.1°
A
m AB = 7.25 cm
B
H
m DH = 5.00 cm
m∠HDE = 46.5°
m∠HED = 39.2°
D
m DE = 7.89 cm
E


Solution b): SSS~
C′
m C′A′ = 10.01 cm
m∠A′C′B′ = 74.2°
m B′C′ = 5.41 cm
m∠C′A′B′ = 31.4°
A′
B′
C
m A′B′ = 10.00 cm
m∠ACB = 74.2°
m BC = 2.70 cm
(m A′B′) / (m AB) = 2.000
m CA = 5.01 cm
m∠CAB = 31.4°
A
B
(m B′C′) / (m BC) = 2.000
m AB = 5.00 cm
(m C′A′) / (m CA) = 2.000
D

SECTION 2.6 — EXAMPLE 1

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Section 2.6, Example 1.

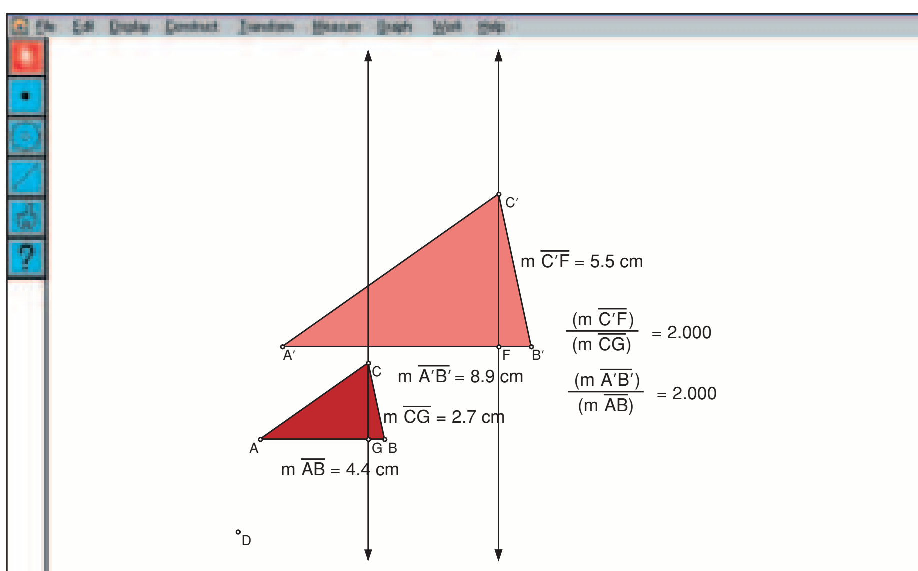
1. One way to construct similar triangles is to use the dilation feature in Geometer's Sketchpad. First, construct any small triangle *ABC*. Next, plot any point *D* below the small triangle. Select point *D* and click on **Mark Centre** in the Transform menu to select this point as the centre of dilation.

2. Now select point *A* and create the image of *A* under a dilation by using the Transform menu, choosing **Dilate**, and entering a dilation factor of any value (Suggestion: use a small factor like 2), and click on OK. This will plot image point *A*′. Repeat this process for the other two vertices and join them with line segments to create a triangle *A*′*B*′*C*′ similar to *ABC*. Verify that the triangles are similar by measuring corresponding angles.

3. Construct the altitude from the base of each triangle by selecting the base and the opposite vertex and constructing a perpendicular. Plot the points of intersection of each perpendicular with the bases of their respective triangles.

4. Measure the lengths of the bases and altitudes of each triangle, and calculate the ratios of the longer length to the shorter length for the bases and the altitudes to verify that they are equal.

5. To show that this property holds true in general, drag any vertex of the original triangle.



SECTION 2.6 — EXAMPLE 2

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Section 2.6, Example 2.

1. Construct two similar triangles using the dilation method as follows:
   i) Construct any small triangle *ABC*.
   ii) Plot any point *D* below the small triangle.
   iii) Select point *D* and click on **Mark Centre** in the Transform menu to select this point as the centre of dilation.

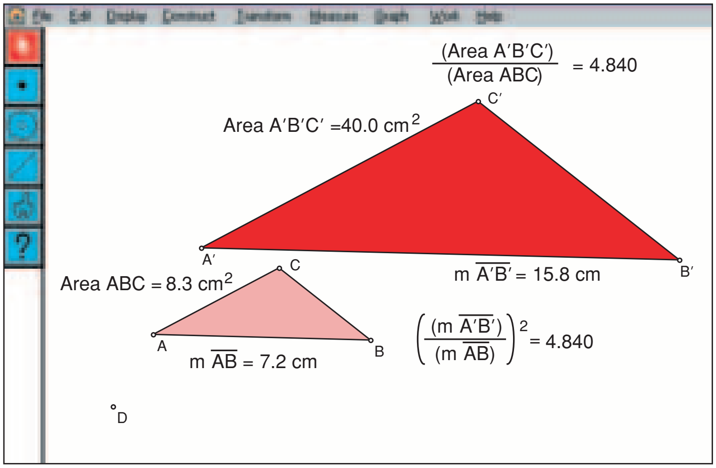
2. Now select point *A* and create the image of *A* under a dilation by using the Transform menu, choosing **Dilate**, and entering a dilation factor of any small value. Click on OK. This will plot image point A'. Repeat this process for the other two vertices and join them with line segments to create a triangle $A'B'C'$ similar to *ABC*. Verify that the triangles are similar by measuring corresponding angles.

3. Construct the polygon interiors of the triangles by selecting each vertex and using the **Polygon Interior** tool in the Construct menu. The interior of each triangle will be shaded. Measure each triangle's area by selecting the interior and using the **Measure/Area** tool. Calculate the ratio of the larger area to the smaller area.

4. Measure the lengths of the bases of each triangle, and calculate the square of the ratio of the longer length to the shorter length using the **Measure/Calculate** tool. Verify that this ratio equals the ratio of the areas.

5. Show that this property holds true in general by dragging any vertex of either triangle.

6. Repeat using a different dilation factor.



QUESTION 2.6.9

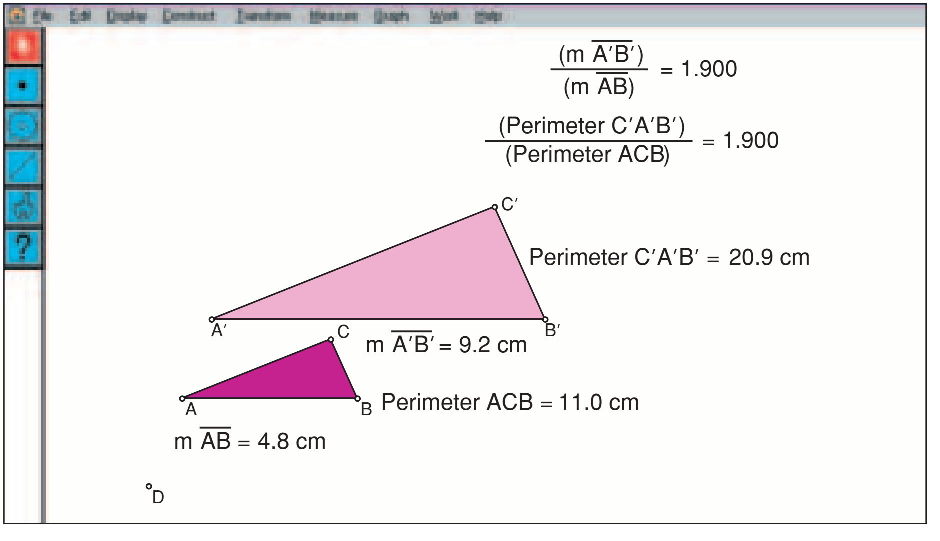
Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Question 2.6.9:

1. Construct a pair of similar triangles using the dilation method as follows:
   i) First construct any small triangle *ABC*.
   ii) Plot any point *D* below the small triangle.
   iii) Select point *D* and click on **Mark Centre** in the Transform menu to select this point as the centre of dilation.

2. Select point *A* and create the image of *A* under a dilation by using the Transform menu, choosing **Dilate**, and entering a dilation factor of any value. Click on OK. This will plot image point *A′*. Repeat this process for the other two vertices, and join them with line segments to create a triangle *A′B′C′* similar to *ABC*. Verify that the triangles are similar by measuring corresponding angles.

3. Construct the polygon interiors of the triangles by selecting each vertex and using the **Polygon Interior** tool in the Construct menu. The interior of each triangle will be shaded. Measure each triangle's perimeter by selecting the triangle's interior and using the **Measure/Perimeter** tool. Calculate the ratio of the larger perimeter to the smaller perimeter using the **Measure/Calculate** tool.

4. Measure the lengths of two corresponding sides of the similar triangles and calculate the ratio of the longer side to the shorter side. Observe that the ratio of the perimeters equals the ratio of the side lengths. Show that this is true in general by dragging any vertex of either triangle.



QUESTION 2.6.11B

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Question 2.6.11b.

1. Construct a right-angled triangle *ABC* by first drawing segment *AB* and constructing a line perpendicular to *AB* through *A*. Plot a point *C* anywhere on this perpendicular and construct segments *AC* and *BC* to complete the right triangle.

2. Construct the altitude *AD* by constructing a perpendicular to *BC* through *A*, plotting the point of intersection of this perpendicular and segment *BC*, and labelling the point *D*.

3. Measure the lengths of all the following segments using the **Measure/Length** tool: *AD*, *BD*, *DC*, *AB*, *AC*, and *BC*.

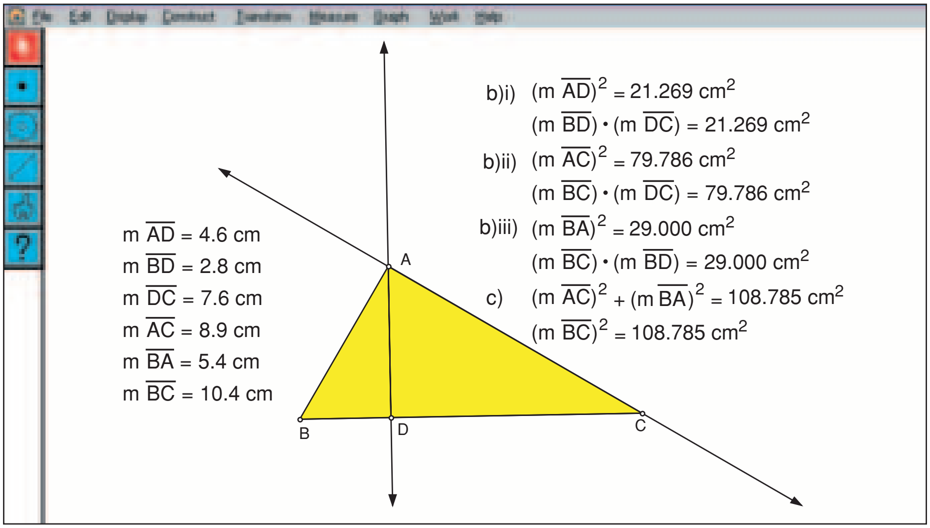
4. Calculate the value of AD^2 and verify that it equals the product (*BD*)(*DC*).

5. Calculate the value of AC^2 and verify that it equals the product (*BC*)(*DC*).

6. Calculate the value of AB^2 and verify that it equals the product (*BC*)(*BD*).

7. Show the Pythagorean theorem by calculating the sum $AC^2 + AB^2$ and comparing it with BC^2

8. Show that all these relationships are true in general by dragging any vertex of the right-angled triangle.



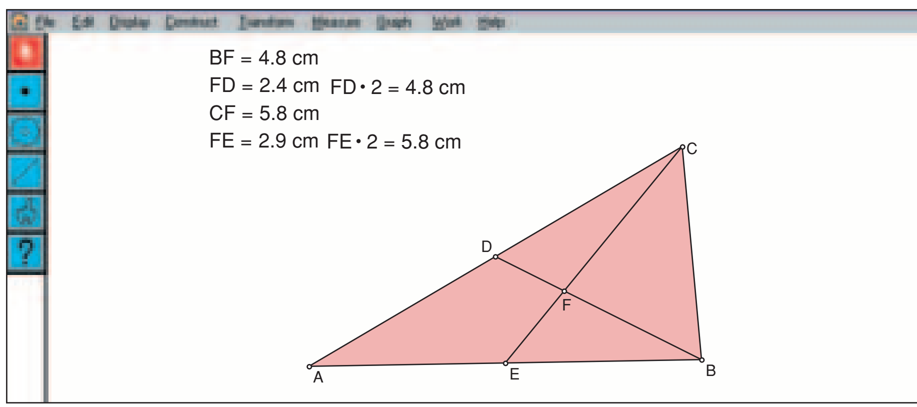
QUESTION 2.6.12

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Question 2.6.12.

1. Construct any triangle *ABC* and midpoints *D* and *E* of sides *AC* and *AB*, respectively. Join points *B* and *D* to form one median *BD*, and join points *C* and *E* to form the other median *CE*.

2. Plot the point of intersection of the medians and label it point *F*. Measure segments *BF* and *FD*, and verify that the product 2(*FD*) equals *BF* using the **Measure/Calculate** tool. Repeat these calculations for the other median.

3. Show that this property is true for many triangles by dragging any vertex of the triangle.



QUESTION 2.6.15

Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in Question 2.6.15.

1. Construct two similar pentagons using the dilation method as follows:
   i) Construct any small pentagon *ABCDE*.
   ii) Plot any point *F* below the small triangle.
   iii) Select the point *F* and click on **Mark Centre** in the Transform menu to select this point as the centre of dilation.

2. Now select point *A* and create the image of *A* under a dilation by using the Transform menu, choosing **Dilate**, and entering a dilation factor of any small

value. Click on OK. This will plot image point A'. Repeat this process for the other four vertices, and join them with line segments to create a pentagon $A'B'C'D'E'$ similar to $ABCDE$.

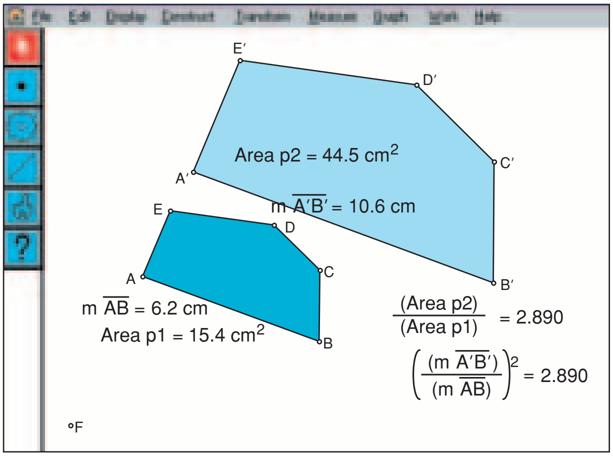
3. Construct the polygon interiors of each pentagon by selecting all the vertices and using Construct, **Polygon Interior**. The interior area will be shaded.

4. Measure the areas of each pentagon by selecting the interior and using Measure, **Area**. Calculate the ratio of the larger area to the smaller area.

5. Measure the length of two corresponding sides of the two pentagons, and calculate the square of the ratio of the longer side to the shorter side. Observe that this result equals the ratio of the areas.

6. Show that this property is true for many similar pentagons by dragging any vertex of either pentagon to change the shape of the pentagon.

7. Repeat using a different dilation factor.



CHAPTER 2 REVIEW EXERCISE, QUESTION 15

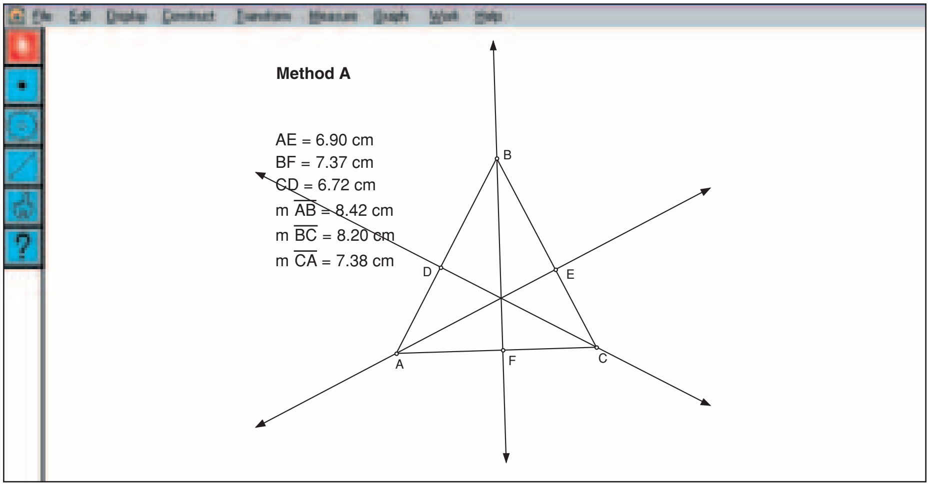
Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in the Chapter 2 Review Exercise, Question 15.

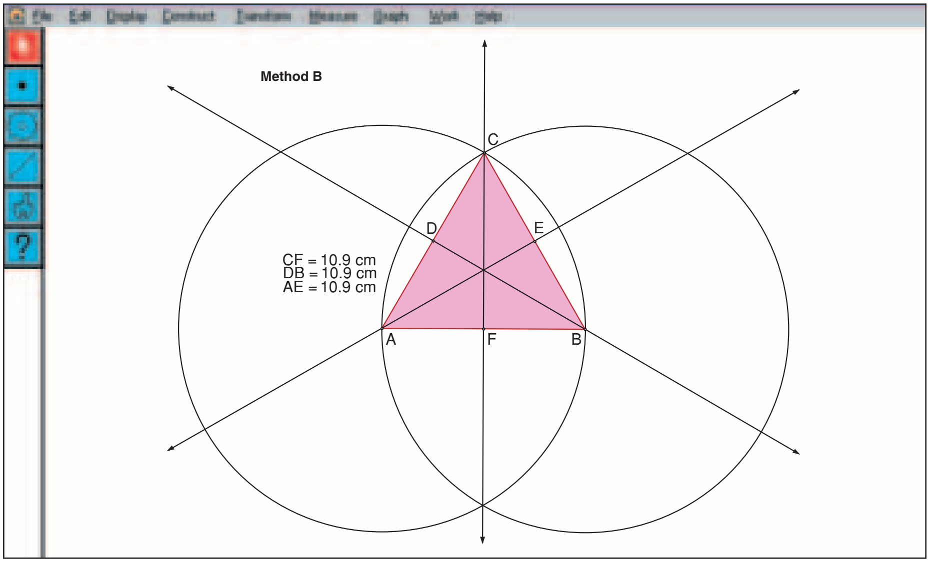
1. Construct any scalene triangle and measure the length of all three sides. Adjust side lengths so that none are equal (if necessary).

2. Construct an altitude to one side by selecting one side and the opposite vertex and using Construct, **Perpendicular**. Plot the point of intersection of the perpendicular line with the opposite side. Measure the length of the altitude using Measure, **Distance**. Repeat for the other two altitudes.

3. Adjust the side lengths of the triangle by dragging one side until all three sides are equal. This may require some careful adjustments of more than one side. Observe that the altitude lengths are now also equal. Conversely, adjust the length of the altitudes by dragging one of the vertices until they are all equal, and observe that the triangle is equilateral.

4. A more accurate method would be to construct an equilateral triangle by the following method, and then to construct and measure the altitudes as described earlier. This method does not demonstrate that the converse is also true. To construct an equilateral triangle, first draw any line segment, then draw a circle with radius equal to the length of the line segment, centred at one endpoint, and then construct another circle with the same radius, centred at the other endpoint. Plot the point of intersection of the two circles. Join this point to the endpoints of the first segment to complete the construction. Repeat step **2** above.





Use Geometer's Sketchpad to construct the following diagram that demonstrates the properties to be proven in the Chapter 2 Review Exercise, Question 17.

1. Construct any triangle *ABC* and midpoints *E* and *D*, of sides *AC* and *BC*, respectively. Join points *A* and *D* to form one median *AD*, and join points *B* and *E* to form the other median *BE*.

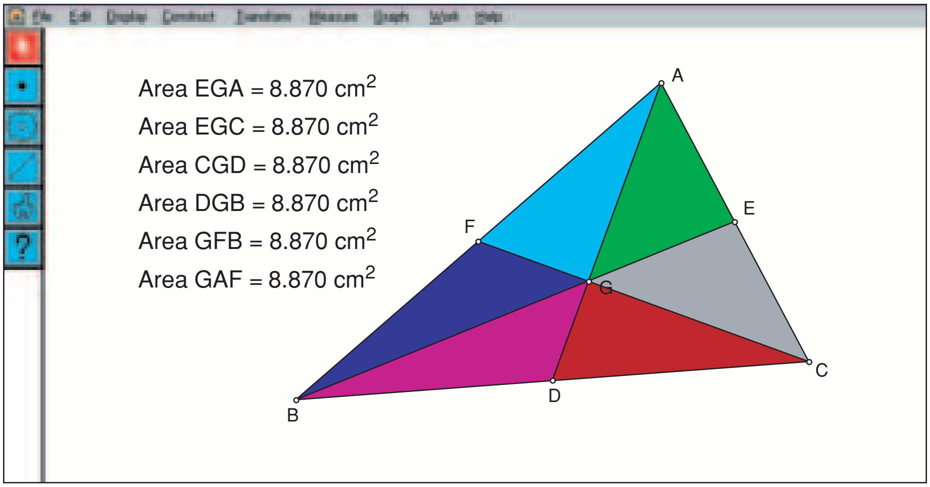
2. Plot the point of intersection of the medians and label it point *G*.

3. Construct the midpoint *F* of side *AB*, and join *F* to *G* and *G* to *C*. This will form the third median and complete the construction of the division of the triangle into six smaller triangles.

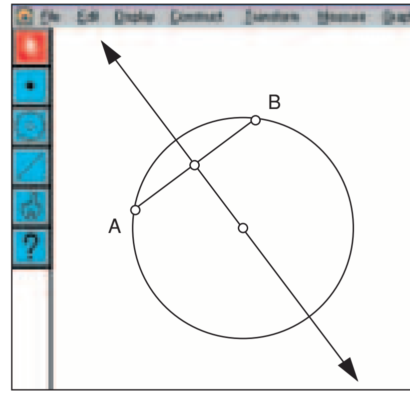
4. Construct the polygon interior of each of the six small triangles by selecting its vertices and using the **Construct/Polygon Interior** tool. The colour of each interior can be changed to a different colour by selecting the interior and using the Display, **Colour**.

5. Measure the areas of the six interior triangles using Measure, **Area**. Observe that all six areas are equal.

6. Show that this property holds true for many triangles by dragging any vertex of the outside triangle.



Use Geometer's Sketchpad to investigate and construct chords in a circle, similar to various examples and exercises in Section 3.1.



1. Use the Circle tool to construct any circle.

2. With the circle selected, choose Construct, **Point on Object**.

3. Select the circle again and choose Construct, **Point on Object** again.

4. Use the Label tool to label the newly created points *A* and *B*.

5. Join these two points using the Segment tool, or by selecting the points using the Arrow tool and the shift key, and then choosing Construct, **Segment**.

6. With the segment selected, choose Construct, **Point at Midpoint**.

7. Select line segment *AB*, hold the shift key, and then select the midpoint and Construct, **Perpendicular Line**.

8. Grab either *A* or *B* and swing it around the circle and witness what happens. (Alternatively, select point *A*, hold Shift, select the circle, and then choose Display, **Animate**, **Slowly** and watch it happen).

9. State a hypothesis illustrating what you have just seen.

10. Develop a formal proof for this phenomenon.

SECTION 5.3 — USING MATRICES TO CALCULATE THE DOT PRODUCT

Remember that the dot product can be calculated in component form as shown below.

Given $\vec{a} = (3, 4, -2)$ and $\vec{b} = (-7, 4, 6)$

then $\vec{a} \bullet \vec{b} = ((2)(-7) + (4)(4) + (-2)(6))$

$= 17$

As we can see, the dot product of two vectors is a scalar.

We can also use matrix multiplication to calculate $\vec{a} \bullet \vec{b}$. Follow the steps below on a TI-83+ calculator.

1. Define matrix [A] to be a 1×3 row vector containing the components of $\vec{a}$. (See “Setting Up a Matrix Using the TI-83+.”). Print [A] on your screen.

   Your screen should have the matrix $[\ 3\ \ 4\ \ -2\ ]$.

2. Define matrix [B] to be a 3×1 column vector containing the components of $\vec{b}$. Print [B] on your screen.

   Your screen should have the matrix $\begin{bmatrix} -7 \\ 4 \\ 6 \end{bmatrix}$.

3. To calculate $\vec{a} \bullet \vec{b}$, multiply the matrices [A][B] by following these steps.

   i) Press **2nd** **x^{-1}** (MATRX). Under NAMES, choose the defined matrix [A] and then press **ENTER**.

   ii.) Press **2nd** **x^{-1}** (MATRX).Under NAMES, choose the defined matrix [B] and then press **ENTER**.

   Your screen should look like this: ... [A] [B]. (Multiplication is implied here.)

   iii.) Press **ENTER** to obtain the product of matrix [A] and [B]. The result is a 1×1 matrix containing the dot product.

EXERCISE

Calculate the dot product using matrix multiplication for each of the following:

1. $\vec{a} = (2, 1, -2)$; $\vec{b} = (0.5, -3.5, 1.75)$
2. $\vec{m} = (-2.75, 0, -1.45)$; $\vec{n} = (5, 7, 0)$

Answers: 1. −6 2. −13.75

SECTION 8.4 — SETTING UP A MATRIX USING THE TI-83+

Defining a matrix

1. To set up your matrix, press [2nd] [x^{-1}] (MATRX) [▶] [▶] to display the MATRX EDIT menu, and then press [ENTER] ([▶] means press the "cursor right" button). You will have Matrix [*A*] followed by its dimensions at the top of your screen.

   (If you wish to define matrix [*B*] or [*C*], etc., press [2nd] [x^{-1}] (MATRX) [▶] [▶] to select EDIT, and then press the [▼] button and choose the matrix to define. Now press [ENTER].)

2. Working with Matrix [*A*]:
   i) Enter the number of rows and then press [ENTER].
   ii) Enter the number of columns and then press [ENTER].

3. Input each element in the matrix by typing a value and then [ENTER]. *Note:* For a negative number, use the white [(–)] key in the bottom row. Input the values for row 1, then row 2, and so on.

4. When you are finished entering values, press [2nd] [MODE] (QUIT) and the screen will clear.

5. To display your matrix, press [2nd] [x^{-1}] (MATRX). Under NAMES, choose your defined matrix by scrolling down and then pressing [ENTER] [ENTER].

Example Define the matrix [*C*] to be the augmented matrix representing the system of equations below.

$$x - 3y - 2z = -9$$
$$2x - 5y + z = 3$$
$$-3x + 6y + 2z = 8$$

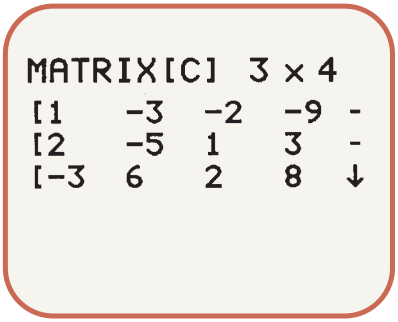
Solution We need to define a 3×4 matrix.

i) Press [2nd] [x^{-1}] (MATRX) [▶] [▶] to select the EDIT option.
ii) Scroll down using the [▼] key to [*C*]. Press [ENTER].
iii) To define the first dimension, type **3** and press [ENTER]. For the second dimension, type **4** and press [ENTER].

iv) Input the values of the matrix.
Begin by typing 1 and pressing ENTER, and then type −3 and press ENTER, and so on.

v) When you have entered your final value, press 2nd and MODE (QUIT).

vi) Now display matrix [*C*] by pressing 2nd x^{-1} (MATRX), then scroll down to [*C*] and press ENTER ENTER. You should have the matrix shown below on your screen.



SECTION 8.4 — REDUCING A MATRIX DIRECTLY TO REDUCED ECHELON FORM OR ROW REDUCED ECHELON FORM

The TI-83+ has built-in commands that allow the user to reduce an augmented matrix directly to *row echelon* form or *reduced row* echelon form. The commands are found under the MATRX MATH menu.

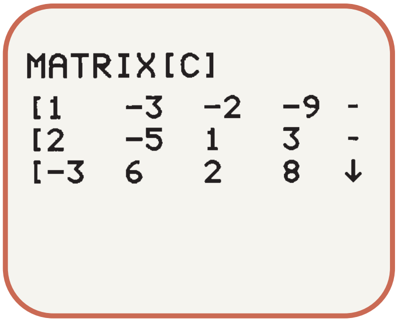
ref(will reduce a matrix to echelon form.

rref(will reduce a matrix directly to reduced row echelon form.

Example

Given the matrix [*C*] below, solve the system with the TI-83+ by

i) writing it in row echelon form and solving by back substitution.
ii) writing it in reduced row echelon form and writing the solution directly.



Solution (part i)

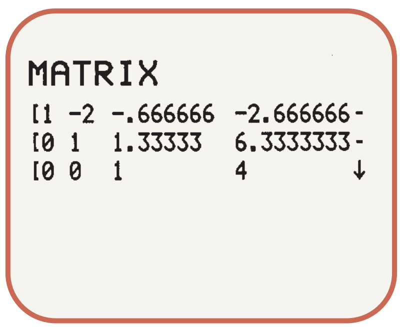
1. Define [*C*] to be the matrix shown above. (See “Setting Up a Matrix Using the TI-83+.”)

2. Press [2nd] [MATRX x^{-1}] [▶] to select MATH.

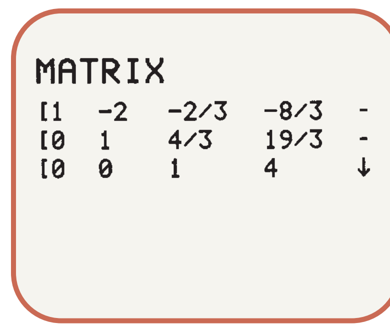
3. Use the cursor down arrow ([▼]) to highlight the command **A:ref(** . Press [ENTER].

4. Now press [2nd] [MATRX x^{-1}], and at the NAMES menu, scroll down to [*C*] and press [ENTER]. Close the brackets. Press [ENTER].

5. The row echelon form of the matrix will be on your screen as shown below. Now write the corresponding equations and complete the solution.



or



To change your decimals to fractions, press [MATH] and select **1: ▶ Frac**, then press [ENTER] [ENTER].

EXERCISE

1. Determine the intersection of the planes using a matrix and the **rref(** or **ref(** command.

a) $3x - 4y + 8 = 0$
$x - y + 3z = 1$
$6x - 5y + 2z = 7$

b) $2x - y - 5 = 0$
$y - 2z = 5$
$3x - 12y - 6z - 15 = 0$

c) $5x - 8y - 9z - 2 = 0$
$x - 3y + 2z = 6$
$10x - 16y - 18z - 4 = 0$

Answers: a) $x = \frac{60}{17}, y = \frac{49}{17}, z = \frac{2}{17}$ b) $x = \frac{25}{9}, y = \frac{5}{9}, z = \frac{-20}{9}$

c) $z = t, y = -4 - \frac{19}{7}t, x = -6 + \frac{-43}{7}t, t \in R$

2. Turn to the Chapter 8 Review Exercise and repeat question 22. For each part, write the row echelon form of the matrix and the reduced row echelon form of the matrix. Give the solution to each and the proper geometrical interpretation.

SECTION 8.4 — SOLVING A MATRIX USING GAUSS-JORDAN ELIMINATION

In Section 8.4 you were introduced to the intersection of two planes and how a matrix could be used to describe the relationship of their intersection. You saw how to make an augmented matrix to represent the planes and how to reduce this matrix through Gauss-Jordan elimination. We will now use the TI-83+ to reduce the matrix and find the relationship between the planes.

First, recall that when performing Gauss-Jordan elimination we may do the following:

- multiply any row by a non-zero constant.
- replace any row by the sum (or difference) of that row and a multiple of another row.
- interchange any rows.

The TI-83+ calculator provides commands to perform each of these operations. These commands are found in the MATRX MATH menu. They are

- **rowSwap(** (Used to interchange rows.)
- **row+(** (Used to add rows together.)
- ***row(** (Used to multiply a row by a non-zero constant.)
- ***row+(** (Used to multiply a row by a non-zero constant and then add that row to another row.)

Let us reduce the matrix given in Example 1 in Section 8.4 using the TI-83+.

EXAMPLE 1

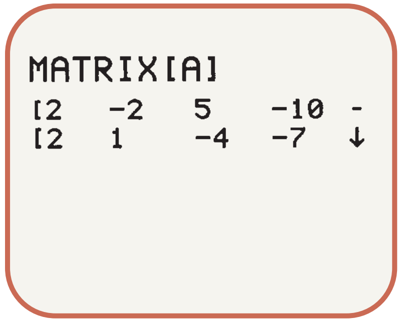
Find the intersection of the two planes

$2x - 2y + 5z + 10 = 0$ and $2x + y - 4z + 7 = 0$.

Solution

Define the matrix $[A]$ as a matrix containing the coefficients from each equation. (See Setting Up a Matrix Using the TI-83+).

Print matrix $[A]$ on your screen. You should have an augmented matrix on your screen like the one that follows.



1. We now want to obtain a zero in position 2, 1 of our matrix. To do this we must multiply row 1 by −1 and then add this row to row 2. We will then replace row 2 with these new values.

Procedure

i) Press [2nd] [x^{-1}] (MATRX), and then cursor right to the MATH menu and down to **F:*row+(**. Press [ENTER].

ii) On your screen you should have ***row+(**.

The format for this command is
***row+(** *constant multiplier, name of the matrix, row to be multiplied, row to be added*).

So, on our screen we want to have ***row+(**−1, [A], 1, 2). This says we will multiply row 1 of matrix [*A*] by −1 and then add this to row 2.

We accomplish this with the following steps:

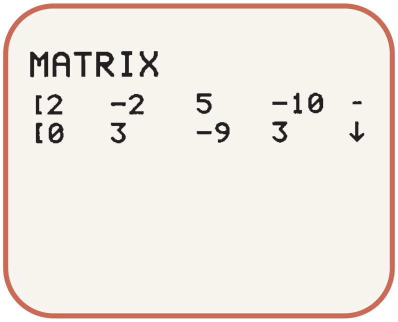
After ***row+ (**, follow these steps:

1. Type -1. (Remember to use the white [(−)] key on the calculator).
2. Press the comma key [,] on the calculator.
3. Press [2nd] [x^{-1}] (MATRX) and select [A], then press [ENTER].
4. Press the comma key on the calculator.
5. Type 1.
6. Press the comma key.
7. Type 2.
8. Close the bracket.

You should now have ***row+(**−1, [A], 1, 2) on your screen.

9. Press [ENTER].

You should have the following matrix on your screen:



iii) This operation does not actually change matrix [*A*]. To do this, press [STO▸] [2nd] [x^{-1}] (MATRX), select [A], and press [ENTER] [ENTER]. Matrix [*A*] is now equal to the matrix above.

2. We now want to divide row 2 by 3. In other words, we multiply row 2 by $\frac{1}{3}$. To do this we will use the ***row(** command. The format for this command is ***row(** *constant multiplier, matrix name, row to be multiplied*)

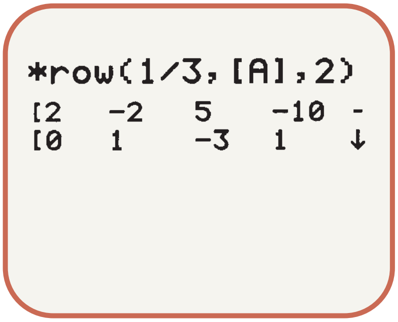
- Press [2nd] [x^{-1}] (MATRX) and select MATH.
- Cursor down to **E:*row(** , and then press [ENTER].
- Type 1, press the [÷] key, and type 3. (This is one third.)
- Press the comma key.
- Press [2nd] [x^{-1}] (MATRX) and select [A], then press [ENTER].
- Press the comma key.
- Type 2.
- Close the bracket.

You should now have ***row(1/3, [A], 2)** on your screen.

- Press [ENTER].

Now store this result in matrix [*A*] by pressing [STO▸] [2nd] [x^{-1}] (MATRX), selecting [A], and then pressing [ENTER] [ENTER].

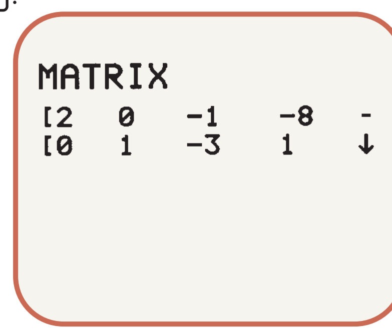
You should have the following matrix on your screen:



3. We now want to make the value in position 1, 2 equal to zero. To do this, we multiply row 2 by 2, add this to row 1, and replace row 1 by this amount.

- Press [2nd] [MATRX x^{-1}] and select MATH.
- Scroll down to **F:*row+(** , and press [ENTER].
- Type 2.
- Press the comma key.
- Press [2nd] [MATRX x^{-1}] and select [A], then press [ENTER].
- Press the comma key.
- Type 2.
- Press the comma key.
- Type 1.
- Close the bracket.
- Press [ENTER].
- Now store this result in matrix [*A*] by pressing [STO▸] [2nd] [MATRX x^{-1}], selecting [A], and then pressing [ENTER] [ENTER].

The matrix on your screen should look like the following:



4. We now multiply row 1 by 1 so we obtain a leading 1 in that row.

- Press [2nd] [MATRX x^{-1}] and select MATH.
- Scroll down to **E:*row(** , and press [ENTER].
- Type 1, press the [÷] key, and type 2. (This is one half.)
- Press the comma key.
- Press [2nd] [MATRX x^{-1}], select [A], and press [ENTER].
- Press the comma key.
- Type 1.
- Close the bracket.
- Press [ENTER].

The matrix on your screen should look like this:

```
MATRIX
[1   0   -.5   -4   -
[0   1   -3    1    ↓
```

You can now write the corresponding equations from the reduced matrix and describe the relationship between the intersection of the planes. In this case, the intersection is a line and we have written it below in parametric form.

$$z = t; x = -4 + \frac{1}{2}t \quad \text{and} \quad y = 1 + 3t; t \in R$$

Extension: Use a 3-D graphing program such as Winplot or Zap-a-Graph and graph these planes to see their line of intersection.

SECTION 8.4 — SUMMARY OF GAUSS-JORDAN COMMANDS USING THE TI-83+

SUMMARY OF GAUSS-JORDAN COMMANDS USING THE TI-83+

1. **rowSwap(** matrix name, row **m**, row **n**)	• used to interchange rows **m** and **n** in the defined matrix.
2. **row+(** matrix name, row **m**, row **n**)	• used to add row **m** to row **n** in the defined matrix, storing the result in row **n**.
3. ***row(** constant, matrix name, row **m**)	• used to multiply row **m** in the defined matrix by a non-zero constant, storing that result in row **m**.
4. ***row+(** constant, matrix name, row **m**, row **n**)	• used to multiply row **m**, in the defined matrix, by a non-zero constant and then to add row **m** to row **n**, storing the result in row **n**.

Remember, in all cases, the defined matrix is not actually changed until after you perform the commands:

STO➡ 2nd MATRX x^{-1} select [*your defined matrix to store in*], then ENTER ENTER.

EXERCISE

1. Return to Exercise 8.4 and complete question 6 using the TI-83+ calculator and the row reducing commands available.

2. Use the TI-83+ calculator to give the geometrical interpretation of the system of equations in question 1 of Exercise 8.4.

3. Using the TI-83+ calculator, complete questions 3, 4, 7, and 8 from Exercise 8.5.

SECTION 8.5 — MATRICES AND DETERMINANTS

In this activity, we will use *determinants* to investigate the conditions in which a system of n linear equations in n variables has a unique solution. Also we will look at using *determinants* to investigate whether sets of vectors are linearly dependent or independent.

For our investigation, we will look at the determinants of matrices having sizes 2×2 or 3×3.

The determinant of a 2×2 matrix $[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Example

Calculate the determinant of the matrix

$$H = \begin{bmatrix} 2 & 4 \\ -1 & 0.25 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -6 & 15 \\ 10 & -25 \end{bmatrix}.$$

Solution 1

$$\det H = (2)(0.25) - (-1)(4) = 4.5 \qquad \det B = (-6)(-25) - (10)(15) = 0$$

Solution 2

Find the determinant for H using the TI-83+ graphing calculator.

1. Set up matrix H and define it as $[A]$.

2. Press **2nd** **MATRX x^{-1}** and select MATH.

3. Select **1:det(** and press **ENTER**.

4. Press **2nd** **MATRX x^{-1}**, select [A], and then press **ENTER**.

5. Close the bracket.

6. Press **ENTER**.

Using the TI-83+, find the determinant for matrix *B*.

Investigation of Dependency Using Determinants

ACTIVITY 1

1. Are the vectors $\vec{a} = (-3, 8)$ and $\vec{b} = (12, -32)$ dependent? Explain.

2. Create a matrix *A* that contains the components of $\vec{a}$ and $\vec{b}$ written horizontally. Calculate the determinant of this matrix.

   Your matrix should look like $\begin{bmatrix} -3 & 8 \\ 12 & -32 \end{bmatrix}$.

3. What is the determinant?

ACTIVITY 2

1. Are the vectors $\vec{m} = (6, 5)$ and $\vec{n} = \left(3, \frac{5}{2}\right)$ dependent? Explain.

2. Create a matrix *A* that contains the components of $\vec{m}$ and $\vec{n}$ written horizontally. Calculate the determinant of this matrix.

   Your matrix should look like $\begin{bmatrix} 6 & 5 \\ 3 & \frac{5}{2} \end{bmatrix}$.

3. What is the determinant?

ACTIVITY 3

1. Are the vectors $\vec{r} = (12, -5)$ and $\vec{t} = (13, 1)$ dependent? Explain.

2. Create a matrix *A* that contains the components of $\vec{r}$ and $\vec{t}$ written horizontally. Calculate the determinant of this matrix.

   Your matrix should look like $\begin{bmatrix} 12 & -5 \\ 13 & 1 \end{bmatrix}$.

3. What is the determinant?

Conclusion

What can be said about the determinant of a matrix that contains dependent vectors?

Construct a theorem that describes the determinant of a matrix containing dependent vector components.

SECTION 8.5 — CRAMER'S RULE FOR 2 × 2 MATRICES

Determinants can be used to determine whether a system of linear equations in two variables has a unique solution. We do this using Cramer's Rule.

Cramer's Rule

Given the system of equations
$$\begin{Bmatrix} ax + by = k_1 \\ cx + dy = k_2 \end{Bmatrix},$$

there is a unique solution if
$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

If $D \neq 0$, then the unique solution is

$$x = \frac{D_1}{D},\ y = \frac{D_2}{D},\ \text{where}\quad D_1 = \begin{vmatrix} k_1 & b \\ k_2 & d \end{vmatrix} \quad \text{and} \quad D_2 = \begin{vmatrix} a & k_1 \\ c & k_2 \end{vmatrix}.$$

EXERCISE

Using the TI-83+ and Cramer's Rule, determine whether the following systems have unique solutions. Where a unique solution does exist, give that solution.

a) $2x - 3y = 7$
$5x + y = 9$

b) $4x - 2y = 7$
$-6x + 3y = 1$

c) $5x - 0.25y = 3$
$-3x - 7y = 0.25$

d) $4x - 3y - 8 = 0$
$2x + 5y - 12 = 5$

e) $y - 5 = 0$
$x + 10 = 7$

Extension: Prove Cramer's Rule or find a resource that has the proof and follow it through.

SECTION 8.5 — CRAMER'S RULE FOR 3 × 3 MATRICES

The determinant of the 3 × 3 matrix, A, where $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, is given by

$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

The three vectors contained in a 3 × 3 matrix are dependent if $\det A = 0$.

We can calculate the determinant using the TI-83+ by defining our 3 × 3 matrix and using the **det(** function under the MATRX MATH menu.

Example

Determine whether the given vectors in R^3 are dependent or independent.

$$\vec{u} = (4, 2, 5); \vec{v} = (-4, 2, 9); \vec{w} = (4, 6, 19)$$

Solution

We will represent the vectors in a 3 × 3 matrix and find the determinant using the TI-83+. If the determinant is equal to zero, then the vectors are dependent.

1. Set up a matrix to represent these vectors and define it as [A].
2. Press 2nd MATRX x^{-1} and select the MATH menu.
3. Select **1:det(** and press ENTER.
4. Press 2nd MATRX x^{-1}, select [A], and then press ENTER.
5. Close the bracket.
6. Press ENTER.

As you can see, the vectors are dependent because the determinant is equal to 0. We knew this to be the case, and we can write the vectors as a linear combination of one another:

$$2(4, 2, 5) + (-4, 2, 9) = (4, 6, 19)$$

As we did with systems of equations in R^2, we can use Cramer's Rule for 3×3 matrices to find the unique solution to a system when it exists.

Cramer's Rule

A system of three equations in three variables has a unique solution if and only if the 3 x 3 determinant of the coefficients is *not* zero. If this is the case, then the unique solution of the system given in

$$\begin{Bmatrix} ax + by + cz = k_1 \\ dx + ey + fz = k_2 \\ gx + hy + iz = k_3 \end{Bmatrix} \quad \text{is} \quad x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D},$$

where $D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ **and** $D_1 = \begin{vmatrix} k_1 & b & c \\ k_2 & e & f \\ k_3 & h & i \end{vmatrix}, D_2 = \begin{vmatrix} a & k_1 & c \\ d & k_2 & f \\ g & k_3 & i \end{vmatrix}, D_3 = \begin{vmatrix} a & b & k_1 \\ d & e & k_2 \\ g & h & k_3 \end{vmatrix}.$

EXERCISE

Using the TI-83+ and Cramer's Rule, determine the unique solution for each system, where it exists. Interpret the unique solution.

a) $\begin{Bmatrix} 2x + 3y + z = 0 \\ x - y - z = 0 \\ 3x + y + z = 0 \end{Bmatrix}$

b) $\begin{Bmatrix} 2x - 3y + z = 7 \\ x + 4y - z = 2 \\ -x + 2z = 5 \end{Bmatrix}$

c) $\begin{Bmatrix} x + y - z = 2 \\ 2x - y + z = 3 \\ 5x - y + z = 8 \end{Bmatrix}$

d) $\begin{Bmatrix} \frac{1}{4}x - 3z = 8 \\ 2y - 5z = 0 \\ 3x - 5y - 2 = 0 \end{Bmatrix}$

e) $\begin{Bmatrix} 0.75x - 8y + z = 0 \\ 6x - 7y + 2z - 2 = 8 \\ 2x - 6y - 3z - 12 = 0 \end{Bmatrix}$

Glossary

Absolute Value: the positive value of a real number, disregarding the sign. Written as $|x|$. For example, $|3| = 3$, $|-4| = 4$, and $|0| = 0$.
Acceleration: the rate of change of velocity with respect to time.
Acute Angle: a positive angle measuring less than 90º.
Algebraic Equation: an equation of the form $f(x) = 0$ where f is a polynomial algebraic function and only algebraic operations are required to solve it.
Algorithm: derived from the name of a ninth-century Persian author, Abu Ja'far Mohammed ibn al Khowarizmi. A step-by-step description of a solution to a problem.
Altitude: the line segment drawn from one vertex of a triangle perpendicular to the opposite side. The three altitudes of a triangle intersect at the orthocentre.
Anagram: a rearrangement of all the letters of a word to form a new word.
Analog: a device that uses physical quantities rather than digits for storing and processing information.
Angle: given two intersecting lines or line segments, the amount of rotation about the point of intersection (the vertex) required to bring one into correspondence with the other.
Angle of Inclination (of a line): the angle α, $0 \leq \alpha \leq 2\pi$, that a line makes with the positive x-axis. Also known as the angle of slope or gradient of a line.
Arc: a portion of a curve. For a circle, an arc is a portion of the curved line (circumference) that encloses the circle. A line drawn through a circle may divide the circumference into two unequal arcs: a major and a minor arc.
Arithmetic Progression (Sequence): an ordering of numbers or terms where the difference between consecutive terms is a constant.
Arithmetic Series: the sum of the indicated terms of an arithmetic sequence.
Assumption: a statement that is to be accepted as true for a particular argument or discussion.
Assymetric: unbalanced, without symmetry.
Asymptote: a straight line is an asymptote of a curve if the curve and the line approach indefinitely close together but never meet.
Augmented Matrix: a matrix made up of the coefficient matrix and one additional column containing the constant terms of the equations to be solved.
Axiom: a statement assumed to be true without formal proof. Axioms are the basis from which other theorems and statements are deduced through proof.
Axis: a line drawn for reference in a coordinate system. Also, a line drawn through the centre of a figure.
Axis of Symmetry: a line that passes through a figure in such a way that the part of the figure on one side of the line is a mirror reflection of the part on the other side.
Basis Vectors: a set of linearly independent vectors such that every vector in that vector space can be expressed as some linear combination of the basis vectors. In the Cartesian coordinate system, the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ form a basis for the two- or three-dimensional spaces in which vectors exist.
Biconditional Statement: a statement in which the truth of either part of the statement depends upon the truth of the other (expressed as $p \leftrightarrow q$). p is true if and only if q is true or q is true if and only if p is true.
Binomial: an algebraic expression with two terms. For example, $2x + 3y$ is a binomial.
Binomial Theorem: the expansion in terms of powers of a and b for the binomial (two-termed) expression $(a + b)^n$. It was first discovered by the Islamic mathematician al-Karaji in the tenth century, and later rediscovered by Newton in the seventeenth century.
Cardioid: a plane curve traced by a point on a circle rolling on the outside of a circle of equal radius.
Cartesian Coordinate System: a reference system in two-dimensional space, consisting of two axes at right angles, or three-dimensional space (three axes) in which any point in the plane is located by its displacements from these fixed lines (axes). The origin is the common point from which each displacement is measured. In two-dimensional space, a set of two numbers or coordinates is required to uniquely define a position; in three-dimensional space, three coordinates are required.

Cartesian (Scalar) Equation of Line: an equation of the form $Ax + By + C = 0$ where the vector $\overrightarrow{(A, B)}$ is a normal to the line. There is no Cartesian Equation of a line in three-dimensional space.

Cartesian (Scalar) Equation of a Plane: an equation of the form $Ax + By + Cz + D = 0$ where the vector $\overrightarrow{(A, B, C)}$ is normal to the plane.

Central Angle (of a circle): an angle subtended by an arc of the circle that has the centre of the circle as its vertex and the radii of the circle as its sides.

Centroid: the centre of mass of a figure. The centroid of a triangle is the point of intersection of the three medians.

Chord: a line segment joining points on a curve. In a circle, the maximum length of a chord is the length of the diameter.

Circle: the locus of a point that moves so that it is always a constant distance (the radius) from a fixed point (the centre). Also, the set of all points in the plane that are equidistant from a fixed point.

Circle, Equation in Standard Form: $(x - a)^2 + (y - b)^2 = r^2$, where the centre of the circle is (a, b) and the radius is r.

Circle, Equation in General Form: $x^2 + y^2 + 2gx + 2fy + c = 0$. The standard form of the equation can easily be derived from the general form and vice versa.

Circumcentre: the point at which the perpendicular bisectors of the sides of a triangle meet, or the centre of the circumscribed circle that passes through the three vertices of the triangle.

Circumference: the boundary line enclosing a figure or the length of that line. Usually applied to the boundary line of a circle, where the length of the circumference or perimeter is $2\pi r$.

Clockwise Rotation: a rotation in the same direction as the movement of the hands of a clock.

Coefficient Matrix: a matrix whose elements are the coefficients of the unknown terms in the equations to be solved by matrix methods.

Collinear: lying in the same straight line. Two vectors are said to be collinear if and only if it is possible to find a non-zero scalar, a, such that $\vec{x} = a\vec{u}$.

Common Difference: the difference between any two consecutive terms in an arithmetic sequence. For example, in the sequence 5, 9, 13, 17, …, the common difference is 4.

Common Ratio: the ratio of consecutive terms in a geometric sequence. For example, in the sequence 3, 12, 48, 192, …. the common ratio is 4.

Combinatorics: the branch of mathematics that deals with systematic ways of counting the number of arrangements in which a set of objects can be arranged.

Commutative: the property that, for certain binary mathematical operations, the order does not matter. Addition and multiplication are commutative operations.

Complement of a Set: If U is the universal set and A is any subset, the complement of A consists of all elements of U not in A. The set A plus its complement equals the universal set U.

Complementary Angle: two angles are called complementary if the two angles add up to a right angle.

Complex Number: a number of the form $z = a + bi$ where a and b are real numbers and $i = \sqrt{-1}$.

Complex Plane: a graphical method of representing a complex number $z = a + bi$, where the real part a is plotted along the horizontal real axis and its imaginary part b along the vertical imaginary axis of a Cartesian coordinate system. The complex number is displayed as the ordered pair (a, b) in the coordinate plane or as a vector drawn from the origin to the point (a, b).

Concurrency: the condition where lines meet together at a common point. In a triangle, each of the medians, altitudes, angle bisectors, and perpendicular bisectors of the sides are concurrent.

Concyclic Points: points that lie on a circle.

Conditional Statement: a statement in which the first part of the statement implies the second part. Expressed as $p \rightarrow q$ (if p is true then q is true).

Congruency: the condition of being equal in size and shape. Two figures are said to be congruent if one of them can be made to coincide at every point with the other by either a translation or rotation in space.

Conjecture: a generalization or educated guess made using inductive reasoning.

Conjugate Axis: the imaginary axis of symmetry of a hyperbola that is perpendicular to the transverse axis.

Conjugate Complex Numbers: two complex numbers of the form $a + bi$ and $a - bi$. The product of the two numbers is a real number.

Converse: a statement formed from another statement by interchanging the subject and the predicate. The result of the interchange may not necessarily be true. For example, if $p \rightarrow q$, the converse, $q \rightarrow p$, may be true or false.
Convex Polygon: a polygon in which each of the interior angles is less than 180º.
Coordinates: a set of numbers that uniquely define the position of a point with respect to a frame of reference. Two coordinates are required in two-dimensional space; three in three-dimensional space.
Coordinate System: a frame of reference used for describing the position of points in space. See *Cartesian Coordinate System.*
Consistent: a linear system is said to be consistent if it has at least one solution. If there are no solutions, the system is said to be inconsistent.
Coplanar: points or lines lying in a plane are said to be coplanar. Three points uniquely define a plane.
Corollary: a theorem that follows so obviously from the proof of some other theorem that virtually no further proof is required.
Cosine Law: a formula relating the lengths of the three sides of a triangle and the cosine of any angle in the triangle. If a, b, and c are the lengths of the sides and A is the magnitude of the angle opposite a, then $a^2 = b^2 + c^2 - 2bc \cos A$. Two other symmetrical formulas exist involving expressions for the other two sides.
Counterclockwise Rotation: a rotation in the opposite direction of the movement of the hands of a clock.
Cross Product (Vector): a vector quantity that is perpendicular to each of two other vectors and is defined only in three-dimensional space.
Cube: the three-dimensional Platonic solid that is also called a hexahedron. The cube is composed of six square faces that meet each other at right angles, and has eight vertices and 12 edges.
Cyclic Polygon: a polygon with vertices upon which a circle can be circumscribed. Since every triangle has a circumcircle, every triangle is cyclic.
Cyclic Quadrilateral: a quadrilateral whose four vertices lie on a circle. The four vertices are concyclic points.
Cylinder: a three-dimensional solid of circular cross-section in which the centres of the circles all lie on a single line of symmetry.
Deductive Reasoning: a method of reasoning that allows us to prove a statement to be true by progressing from the general to the particular.
Degree: the unit of angle measure defined such that an entire rotation is 360º. The degree likely derives from the Babylonian year, which was composed of 360 days (12 months of 30 days each). The degree is subdivided into 60 minutes per degree and 60 seconds per minute since the Babylonians used a base 60 number system.
Diagonal: a line connecting two non-adjacent vertices of a polygon.
Diameter: a line segment joining two points on the circumference of a circle or sphere and passing through the centre.
Dilatation: a transformation that changes the size of an object.
Direct Proportion: two quantities x and y are said to be in direct proportion if $y = kx$ where k is a constant. This relationship is commonly written as $y \infty x$.
Direction Angles (of a vector): the angles that a vector makes with the x-, y-, and z-axes, respectively, where the angles lie between 0º and 180º.
Direction Cosines (of a vector): the cosines of the direction angles of a vector.
Direction Numbers (of a line): the components of the direction vector of a line. If the direction vector is normalized into a unit vector, the resulting components represent the direction cosines of the line.
Direction Vector (of a line): a vector that determines the direction of a particular line.
Discriminant: in the quadratic formula, the value under the square root sign: $b^2 - 4ac$. It is used to determine the nature of the roots of an equation.
Disjoint Sets: two sets that have no elements in common. If set A and set B have no elements in common, then $A \cap B = 0$.
Displacement: a translation from one position to another, without consideration of any intervening positions. The minimal distance between two points.
Distance: the separation of two points measured in units of length, or the length of the path taken between two points, not necessarily the minimal distance (displacement).
Dot (Scalar) Product: the multiplication of two vectors resulting in a scalar quantity. It is calculated by multiplying the magnitude of each of the two vectors by the cosine of the angle between the vectors with the tails of the two vectors joined together.

Element of a Set: any member of the set.

Equiangular Triangles: three-sided figures where the three contained angles are equal to each other. (See *Similar Triangles*).

Equilibrant Force: a force equal in magnitude but acting in the opposite direction to the resultant force. It exactly counterbalances the resultant force, resulting in a state of equilibrium.

Equilibrium, State of: a state of rest or uniform motion of an object that will continue unless the object is compelled to change position by the action of an outside force.

Equivalent: equal in value.

Explicit: precisely and clearly expressed or readily observable; leaving nothing to be implied.

Exponent: the notation b^p means the product of p factors of b where b is the base and p the exponent of the power b^p.

Factorial: for any positive integer n, the product of all the positive integers less than or equal to n. Factorial n is denoted by $n!$.
$n! = n(n-1)(n-2) \ldots 3.2.1.$ 0! is defined as 1.

Fibonacci Numbers: the sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, … in which each number is the sum of the two previous ones. $x_1 = 1, x_2 = 1, x_{n+1} = x_n + x_{n-1}\ n \geq 2$.

Force: a physical influence that causes a change in the direction of a physical object.

Formula: a mathematical equation relating two or more quantities.

Fractal: a curve of surface that contains more but similar complexity the closer one looks.

Frame of Reference: a fixed arrangement used for describing the position of points or objects.

Frequency: the number of occurrences within a given time period (usually 1 second) or the ratio of the number of observations in a statistical category to the total number of observations.

Gaussian Elimination: a matrix method used to solve a system of linear equations, in which all elements below the main diagonal are made 0 by row reduction, and the resulting lines are considered as equations.

Gauss-Jordan Elimination: a matrix method used to solve a system of linear equations, in which all elements on the main diagonal are made 1, and other elements above or below the main diagonal are made 0 using row reduction.

Geographic Profiling: a field developed by mathematicians to help police determine where perpetrators of crime are likely to live.

Geometric Progression (Sequence): a succession of numbers in which each consecutive number is found by multiplying the previous number by a fixed multiplier (the common ratio).

Geometric Series: the indicated sum of the terms of a geometric sequence.

Geometry: the branch of mathematics that deals with the shape, size, and position of figures in space.

Golden Ratio: the division of a line segment AB by an interior point P so that $\frac{AB}{AP} = \frac{AP}{PB}$. It follows that $\frac{AP}{PB} = \frac{1 + \sqrt{5}}{2}$, which is a root of $x^2 - x - 1 = 0$.

Gradient: the slope or steepness of a line or curve.

Gravity: the force of attraction exerted by one object on another.

Hexagon: a six-sided polygon.

Hypotenuse: the side opposite the right angle in a right-angled triangle. It is always the longest of the three sides.

Hypothesis: a concept that is not yet verified but that, if true, would explain certain facts or phenomena.

Identity: a mathematical statement of equality that is true for all values of the variables. For example, $\sin^2\theta + \cos^2\theta = 1$ is an identity, true for all values of the variable.

Implicit: implied but not directly expressed.

Inconsistent: a linear system of equations that has no solution.

Indirect Proof (Proof by Contradiction): an approach where all possible outcomes are listed and all but one are eliminated through an intelligent reasoning process.

Inductive Reasoning: a method of reasoning that allows us to prove a statement to be true by progressing from specific examples of data or collected evidence to a general conclusion.

Intercept: the directed distance along an axis from the point of origin to a point of intersection of the graph of a curve with that axis.

Intersection (of sets): a subset corresponding to the elements common to two sets. The intersection of set A and set B is denoted by $A \cap B$ and contains only elements present in both sets.

Irrational Number: a real number that cannot be expressed as the ratio of two integers.

Isosceles: having two sides of equal length
Iteration: a method of evaluating a function where an initial value is calculated, and each subsequent term is calculated based on the output from the previous term.
Lever Arm: the distance along the shaft from the axis of rotation to the point at which the force is applied.
Linear Combination (of vectors): an expression that consists only of scalar multiples of vectors (for example, $a\vec{u} + b\vec{v} + c\vec{w}$, $a, b, c \in R$).
Linear Dependence (of vectors): a set of vectors $\vec{u}, \vec{v}, \vec{w}, \vec{x}$... is linearly dependent if a linear combination of them (for example, $a\vec{u} + b\vec{v} + c\vec{w} + d\vec{x}$ ...) produces the zero vector $[\vec{0}]$ and *not all* of a, b, c and d… are zero.
Linear Independence (of vectors): a set of vectors $\vec{u}, \vec{v}, \vec{w}, \vec{x}$... is linearly independent if the only linear combination $a\vec{u} + b\vec{v} + c\vec{w} + d\vec{x}$ ... that produces the zero vector $[\vec{0}]$ is the one in which *all* of the scalars a, b, c, d… are zero.
Linear System (of equations): a set of two or more linear equations. A system of linear equations may have a unique solution, an infinite number of solutions, or no solution.
Locus: a set of points that satisfy a given condition or the path traced out by a point that moves according to a stated geometric condition. See *Circle*.
Magnitude: the property of relative size or extent. The magnitude of a vector is the length of the vector from the tail to the head.
Mathematical Induction: a system of reasoning applied to certain theorems about integers, leading from specific facts to general conclusions.
Matrix: a rectangular (or square) array of numbers set out in rows and columns. The numbers are called elements. The number of elements is the product of the number of rows multiplied by the number of columns.
Median: the middle term of a sequence of numbers arranged in ascending order. If the sequence has an even number of terms, the median is the average of the two middle terms.
Median Line of a Triangle: the line in a triangle drawn from a vertex to the midpoint of the opposite side. The three medians of a triangle intersect at the centroid.
Newton's First Law of Motion: an object will remain in a state of rest or equilibrium unless it is compelled to change that state by the action of an external force.
Normal: perpendicular; any vector that is perpendicular to a line is called the normal to the line.
Obtuse Angle: an angle that measures greater than 90º and less than 180º.
Origin: the point of intersection of the coordinate axes drawn in a Cartesian coordinate system.
Orthocentre: the intersection of the three altitudes drawn in a triangle.
Orthogonal: meeting at right angles.
Palindrome: a sequence of symbols that reads the same from either end (for example, the number 1331 or the word *level*).
Parallel: being everywhere equidistant but not intersecting.
Parallelogram: a quadrilateral with opposite sides that are parallel.
Parallelepiped: a box-like solid, the opposite sides of which are parallel and congruent parallelograms.
Parameter: a variable that permits the description of a relation among other variables (two or more) to be expressed in an indirect manner using that variable.
Parametric Equation: an equation in which the coordinates are each expressed in terms of quantities called parameters (for example, $x = r\cos\theta$, $y = r\sin\theta$ $\theta \geq 0$). θ, the parameter, may assume any positive value.
Pascal's Triangle: a triangular array of numbers where each number in a particular row is equal to the sum of the two numbers in the row immediately above it. The triangle was studied by Pascal (1623–1662), although it had been described 500 years earlier by Chinese mathematician Yanghui and the Persian astronomer-poet Omar Khayyám. It is known as the Yanghui triangle in China.
Pentagon: a five-sided polygon.
Perfect Square: a number that can be expressed as the product of two equal factors.
Perimeter: the length of the boundary enclosing a figure.
Permutation: an ordered arrangement or sequence of a set of elements or objects.
Perpendicular: a straight line at right angles to another line.

Plane: a flat surface, possessing the property that the line segment joining any two points in the surface lies entirely within the surface.

Polygon: a closed plane figure consisting of n points (vertices) where $n \geq 3$ and corresponding line segments. A polygon of three sides is a triangle; of four sides, a quadrilateral; and so on.

Polyhedron: a solid bounded by plane polygons.

Position Vector: a vector drawn from the origin to the point marking the head of the vector.

Prime Number: a natural number (counting number) having no factors except itself and 1. The first primes are 2, 3, 5, 7, 11, 13, … . It is not common to include 1 among the prime numbers.

Prism: a polyhedron with two congruent and parallel faces.

Probability: the ratio of the number of favourable outcomes to the total number of possible outcomes.

Projection: a mapping of a geometric figure formed by dropping a perpendicular from each of the points onto a line or plane.

Proof by Contradiction (Indirect proof): an approach where all possible outcomes are listed and all but one are eliminated.

Pyramid: A polyhedron with one face (the base) as a polygon and all the other faces as triangles meeting at a common vertex (the apex). A right pyramid is a pyramid for which the line joining the centroid of the base and the apex is perpendicular to the base.

Pythagorean Theorem: in any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Quadrant: any one of the four areas into which a plane is divided by two orthogonal coordinate axes.

Radius: a line segment that joins a point on the circumference of a circle to the centre.

Ratio: a number or quantity compared with another. It is usually written as a fraction or with the symbol [:].

Rational Number: a number that can be expressed as an integer or as a quotient of integers (a fraction).

Real Number: any rational or irrational number.

Rectangle: A parallelogram in which the angles are right angles.

Recursion: a method of defining sequences in which the first term is defined and each subsequent term is determined by a process applied to preceding terms. See *Fibonacci Numbers* for an example.

Reduced Row-Echelon Form: a matrix derived by the method of Gauss-Jordan elimination that permits the solution of a system of linear equations.

Reflection: a transformation of a point, line, or figure that results in a mirror image of the original.

Resultant Force: the single force that has the same net effect of a group of several forces.

Rhomboid: a parallelogram with adjacent sides not equal.

Rhombus: a parallelogram having equal sides. The diagonals of a rhombus are at right angles to each other

Scalar: a quantity having magnitude only. Quantities having magnitude and direction are called vectors.

Scalar Dot Product: the multiplication of two vectors resulting in a scalar quantity. It is the multiplication of the magnitude of each of the two vectors by the cosine of the angle between them as they are joined at their tails.

Scalene Triangle: a triangle with no sides equal.

Secant: a line segment that cuts through a circle or other figure. In a circle, the portion of the secant inside the circle is called a chord.

Sector: the part of a circle that is bounded by two radii and the included arc.

Segment of a Circle: the part of a circle bounded by a chord and the arc subtending the chord.

Semicircle: the part of a circle bounded by the diameter and an arc.

Sequence: a set of numbers arranged in order according to some rule.

Series: the sum of the terms of a sequence.

Set: a collection of objects. The individual members of a set, called elements, share some property or rule that determines whether each element is in the set. The elements of a set are unique.

Set Theory: the systematic study of the properties of sets.

Sierpinski's Triangle: a simple fractal resulting from the recursive manipulation of an equilateral triangle.

Sigma Notation: a convenient method to express the sum of the terms of a sequence. For example, $S_n = \sum_{i=1}^{n} t_i$, where Σ is the upper-case Greek S indicating sum. The i is called the index and the values 1 and n give the range (inclusive) of the index in summation.

Similarity: two plane figures are similar if the angles of one, taken in order, are respectively equal to the angles of the other, in the same order, and the corresponding sides are proportional.

Similar Triangles: two triangles are similar if the angles of one, taken in order, are respectively equal to the angles of the other, in the same order, and the corresponding sides are proportional.

Sine Law: the theorem that relates the lengths of sides of a triangle to the sines of the angles opposite those sides. In a triangle with sides of lengths a, b, and c and angles opposite those sides A, B, and C, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Skew Lines: non-intersecting, non-parallel lines in space. Two lines are skew if and only if they do not lie in a common plane.

Slope: the steepness of a line or curve. In the plane, the slope is equal to the tan θ, where θ is the angle of inclination.

Sparse System: a linear system of equations involving a large number of equations, many having coefficients equal to zero.

Speed: the rate of change of distance with respect to time but without reference to direction. The average speed is the distance travelled divided by the travel time. Velocity is the quantity used when direction is indicated.

Sphere: the set of points in space at a given distance (the radius) from a fixed point (the centre). In Cartesian coordinates, the equation of a sphere is $x^2 + y^2 + z^2 = r^2$.

Square: a rectangle having all sides equal.

Statistics: a branch of mathematics dealing with the systematic collection and arrangement of large numbers of observations together with ways of drawing useful conclusions from such data.

Subset: a subset A of a set B is a set whose elements are all elements of B. A is called a proper subset if it does not contain all the elements of B. If it contains all the elements of B, it is called an improper subset of B.

Supplementary Angles: two angles whose sum is two right angles or 180º.

Symmetric Equation (of a line): the equation of a line determined by eliminating the parameter from the parametric equations of a line.

Symmetry: an attribute of a shape; exact correspondence of form on opposite sides of a dividing line (axis of symmetry) or plane.

Tangent: a line segment drawn to a figure that touches that figure at one and only one point.

Theorem: a statement that has been proved to be true, provided certain hypotheses (axioms) are true. Such a statement might not be deemed to be a theorem unless it is considered worthy of special attention by the mathematics community.

Torque: the action of a force that causes an object to turn rather than to change position.

Transformation: a change in the size, shape, or position of a figure. Examples of transformations are translations, reflections, rotations, and dilatations.

Translation: a transformation that changes only the position of a figure. A transformation that maps each point (x, y) on the figure to a new point $(x + a, y + b)$ where a and b are components of the translation vector $\overrightarrow{(a, b)}$.

Transpose (of a matrix): the interchange of rows and columns in a matrix. Row 1 becomes Column 1, and so on.

Transverse Axis: the real axis of symmetry of a hyperbola. See *Conjugate Axis*.

Trapezium: a quadrilateral with neither pair of opposite sides parallel. See *Trapezoid*.

Trapezoid: a four-sided planar figure in which two of the opposite sides are parallel.

Trigonometry: the study of the properties of trigonometric functions and their applications to various mathematical problems.

Trigonometric Functions: the sine (sin), cosine (cos), tangent (tan), and their inverses, cosecant (csc), secant (sec), and cotangent (cot). Also called circular functions.

Union (of sets): the set of elements made up of the elements of a pair of sets. The union of set A and set B is denoted by $A \cup B$ and contains all elements present in both sets. Elements found in each set are found only once in the resulting set.

Unit Vector: a vector with a magnitude of 1. Such vectors are denoted with a carat [ˆ] sign placed over the symbol. For example, $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors in the direction of the x-, y-, and z-axes.

Universal Set: the set of all possible elements of the type being counted. The universal set is designated as U.

Variable: a quantity, represented by an algebraic symbol, that can take on any one of a set of values.

Varignon Parallelogram: a parallelogram formed by joining the midpoints of the four sides of a quadrilateral.

Vector: a quantity possessing magnitude and direction. A directed line segment consisting of two points: the tail (initial point) and the head (end point). The distance between the tail and the head is the *magnitude* of the vector. The *direction* of the vector is the direction of the arrow drawn from the tail to head in reference to the basis vectors of the coordinate system.

Vector Cross Product: a vector quantity that is perpendicular to each of two other vectors and is defined only in three-dimensional space.

Vector Space: an abstract system, first developed by Peano, to enable the study of common properties of many different mathematical objects, including vectors.

Velocity: the distance travelled per unit time where the direction as well as the magnitude (speed) is important.

Velocity (Relative): the velocity of an object that an observer measures when he perceives himself to be stationary (at rest).

Venn Diagram: a picture used to display the universal set and the relationship between selected subsets. Sets and subsets are represented as circles; the boundary of the universal set is a rectangle.

Weight: the vertical force exerted by a mass (of a body) as a result of the force of gravitation.

Work: the action of a force on an object causing a displacement of the object from one position to another.

Zero of a Function: the value(s) of x for which the function $f(x) = 0$. A polynomial of the nth degree has n zeros or roots.

Zero Matrix: a matrix in which all the elements are zero.

Zero Vector: the zero vector $[\vec{0}]$ has zero magnitude. Its direction is undefined.

Answers

CHAPTER 1

Exercise 1.1

1. true **2.** $f(5) = 31$ is prime **3.** true **4. a.** not prime for $n = 5$ **b.** not prime for $n = 11$ **c.** not prime for $n = 41$ **5.** true **6.** not true **7.** true **8.** Not true

Exercise 1.3

9. a. 540° **b.** $180(n - 4)°$

Exercise 1.4

3. a. $3x - y = 11$ **b.** $x + 6y = 10$ **e.** they are concurrent
4. a. $3x + 4y = 12$ **b.** $x - 2y = -6$ **c.** (0, 3)
e. they are concurrent

Chapter 1 Test

4. 29°

CHAPTER 2

Review of Prerequisite Skills

1. a. 120 **b.** 60 **2. a.** 204 **b.** $15\frac{9}{13}$ **3.** 168 **4.** 96 **6.** 15° **7.** $3\frac{3}{5}$

Exercise 2.1

2. a. $AE = AD$ **3. a.** $\angle DEF$ **b.** AC

Exercise 2.2

2. $||^{gm}TQVU = ||^{gm}TVRU = ||^{gm}TPUV$

Exercise 2.3

2. a. 1*a*, *e*, *f* are true **b.** converse of 1*a*, *c* are true **c.** 1*a*
3. *b*, *e* are true

Exercise 2.5

1. a. 1:3 **b.** 1:4 **c.** 1:2 **d.** 1:3 **e.** 1:2 **f.** 1:4 **2. a.** $2\frac{2}{5}$ **b.** $6\frac{2}{3}$ **c.** 4
3. $a = 3\frac{3}{4}, b = 5\frac{1}{3}$ **4.** $PT:SQ = 4:3$ **7.** 5:4 **9.** 1:36

Exercise 2.6

1. *c*, *e*, *f*, *g*, *h* **2.** $AC = 24, BC = 30$ **3.** 40, 50, 65 cm **4.** 3.5 m
5. $XY = 6$ **6. a.** 441 **b.** 360 **7.** 2:7 **16.** $\frac{AD}{DB} = \frac{1}{4}$

Review Exercise

1. *c*, *d* **2.** $P = 2L + \frac{2A}{L}$ **3.** 3:4 **4.** 72 **5. a.** 2:5 **b.** 4:25
7. a. 2:25 **b.** 2:15 **9.** $y = 6.0$ **10. a.** 1:2 **c.** ΔADE:rect $ABCD = 1:8$, ΔABF:rect $ABCD = 1:6$ **11.** 45 **14.** $12\frac{1}{4}$
18. 3:11 **19.** 1:2 **21.** 21

Chapter 2 Test

2. a. 120 **b.** 6 **c.** 40 **d.** 12 **3.** $MY = 4\frac{8}{13}$ **4.** 288 **6.** 100

CHAPTER 3

Review of Prerequisite Skills

1. a. $36\pi, 9\pi$ **b.** $144\pi, 24\pi$ **c.** $25\pi, \frac{50\pi}{9}$ **d.** $100\pi, \frac{100\pi}{3}$ **e.** $49\pi, \frac{49\pi}{2}$ **f.** $64\pi, \frac{80\pi}{9}$

2. For sector angle of $s°$, $\frac{\pi s r^2}{360}$ **3. a.** 3:4 **b.** 4:9 **c.** 7:10
4. a. 9:16 **b.** 1:9 **c.** 1:8 **d.** 1:9 **5. a.** 4 cm **b.** 8π cm **6.** $\frac{81\pi}{4}$ cm^2
7. 2 cm **8.** 36

Exercise 3.1

1. 8 **2.** 12 **3.** 7 **4.** 16 **5.** $3 + 4\sqrt{3}$ **6.** $2\sqrt{13}$ **7.** $4\sqrt{21}$ **14.** 24

Exercise 3.2

1. a. 62.5° **b.** 60° **c.** 42°, 118° **d.** 90° **8.** 16

Exercise 3.3

2. $\angle ACD = 50°, \angle BCA = \angle ADB = 30°, \angle DAC = \angle DBC = 70°$, $\angle CED = \angle BEA = 100°, \angle BEC = 80°, \angle BAC = \angle BDC = 30°$
3. *A*, *D*, *F*, *E*; *B*, *D*, *E*, *C* **5.** 100° **12.** 1260°

Exercise 3.4

1. $AC = 12, OA = 13$ **2. a.** 24 **b.** 65° **c.** 120° **d.** $x = 65°$, $y = 115°, z = 50°$ **7.** 20 **9.** $2\sqrt{2}$ **12.** 6

Exercise 3.5

1. a. 9 **b.** $\frac{55}{3}$ **c.** $3\sqrt{2}$ **d.** $x = 50°, y = 80°, z = 40°$
e. $x = y = 30°$ **f.** $x = 70°, y = 220°$ **2.** $\angle DEF = 50°$, $\angle EDF = 66°, \angle DFE = 64°$ **4. a.** 3 **b.** 6 **c.** 22 **5. a.** 4
b. 27 **c.** 9 **d.** 4

Review Exercise

1. a. $x = 60°, y = 100°$ **b.** $x = 45°, y = 135°$ **2. a.** $\frac{24}{5}$ **b.** $\frac{7}{3}$
c. $2\sqrt{15}$ **d.** 50° **e.** 105° **f.** 38° **5.** $\angle ACD = 60°$, $\angle AEB = \angle DEC = 95°, \angle BEC = 85°$, $\angle BAC = \angle BDC = 25°, \angle ADB = \angle BCA = 20°$, $\angle DBC = \angle DAC = 75°$ **9.** $12 + \sqrt{39}$ **12.** $\angle HED = 60°$, $\angle HDE = 52\frac{1}{2}°, \angle EHD = 67\frac{1}{2}°$ **14.** 20 **15.** $\frac{90° - x}{2}$
16. 20° **18.** 30 **21.** $\angle x = 60°, \angle y = 65°, \angle z = 55°$

Chapter 3 Test

1. a. 50 **b.** $4\sqrt{3}$ **c.** $\frac{16}{3}$ **d.** $x = 120°, y = 80°$ **2.** 40° **3.** 9

Cumulative Review Chapters 1–3

1. 9.45 **3.** $2x^2 - 3x - 5$ **5. a.** $\Delta EBF = 8$ **b.** quad $AEFD = 16$
6. partial circle **15.** 4:9 **18.** 6 **20.** 60° **24.** 4.6

CHAPTER 4

Review of Prerequisite Skills

1. a. $\frac{\sqrt{3}}{2}$ **b.** $\frac{1}{2}$ **c.** $\frac{1}{\sqrt{2}}$ **d.** $-\sqrt{3}$ **e.** $\frac{\sqrt{3}}{2}$ **f.** 1 **2.** $\frac{4}{3}$ **3.** 7.36, 6.78, 50° **4.** 34°, 44°, 102° **5.** 5.8 km **6.** 8.7 km

Exercise 4.1

2. Vector: a, c, g, j, l **3. a.** $\overrightarrow{EF} = \overrightarrow{CB}$ **b.** $\overrightarrow{FE}, \overrightarrow{AD}$ **c.** $\overrightarrow{AB}, \overrightarrow{DE}$
d. $\overrightarrow{AB}, \overrightarrow{BC}$ **e.** $\overrightarrow{FD}, \overrightarrow{EB}$ **5. a.** 45° **b.** 135° **c.** 90° **8. a.** $\overrightarrow{AG}$
b. $\overrightarrow{GF}$ **c.** $\overrightarrow{AF}$ **d.** $\overrightarrow{CA}$ **e.** $\overrightarrow{GA}$ **9. a.** $\overrightarrow{AB} = \overrightarrow{DC}$ **b.** $\overrightarrow{AD} = -\overrightarrow{CB}$
c. $\overrightarrow{BD} = 2\overrightarrow{PD}$ **d.** $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AC}$ **10.** 10.9° **11.** $|\vec{a}| = 10$, 37°; $|\vec{b}| = 5$, 180°; $|\vec{c}| = \sqrt{29}$, 112°; $|\vec{d}| = 5$, 217°; $|\vec{e}| = 3$, 90°

12. a. 300 km, N20°W; 480 km, N80°E; 520 km SW. **b.** 1300 km, 5 h 25 min **13.** $-2 < k < 6$

Exercise 4.2

1. a. $\overrightarrow{DB}, \overrightarrow{AC}$ **b.** $\overrightarrow{CA}, \overrightarrow{BD}$ **c.** $\overrightarrow{DK} = \overrightarrow{AD}, \vec{u} + \vec{v} = \overrightarrow{BK}, \overrightarrow{AB}$ **2. a.** $\overrightarrow{BC} + \overrightarrow{CE}, \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}, \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DF} + \overrightarrow{FE}$ **b.** $\overrightarrow{BG} - \overrightarrow{EG}, \overrightarrow{BC} - \overrightarrow{EC}$ **3. a.** $\overrightarrow{PQ}$ **b.** $\overrightarrow{AG}$ **c.** $\overrightarrow{EC}$ **d.** $\overrightarrow{PR}$ **4. a.** 27.5, 24° to $\vec{v}$ **b.** 11.6, 51° to $\vec{u}$ **5.** 36.5 km, S54°E **6. a.** 4.4 **b.** 9.8 **7. a.** $\vec{u} \perp \vec{v}$ **b.** $0° \leq \theta < 90°$ **c.** $90° < \theta \leq 180°$ **9.** 7.7, 37° to $\vec{a}$ **10. a.** $3\vec{x} + \vec{y}$ **b.** $-2\vec{x} + 4\vec{y}$ **c.** $76\vec{y}$ **d.** $-7\vec{x} + 2\vec{y}$ **11. a.** $5\hat{i} - 2\hat{j} - \hat{k}$ **b.** $-2\hat{i} - \hat{j} + 12\hat{k}$ **c.** $5\hat{i} - 3\hat{j} - 15\hat{k}$ **12.** $\vec{x} = \frac{2}{11}\vec{a} + \frac{1}{11}\vec{b}, \vec{y} = \frac{5}{22}\vec{a} - \frac{3}{22}\vec{b}$ **17.** 6, 60° to $\overrightarrow{AB}$ **18.** 20 **20. a.** $\hat{j} - \hat{k}$ **b.** $\overrightarrow{BH} = \hat{j} + \hat{k}, \overrightarrow{DH} = \hat{i} + \hat{j}, \overrightarrow{FE} = -\hat{i} + \hat{j}, \overrightarrow{CH} = \hat{i} + \hat{k}, \overrightarrow{EG} = \hat{i} - \hat{k}$ **c.** $-\hat{i} + \hat{j} + \hat{k}$ **d.** $\overrightarrow{AH} = \hat{i} + \hat{j} + \hat{k}, \overrightarrow{CF} = \hat{i} - \hat{j} + \hat{k}, \overrightarrow{GD} = -\hat{i} - \hat{j} + \hat{k}$ **e.** $\sqrt{2}, \sqrt{3}$

Exercise 4.3

2. a. 173.2N, 100N **b.** 52.1N, 15.3N **c.** 58.3N, 47.2N **d.** 0N, 36N **3. a.** 5N, W **b.** 13.9N, N30°E **c.** 10N, N82°W **d.** 2N, NW **4.** 5N **5.** $\sqrt{|\vec{F}_1|^2 + |\vec{F}_2|^2}, \theta = \tan^{-1}\left(\frac{|\vec{F}_1|}{|\vec{F}_2|}\right)$ **6. a.** 9.8N, 15° to 8N **b.** 11.6N, 32° to 15N **7. a.** 57.7N, 146° to 48N **b.** 25.9N, 174° to 10N **9.** 87.9N, 71.7N **10.** $10\sqrt{3}$N **11.** b. c. **12. b.** 98° **13.** 911.6N, 879.3N **14.** 375N, 0N **15.** 937.9N, 396.4N **16.** $|\vec{u}_x| = 0, |\vec{u}_y| = 5; |\vec{v}_x| \cong 6.9, |\vec{v}_y| \cong 5.8; |\vec{w}_x| \cong 10.9, |\vec{w}_y| \cong 5.1$ **17.** 1420N **18. a.** 92N, 173N **19. a.** 108N **b.** 360N **20.** 54.5kN, 7.7kN **21.** 1035N **22.** 238N **23.** 19N, 58°, 38° **24.** 10° off the starboard bow **25. b.** Yes

Exercise 4.4

1. a. greater, south **b.** greater, north **c.** less, north **d.** less, south **2. a.** 60° **b.** not possible **3. a.** 15 km/h south **b.** 77 km/h north **c.** 92 km/h north **d.** 77 km/h south **4. a.** 0.6 km **b.** 6 min **5. a.** 1383 km **b.** N13°E **6.** 167 km/h, N5°W **7.** 2.5 m/s, N56°W **8.** 290 km/h, S81°E **9. a.** 204 km/h, 66 km/h **10. a.** S25°E **b.** 510 km/h **11.** N62°E **12.** b **13.** 12 m/s **14.** 94.3 km/h, N32°E

Review Exercise

1. a. $\vec{0}$ **b.** 1 **c.** 0 **5. a.** 5 **b.** 25 **c.** $\sqrt{a^2 + b^2}$ **7. a.** 32N **b.** 22° **8. a.** 79N **b.** 32N **9.** 605N, 513N **10.** 18N, 8° with 12N and 32° with 5N forces **11.** 94N, 80N **12. a.** N86°E **b.** 1 h 5 min **13.** 140 km/h **14. a.** 66 m **b.** 100 s **15. a.** N69°E **b.** 451 km/h **c.** 47 min **16.** 7.9 knots, N54°E **17.** 320 km, S70°E **18.** $a = k|\vec{v}|, b = k|\vec{u}|, k \in R$ **19.** $\vec{u} = -\vec{v}$

Chapter 4 Test

3. $7\vec{u} + 6\vec{v}$ **4.** distributive property **5.** 12.5 N **6.** 294N, 392N **7.** 68° upstream to the bank, 2 min 55 sec **8.** 640 knots, S44°E

CHAPTER 5

Exercise 5.1

2. a. $-5\hat{i} + 2\hat{j}$ **b.** $6\hat{j}$ **c.** $-\hat{i} + 6\hat{j}$ **3. a.** (2, 1) **b.** (−3, 0) **c.** (5, −5) **4. a.** $-2\hat{i} + \hat{j} + \hat{k}$ **b.** $3\hat{i} + 4\hat{j} - 3\hat{k}$ **c.** $4\hat{j} - \hat{k}$ **d.** $-2\hat{i} + 7\hat{k}$ **5. a.** (3, −8, 1) **b.** (−2, −2, −5) **c.** (0, 2, 6) **d.** (−4, 9, 0) **6. a.** $(-6\sqrt{2}, 6\sqrt{2})$ **b.** $(18\sqrt{3}, -18)$ **c.** (−15.8, −2.8) **d.** (0, −13) **7. a.** 12, 150° **b.** $8\sqrt{3}$, 240° **c.** 5, 37° **d.** 8, 90° **8. a.** (−4, −3) **b.** (5, −2) **c.** (−5, 0, 6) **d.** (4, −7, 0) **e.** (−6, 2, 6) **f.** (11, 12, 3) **10. a.** on the *z*-axis **b.** *yz*-plane **c.** *xz*-plane **d.** *xy*-plane **e.** a line parallel to the *z*-axis through (1, 3.0) **f.** a line with $x = y = z$ **11. a.** *xy*-plane **b.** *x*-axis **c.** *yz*-plane **d.** *z*-axis **e.** *xz*-plane **f.** *y*-axis **13. a.** $2\sqrt{5}$, 82° **b.** 6, 270° **c.** 15, 127° **d.** 1, 120° **e.** $\sqrt{2}$, 309° **f.** $\sqrt{6}$, 180° **14. a.** 14 **b.** 35.1 **c.** 1 **d.** 4 **16. a.** $\sqrt{17}$ **b.** $\left(\frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}\right)$, yes **17. a.** 7 **b.** $\left(\frac{2}{7}, \frac{-3}{7}, \frac{-6}{7}\right)$ **18.** $\left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right)$ **20. a.** (5, 9) **b.** (9, −6) **c.** (−5, 6, 0) **d.** (4, −9, 11) **22.** 55°, 125° **24.** 7

Exercise 5.2

2. a. (3, 3) **b.** (5, 20) **c.** (0, 0) **d.** (1, −7) **e.** (0, 0, 6) **f.** (2, 2, −8) **g.** (6, −2, 0) **h.** (−8, 11, 3) **i.** (0, 2, 5) **j.** (4, −6, 8) **k.** (−12, −42, −20) **l.** (21, 6, 32) **3. a.** $6\hat{i} - \hat{j}$ **b.** $6\hat{i} - 18\hat{j} + 18\hat{k}$ **c.** $-5\hat{i} + 2\hat{k}$ **d.** $90\hat{i} - 35\hat{j} - 35\hat{k}$ **4. a.** (1, −10, 14) **b.** (−1, −9, 10) **c.** (−5, −15, 16) **d.** (13, 11, 0) **e.** (−5, 8, −15) **f.** (6, −18, 29) **5. a.** $6\hat{i} - \hat{j} + 7\hat{k}$ **b.** $2\hat{i} + \hat{j} + 5\hat{k}$ **c.** $2\hat{i} - 3\hat{j} - 3\hat{k}$ **d.** $-2\hat{i} + 3\hat{j} + 3\hat{k}$ **6. a.** $\sqrt{11}$ **b.** $3\sqrt{3}$ **c.** $\sqrt{149}$ **7. a.** $5\sqrt{2}$ **b.** $\sqrt{30}$ **c.** (−5, −3, 0) **d.** $\sqrt{34}$ **e.** (5, 3, 0) **f.** $\sqrt{34}$ **9. a.** $\overrightarrow{AB} \nparallel \overrightarrow{CD}, |\overrightarrow{AB}| = |\overrightarrow{CD}|$ **b.** $\overrightarrow{AB} \nparallel \overrightarrow{CD}, |\overrightarrow{AB}| \neq |\overrightarrow{CD}|$ **c.** $\overrightarrow{AB} \parallel \overrightarrow{CD}, |\overrightarrow{AB}| = |\overrightarrow{CD}|$ **10.** (−13, −5) **11.** (−3, −7), (−7, 13), (17, −9) **12.** (7, 6, 0), (6, 4, −3), (5, 10, −1), (9, 10, −2) **13. a.** (4, 3) **b.** $\left(\frac{3}{2}, \frac{-7}{2}\right)$ **c.** (2, 6, 0) **d.** $\left(\frac{9}{2}, \frac{9}{2}, -3\right)$ **14. a.** 2, 1 **b.** $-5, \frac{-1}{3}, -20$ **15.** $\left(\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$ **16. a.** (0, 1, 0) **b.** (1, 0, 2) **17. a.** $\frac{5}{2}$ **b.** $\frac{5}{3}$ **18. a.** $\left(1, \frac{-1}{3}\right)$ **b.** $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ **c.** $\left(\frac{15}{4}, 1\right)$ **d.** $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ **19. a.** $\left(\frac{18}{11}, \frac{-41}{11}\right)$ **b.** $\left(\frac{2}{11}, \frac{-17}{11}, \frac{72}{11}\right)$

Exercise 5.3

1. a. $|\vec{a}||\vec{b}|, 0, -|\vec{a}||\vec{b}|$ **b.** acute, obtuse, 90° **2. a.** $6\sqrt{2}$ **b.** 15 **c.** $\frac{-27\sqrt{2}}{2}$ **d.** 0 **3. a.** 0, perpendicular **b.** −29, not perpendicular **c.** 0, perpendicular **d.** −28, not perpendicular **4. a.** 0 **b.** −14 **c.** 0 **d.** 25 **e.** 6 **f.** 130, a and c are perpendicular **5. a.** (3, 2), (−6, −4), $\left(\frac{3}{13}, \frac{2}{13}\right)$ **b.** 2 **6. a.** (0, 1, 3), (3, 2, 0), (3, 4, 6) **b.** infinite number **7. a.** 0.9931 **b.** −0.1750 **8. a.** 107° **b.** 89° **c.** 55° **d.** 73° **9. a.** −6 **b.** $\frac{106}{3}$ **10.** (0, 4, −3) **11.** $y = \frac{-4}{3}z - -\frac{20}{3}$ **14. a.** −1 **b.** −3 **c.** 17 **15. a.** $15|\vec{a}|^2 + 38\vec{a} \bullet \vec{b} + 24|\vec{b}|^2$ **b.** $4|\vec{a}|^2 - |\vec{b}|^2$ **16.** −80 **17.** 60° **18. a.** $\frac{-3}{2}$ **b.** $\frac{-11}{2}$ **19.** $\frac{5\sqrt{3}}{2}$, 72°, 108° **20. a.** Pythagorean Theorem **b.** $|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$ cosine law **22.** $\left(\frac{3}{5}, \frac{4}{5}, 0\right)$ **23.** $\frac{-29}{2}$ **24.** 71°

Exercise 5.4

2. a. 55.6, into **b.** 389.7, out **c.** 3.1, into **3.** vectors are f, h, i, j; scalars are a, c, e, k **4. a.** (0, 1, 0) **b.** (1, 0, −1) **c.** (−10, 7, 9) **d.** (45, 20, 8) **5.** $\left(0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$ **6.** (4, 2, 0), (−2, −1, 0) **10. a.** 19 **b.** 19 **c.** 19 **d.** (−5, 1, 21) **e.** (−1, 2, −23) **f.** (6, −3, 2) **15. a.** −4, 2 **b.** (−2, −1, 1)

Exercise 5.5

1. a. $\left(\frac{48}{13}, \frac{32}{13}\right); \frac{16\sqrt{13}}{13}$ **b.** $\left(\frac{-42}{13}, \frac{28}{13}\right); \frac{14\sqrt{13}}{13}$
c. $\left(\frac{40}{89}, \frac{-30}{89}, \frac{-80}{89}\right); \frac{10\sqrt{89}}{89}$ **d.** $(0, 0, -4)$; 4 **3.** $2\hat{i}, 3\hat{j}, -4\hat{k}$
4. $-3\hat{i}, -\hat{j}, \vec{0}$ **5. a.** $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ **b.** $(1, 0, 0)$ **6. a.** $\sqrt{65}$ **b.** 0
7. a. $\frac{49}{2}$ **b.** $\frac{\sqrt{3}}{2}$ **8.** 29 **9. a.** 2165 J **b.** 1.0 J **c.** −29 J **d.** 0 J
10. greater than 225 J **11.** 7240 J **12.** 32819 J **13.** 80 J
14. 2114 J **15. a.** 10 **b.** 22 **c.** 46×10^3 **d.** −88 **16.** $60\sqrt{2}$ J
17. $-19\sqrt{30}$ J **18. a.** 5 N **b.** 10 J, $\theta = 90°$

Review Exercise

1. a. $\hat{i} + 3\hat{j} + 2\hat{k}$ **b.** $\hat{i} + 5\hat{k}$ **c.** $-6\hat{i} - 8\hat{j} + 11\hat{k}$
d. $9\hat{i} - 6\hat{j} + 2\hat{k}$ **2. a.** $(3, -2, 7)$ **b.** $(-9, 3, 14)$ **c.** $(1, 1, 0)$
d. $(2, 0, -9)$ **3. a.** −3 **b.** 96° **4.** $(-16, 2, 3)$
5. $|\vec{a}|^2 - |\vec{b}|^2$ **6.** $\vec{a}\cdot\vec{c} + \vec{a}\cdot\vec{d} + \vec{b}\cdot\vec{c} + \vec{b}\cdot\vec{d}$ **7.** 4 **8.** 1 **9.** 84
12. b. $\sqrt{82}$ **c.** 16.2 **d.** $(1, 5, 4)$ **13. a.** $17\hat{i}, -3\hat{j}, 8\hat{k}$
b. $(17, -3, 0), (0, -3, 8), (17, 0, 8)$ **14.** 36 **15. a.** $(0, 0, 0)$,
$(0, 1, 0), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right), \left(\frac{\sqrt{3}}{6}, \frac{1}{2}, \frac{\sqrt{6}}{3}\right)$ **b.** $\left(\frac{\sqrt{3}}{6}, \frac{1}{2}, \frac{\sqrt{6}}{12}\right)$
c. $\frac{\sqrt{6}}{4}$ **17.** $\frac{197}{3}$

Chapter 5 Test

1. a. $\vec{u} \perp \vec{v}$ **b.** $\vec{u} = k\vec{v}, k > 0$ **c.** $\vec{u} = k\vec{v}$ **d.** $\vec{u} \perp \vec{v}$ **e.** nothing **f.** $\vec{u} = k\vec{v}$
2. a. $33\hat{i} + 5\hat{k}$ **b.** −4 **c.** $-5\hat{i} - 12\hat{j} + 33\hat{k}$
d. $\frac{5}{\sqrt{1258}}\hat{i} + \frac{12}{\sqrt{1258}}\hat{j} - \frac{33}{\sqrt{1258}}\hat{k}$ **3. b. ii.** 5 **iii.** $\sqrt{13}$
4. a. $(0, -2, 1)$ **b.** 129° **c.** $\sqrt{185}$ **5.** 1562.5 J
6. a. perpendicular to the axis of the wrench **b.** 9 J, perpendicular to the plane of the wrench and the applied force **c.** 30°
7. $\frac{|\vec{a}|^2 - |\vec{b}|^2}{|\vec{a}|^2 + |\vec{b}|^2}$ for $|\vec{a}| > |\vec{b}|$

CHAPTER 6

Exercise 6.1

4. a. $\vec{p} = -4\hat{i} + 5\hat{j}$ **b.** $\overrightarrow{OA} = 8\hat{i} - 3\hat{j}$ **c.** $\hat{i} + \hat{j}$ **d.** $-3\sqrt{3}\hat{i} + 3\hat{j}$
5. a. yes **b.** $-321\vec{u} + 123\vec{v}$ **6.** No **7.a.** yes **b.** no **8. a.** parts (i), (ii), (iii) **9.** 4, −2 **10. a.** 2, 2, −1 **b.** −1, 1, 3 **11.**
b. $\hat{i} = \frac{1}{14}\vec{u} + \frac{1}{3}\vec{v} + \frac{5}{42}\vec{w}, \hat{j} = \frac{3}{14}\vec{u} - \frac{1}{3}\vec{v} + \frac{1}{42}\vec{w}$,
$\hat{k} = \frac{1}{7}\vec{u} + \frac{1}{3}\vec{v} - \frac{2}{21}\vec{w}$ **12. a.** $\vec{x} = 2\vec{u} + 3\vec{v} + 8\vec{w}$
b. $\vec{x} = 6\sqrt{6}\hat{u} + 27\hat{v} + 24\hat{w}$ **13. a.** $\vec{x} = -1.61\vec{u} + 0.54\vec{v}$
b. $\vec{x} = -7.88\hat{u} + 7.52\hat{v}$

Exercise 6.2

3. a. 0, 0 **b.** −5, 3 **c.** 2, −1 **d.** 5, −7 **4. a.** $0, \frac{1}{2}, \frac{-1}{2}$
b. $2, -3, \frac{2}{3}$ **5. a.** not possible **b.** $\frac{2}{3}$ **c.** −2 **d.** not possible
10. a. yes, $(-19, 9)$ **b.** no **11. a.** $(1, 4, -2)$ **b.** no
c. $(-1, -1, 3)$ **15.** −1, 2

Exercise 6.3

2. a. 7:−3 **b.** 2:3 **c.** −2:5 **d.** 5:−3 **e.** −1:2 **4. a.** 1:1 **b.** 3:−1
c. −3:4 **d.** 4:−1 **e.** −2:3 **7.** a and c **8. a.** 7:2 **b.** 13:−4
c. −4:5 **d.** −2:9 **9. a.** $\overrightarrow{OA} = \frac{2}{5}\overrightarrow{OB} + \frac{3}{5}\overrightarrow{OC}$
b. $\overrightarrow{OA} = 3\overrightarrow{OB} - 2\overrightarrow{OC}$ **c.** $\overrightarrow{OA} = -2\overrightarrow{OB} + 3\overrightarrow{OC}$ **10.** $\left(5, 6, \frac{3}{2}\right)$
11. $(4, 4, 5), (5, 2, 2)$ **12. a.** $\left(\frac{11}{6}, \frac{15}{2}\right)$ **b.** $(0, -20)$ **c.** $\left(\frac{10}{3}, 30\right)$
d. $\left(\frac{9}{5}, 7\right)$ **13. a.** −4:7 **b.** 1:2 **14.** $a:b$

Exercise 6.4

5. a. 1:1 **b.** 2:1 **14.** 1:3 **15. a.** 2:1, 5:4 **b.** 22:45 **16.** 1:2, 1:2

Review Exercise

1. b. $(3, -1) = \frac{5}{18}(2, 3) - \frac{11}{18}(-4, 3)$ **2.** a, b, c are linearly dependent **3.** $\frac{3}{2}$ **5. a.** $\left(\frac{5}{2}, -3\right), (1, 0), \left(\frac{7}{2}, -3\right), \left(\frac{7}{3}, -2\right)$
b. $\left(5, 4, \frac{1}{2}\right), (2, 3, 3), \left(3, 0, \frac{3}{2}\right), \left(\frac{10}{3}, \frac{7}{3}, \frac{5}{3}\right)$ **6. a.** 1:1 **b.** 5:−4
8. b. $\overrightarrow{OM} = \frac{-9}{2}\overrightarrow{ON} + \frac{11}{2}\overrightarrow{OQ}$ **9.** 14:9, 21:2 **10.** 20:3, 4:19

Chapter 6 Test

2. a. $\overrightarrow{OP} = \frac{-3}{7}\overrightarrow{OQ} + \frac{10}{7}\overrightarrow{OR}$ **b.** $\overrightarrow{OR} = \frac{7}{10}\overrightarrow{OP} + \frac{3}{10}\overrightarrow{OQ}$
3. b. $2, \frac{3}{2}$ **4.** $\vec{u} \cong 0.17\vec{v} + 0.19\vec{w}$ **5. b.** 5:2
6. b. $\overrightarrow{OP} = \frac{-2}{3}\overrightarrow{OA} + \frac{5}{3}\overrightarrow{OB}$

CHAPTER 7

Exercise 7.1

2. a. $(-3, 1)$ **b.** $(4, 5)$ **c.** $(1, 3)$ **d.** $(4, 3)$ **e.** $(1, 0)$ **f.** $(0, 1)$
3. a. $(3, 0), (11, -4)$ **b.** $(4, 0), (4, 10)$
4. a. $x = 2 + 2t, y = t; \vec{r} = (2, 0) + t(2, 1)$
b. $x = 3 - t, y = t; \vec{r} = (3, 0) + t(-1, 1)$
6. a. $x = 1 + t, y = 1 - t$; $(2, 0), (-2, 4)$ **b.** $x = 5 + t, y = 3t$; $(6, 3), (3, -6)$ **7. a.** $x = t, y = 0$ **b.** $x = t, y = 5$
8. a. $\vec{r} = (-2, 7) + t(3, -4)$ **b.** $\vec{r} = \left(2, \frac{3}{4}\right) + t(1, 9)$
c. $\vec{r} = (1, -1) + t(-\sqrt{3}, 3)$ **d.** $\vec{r} = (0, 0) + t(-2, 3)$
9. a. $\vec{d} = (3, -1)$ **b.** $\vec{d} = (4, 3)$; $(5, 4)$ **c.** $\vec{d} = (1, -10)$; $(-1, 18)$
10. a. perpendicular **b.** parallel **11.** $\vec{r} = (4, 5) + t(7, -3)$
12. a. $(6, 0)$ **b.** $(-7, 0), (0, 35)$ **c.** $(11, 0), \left(0, \frac{11}{3}\right)$ **13.** 87°
14. a. (i) 127° **(ii)** 11° **15. a.** $x = 24 + 85t, y = 96 - 65t$
b. 1h 12 min **c.** $(126, 18)$ **16. b. (i)** $\frac{x-5}{-8} = \frac{y+3}{5}$
(ii) $\frac{x}{4} = \frac{y+4}{1}$ **c.** $\frac{x-7}{6} = \frac{y+2}{1}$ **17. b.** $x = 7 + 2t$,
$y = 3 - 5t, -1 \le t \le 5$ **18. b.** $0 < t < 1$ **d.** $t > \frac{1}{2}$
19. a. $\vec{r} = (5, 2) + t(2, -1); \vec{r} = (5, 2) + s(1, 2)$ **c.** yes

Exercise 7.2

2. a. $(2, 1)$ **b.** $(1, -2)$ **c.** $(2, -1)$ **d.** $(1, 2)$
3. a. $2x + 7y + 6 = 0$ **b.** $2x - 1 = 0$ **c.** $x + y - 6 = 0$
d. $x - y = 0$ **4. a.** $\vec{n} = (4, 3), \vec{d} = (3, -4), P(3, 0)$
b. $\vec{n} = (1, -2); \vec{d} = (2, 1), P\left(\frac{14}{3}, 0\right)$ **c.** $\vec{n} = (1, 0)$;
$\vec{d} = (0, 1), P(5, 6)$ **d.** $\vec{n} = (3, -1); \vec{d} = (1, 3), P(2, -4)$
6. a. $x - 4y - 28 = 0$ **b.** $x - 2y + 5 = 0$ **c.** $7x - 2y = 0$
d. $y + 2 = 0$ **7. a.** $2x - 3y - 22 = 0$ **b.** $2x + 3y + 14 = 0$
c. $3x + 2y + 6 = 0$ **d.** $3x - 2y - 18 = 0$
8. $3x + 5y - 14 = 0$ **9. a.** $\vec{r} = (0, 5) + t(3, 5); x = 3t$,
$y = 5t + 5; \frac{x}{3} = \frac{y-5}{5}$ **b.** $\vec{r} = \left(0, \frac{-3}{2}\right) + t(3, 2); x = 3t$,
$y = 2t - \frac{3}{2}; \frac{x}{3} = \frac{y + \frac{3}{2}}{2}$ **11. a.** $\frac{7}{\sqrt{13}}$ **b.** $\frac{12}{\sqrt{53}}$ **c.** 0 **d.** 8
12. a. $\frac{7\sqrt{5}}{3}$ **b.** $\frac{2\sqrt{5}}{3}$ **c.** $\frac{\sqrt{5}}{3}$ **d.** 0 **14. b.** 153°
c. $\sqrt{3}x + y + 4 - 6\sqrt{3} = 0$
15. a. $x^2 + y^2 - 10x - 12y + 36 = 0$

Exercise 7.3

2. a. $(4, -2, 5)$ **b.** $(7, -2, 3)$ **c.** $(-1, 2, 4)$
3. a. $(4, 0, 1), (-5, 3, 4)$ **b.** $(4, -2, 5), (2, 3, 9)$
c. $(4, -5, -1), (7, -1, -2)$ **4. a.** $\vec{r} = (2, 4, 6) + t(1, 3, -2)$;
$x = 2 + t, y = 4 + 3t, z = 6 - 2t$;
$\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{-2}$
b. $\vec{r} = (0, 0, -5) + t(1, -4, -1)$; $x = t, y = -4t, z = -5 - t$;
$\frac{x}{1} = \frac{y}{-4} = \frac{z+5}{-1}$ **c.** $\vec{r} = (1, 0, 0) + t(0, 0, -1)$;
$x = 1, y = 0, z = -t$ **5.** $(-20, 10, -27), (-14, 8, -17)$,
$(-8, 6, -7), (-2, 4, 3), (4, 2, 13), (10, 0, 23), (16, -2, 33)$
6. a. $P(2, 4, 2)$ **b.** $a = -8, b = -1$ **7.** $x = 6t, y = -1 + 4t$,
$z = 1 + t$ **8.** $\frac{x}{6} = \frac{y}{7} = \frac{z}{-2}$ **9. a.** parallel **b.** neither **c.** same
12. $\frac{x+6}{6} = \frac{y-4}{-5} = \frac{z-2}{-2}$ **13. b.** $x = 3t, y = t, z = 2 + 6t$;
$-3 \le t \le 2$ **14.** $\vec{r} = (4, 5, 5) + s(1, 5, 2)$ **15. b.** $\frac{\sqrt{66}}{6}$
c. $\sqrt{\frac{1555}{74}}$

Exercise 7.4

2. a. $(-5, -1)$ **b.** $(1, -2)$ **3. a.** coincident **b.** neither **c.** neither
d. parallel and distinct **4. a.** $(8, 2, 3)$ **b.** lines are coincident
c. skew **d.** parallel and distinct **e.** $(-1, 1, 1)$
5. a. $(-2, -3, 0)$ **b.** $\vec{r} = (-2, -3, 0) + s(1, -2, 1)$ **6.** $(2, 3, 1)$
7. x intercept is -4 **8.** $\left(\frac{21}{2}, -1\right)$
11. $\vec{r} = (-5, -4, 2) + t(14, -5, 2)$;
$(9, -9, 4)$ **12.** $(2, -1, -1), (1, 2, 1)$, No
13. $\vec{r} = s(17, -15, -20)$ **14. a.** $\left(\frac{-AC}{A^2 + B^2}, \frac{-BC}{A^2 + B^2}\right)$
b. $\frac{|C|}{\sqrt{A^2 + B^2}}$ **15.** $(0, 1, 2), (1, 1, 1)$ **16. a.** $\sqrt{3}$ **b.** 6

Review Exercise

2. a. $\vec{r} = (3, 9) + t(1, 1)$ **b.** $\vec{r} = (-5, -3) + t(1, 0)$
c. $\vec{r} = (0, -3) + t(2, -5)$ **3. a.** $x = -9 + 3t, y = 8 - 2t$
b. $x = 3 + 2s, y = -2 - 3s$ **c.** $x = 4 + 2t, y = t$
4. a. $\vec{r} = (2, 0, -3) + t(5, -2, -1)$ **b.** $\vec{r} = (-7, 0, 0) + t(7, 4, 0)$
c. $\vec{r} = (0, 6, 0) + t(4, -2, 5)$ **5. a.** $x = 3t, y = 2t, z = -t$
b. $x = 6, y = -4 + t, z = 5$ **c.** $x = t, y = -3t, z = -3 + 6t$
6. a. $3x - 4y - 5 = 0$ **b.** $x + 2y + 1 = 0$ **c.** $4x - y = 0$
7. a. $x = 6 + 5t, y = 4 - 2t, z = -3t$ **b.** $m = 8, n = -2$
8. a. coincident **b.** perpendicular **c.** parallel and distinct
d. neither parallel nor perpendicular **9.** $\left(-3, \frac{11}{2}, 0\right), (8, 0, 22)$,
$(0, 4, 6)$ **10. a.** $5x + y - 13 = 0$ **b.** $\vec{r} = (0, 5) + t(2, 5)$
c. $\vec{r} = (2, 2) + t(4, 3)$ **11. a.** $(6, 0, 0), (0, 8, 4)$
b. x intercept is 6 **12. a.** $\cos\alpha = \frac{5}{\sqrt{30}}, \cos\beta = \frac{2}{\sqrt{30}}$,
$\cos\gamma = \frac{-1}{\sqrt{30}}$; $\alpha \cong 24°, \beta \cong 69°, \gamma \cong 101°$
b. $\cos\alpha = \frac{8}{9}, \cos\beta = \frac{-1}{9}, \cos\gamma = \frac{-4}{9}$; $\alpha \cong 27°, \beta \cong 96°$,
$\gamma \cong 116°$ **c.** $\cos\alpha = \frac{4}{\sqrt{17}}, \cos\beta = \frac{1}{\sqrt{17}}, \cos\gamma = 0$;
$\alpha \cong 14°, \beta \cong 76°, \gamma = 90°$ **13. a.** $(0, 0, 2)$ **14. a.** $3\sqrt{5}$
b. $\frac{22\sqrt{13}}{13}$ **c.** $\sqrt{2}$ **d.** $\frac{5\sqrt{3}}{3}$ **15.** $(4, -1, 5)$

Chapter 7 Test

1. a. $\vec{r} = (9, 2) + t(3, -1)$ **b.** $x = 9 + 3t, y = 2 - t$
c. $\frac{x-9}{3} = \frac{y-2}{-1}$ **d.** $x + 3y - 15 = 0$ **2.** $3x + 2y - 2 = 0$
3. $(-2, 2, 0), (0, 3, -1)$ **4.** $3\sqrt{2}$ **5.** $\vec{r} = (1, -1, \sqrt{2})s$;
$\vec{r} = (-1, -1, \sqrt{2})t$ **6.** $(8, 2, 3)$ **7. b.** $P_1(-10, 1, 2)$
c. $P_2(-1, -2, -3)$

CHAPTER 8

Exercise 8.1

2. a. $(1, 0, 0), (4, 0, -3)$ **b.** y component is 0
3. a. $(-3, 5, 2), (-6, 1, 2)$ **b.** $(5, -5, 3), (1, 6, -2)$
c. $(4, -2, 1), (-1, 5, 2)$ **4. a.** $(9, 4, -3), (7, 4, 4)$
b. $(1, 1, 2), (1, 1, -2)$ **c.** $(3, -2, -2), (9, -1, -1)$
d. $(5, 0, 1), (-3, 0, 2)$ **5. a.** $x = -4 + 5s - 4t$
$y = -6 + 2s - 6t,\ z = 3 + 3s + 3t$
b. $x = 3t, y = 2s, z = 1$ **c.** $x = s, y = 0, z = t$
6. a. $\vec{r} = (-4, -1, 3) + s(1, 3, 4) + t(3, -4, -1)$
b. $\vec{r} = (0, 4, 0) + s(7, 0, 0) + t(0, 0, -2)$
c. $\vec{r} = s(1, 0, 0) + t(0, 0, 1)$
7. a. $\vec{r} = (-4, 5, 1) + s(-3, -5, 3) + t(2, -1, -5)$
b. $\vec{r} = (4, 7, 3) + s(1, 4, 3) + t(-1, -1, 3)$
c. $\vec{r} = (8, 3, 5) + s(5, 2, -3) + t(11, -1, -1)$
d. $\vec{r} = (0, 1, 3) + s(2, 1, -2) + t(4, -4, 7)$
e. $\vec{r} = (2, 6, -5) + s(5, 5, -1) + t(4, -8, 7)$
8. a. $x = 7 + 4s - 3t, y = -5 - s + 4t,\ z = 2 + s + 4t$
b. $x = 5 + 2s + 4t, y = 4 - 2t, z = 2 - 9s + t$
c. $x = 8 + 5s + 2t, y = 3 - 2s + 2t, z = 5 + 11s - 5t$
d. $x = 3 + s + 3t, y = 2 - 2s - 2t, z = 2 + 4s + 2t$
e. $x = 2 + 5s + 4t, y = 6 + 5s - 8t, z = -5 - s + 7t$
9. a. $\vec{r} = (6, 4, 2) + s(0, 1, 0) + t(0, 0, 1)$
b. $\vec{r} = s(1, 1, 1) + t(8, -1, -1)$ **c.** $\vec{r} = s(1, 0, 0) + t(1, 4, 7)$
10. a. the three points are collinear **b.** the point is on the line
11. $\vec{r} = (7, 0, -7) + s(0, 0, 1) + t(1, 2, -1)$; $x = 7 + t, y = 2t$,
$z = -7 + s - t$ **13. b.** All points in and on the parallelogram whose vertices have position vectors $\vec{a}, \vec{b}, -\vec{a} + \vec{b} + \vec{c}$ and $\vec{c}$
14. b. all points on and between the parallel lines

Exercise 8.2

1. a. $7x + y - z - 18 = 0$ **b.** $x - 5 = 0$ **c.** $2x + 3z + 6 = 0$
d. $2x - y + 4z = 0$ **2. a.** $y + 2 = 0$ **b.** $z - 3 = 0$
c. $x - y - 2z + 3 = 0$ **3. a.** $Ax + By + Cz = 0$ **b.** $D = 0$
5. a. $12x + 8y + 13z = 0$ **b.** $3x - 8y + z - 15 = 0$ **c.** $x - 2 = 0$
d. $3x + 10y - 4z + 4 = 0$ **e.** $x - 2 = 0$
f. $12x + 8y + 13z = 0$; a and f, c and e are coincident
6. a. $4x - 13y - 20z + 30 = 0$ **b.** $9x - 6y - 2z + 22 = 0$
7. a. $11x + 8y - 2z - 21 = 0$ **b.** $x + 3y + z = 0$ **c.** $y - 1 = 0$
d. $6x - 2y + 5z = 0$ **8.** $y + 2z = 0$
9. $10x + 11y - 10z - 50 = 0$ **10. a.** parallel and distinct
b. neither **c.** coincident **d.** coincident
11. a. $\vec{r} = (0, -24, 0) + s(1, 2, 0) + t(0, 3, 1)$
b. $\vec{r} = (0, 0, 3) + s(5, 0, 3) + t(0, 1, 0)$
12. a. parallel to and on the plane **b.** parallel to the plane, not on
c. not parallel **13. a.** 17° **b.** 90° **15. a.** $6x - 4y - 4z - 3 = 0$
b. a plane passing through the mid point of AB and having normal $\overrightarrow{AB}$. **16. b.** $38x + 33y + 111z - 103 = 0$
17. $x = 3t, y = -2t, z = 0$ **18.** $|D|$ will be the distance from the origin to the plane **20.** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Exercise 8.3

1. a. (4, 6, −2) **b.** (1, 1, 2) **c.** no intersection **d.** $(x, y, z) = (2 - t, 14 - t, 1 + t)$ **e.** (5, 15, −5)
2. a. yes **b.** no **3. a.** (2, 0, 0) **b.** (0, −3, 0) **c.** $\left(0, 0, \frac{-6}{7}\right)$
4. a. (i) $\left(\frac{28}{27}, 0, 0\right)$ **(ii)** $\left(0, \frac{56}{55}, 0\right)$ **(iii)** $\left(0, 0, \frac{-7}{2}\right)$
b. (i) $\vec{r} = (-1, 2, 0) + k(-55, 54, 0)$
(ii) $\vec{r} = (0, 8, 24) + p(0, 16, 55)$
(iii) $\vec{r} = (4, 0, 10) + u(8, 0, 27)$ **5. a.** (9, 14, 0)
b. $\left(\frac{-3}{2}, 0, \frac{7}{2}\right)$ **c.** (0, 2, 3) **6. a.** one point **b.** infinite number of points **c.** no points, one point, or an infinite number of points
7. $\left(\frac{3}{2}, -1, \frac{7}{2}\right)$ **8.** $\left(\frac{14}{5}, \frac{-2}{5}, -6\right)$ **9. a.** 1, 4, 3 **b.** 5, $\frac{-5}{2}$, −5 **c.** −4, 8, −8 **d.** 4, −16, 8 **10. a.** x-intercept is 4, y-intercept is 4; intersection with: xy plane is $\vec{r} = (u, 4 - u, 0)$, xz plane is $\vec{r} = (4, 0, s)$, yz plane is $\vec{r} = (0, 4, t)$ **b.** x-intercept is 3; intersection with: xy plane is $\vec{r} = (4, t, 0)$, xz plane is $\vec{r} = (4, 0, u)$
c. y-intercept is $-\frac{1}{2}$; intersection with: xy plane is $\vec{r} = \left(t, \frac{-1}{2}, 0\right)$, yz plane is $\vec{r} = \left(0, \frac{-1}{2}, s\right)$
d. x-intercept is 2, z-intercept is 6; intersection with: xy plane is $\vec{r} = (2, t, 0)$, xz plane is $\vec{r} = (u, 0, 6 - 3u)$, yz plane is $\vec{r} = (0, s, 6)$
e. y-intercept is 0, z-intercept is 0; intersection with: xy and xz plane is $\vec{r} = (t, 0, 0)$, yz plane is $\vec{r} = (0, 2u, u)$, **f.** x, y, is and z-intercepts are each 0; intersection with: xy plane is $\vec{r} = (t, -t, 0)$, xz plane is $\vec{r} = (s, 0, s)$, yz plane is $\vec{r} = (0, u, u)$
11. a. no value **b.** $k = 9$ **c.** $k \neq 9$

Exercise 8.4

2. a. yes **b.** no **c.** no **d.** yes **3. a.** $x = 7 + 5t$, $y = -3 - 2t$, $z = t$ **b.** parallel **c.** $x = 8 - 7t$, $y = t$, $z = 11 - 10t$
d. $x = 0$, $y = 1 - t$, $z = t$ **e.** parallel
4. a. $\left[\begin{array}{ccc|c} 3 & -7 & 1 & 12 \\ 1 & 1 & -2 & -3 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} -4 & 3 & 2 & 4 \\ 0 & 2 & -5 & 5 \end{array}\right]$
c. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 16 \\ 0 & 1 & -8 & -2 \end{array}\right]$ **d.** $\left[\begin{array}{ccc|c} 6 & 5 & -2 & 4 \\ -2 & 5 & 3 & -4 \end{array}\right]$
5. a. $x + 4z = 9$, $y - 6z = 4$
b. $8x - 2y + 3z = -6$, $2x - 6y - 6z = 9$
c. $5x - 10y = 8$, $3y - 4z = 6$ **d.** $x + 4z = 0$, $y + 9z = 0$
6. a. $\vec{r} = (10, -3, 0) + t(-15, 4, 1)$
b. $\vec{r} = \left(\frac{5}{9}, \frac{-10}{9}, 0\right) + t(-3, 0, 1)$
c. $\vec{r} = \left(\frac{13}{4}, 0, \frac{1}{4}\right) + t(-4, 1, 0)$
d. $\vec{r} = (0, 0, -2) + t(2, 1, 0)$
e. $\vec{r} = (7, -8, 0) + t(0, 3, 1)$ **f.** $\vec{r} = (0, 0, -3) + t(1, 0, 2)$
7. a. 3 planes intersect at the point $\left(\frac{-253}{30}, \frac{106}{15}, \frac{154}{15}\right)$
b. no solution, the 3 lines are not concurrent
c. the 4 planes have no common intersection
8. b. $4x + 5y - 14z = 0$ **c.** $8x - 8y - 12z + 15 = 0$
9. $x - z - 14 = 0$

Exercise 8.5

2. a. coplanar **b.** coplanar **c.** coplanar **d.** collinear
3. a. $\left(4, \frac{1}{2}, -3\right)$ **b.** (0, 2, 0) **c.** (−1, 1, −1) **4.** $\left(\frac{47}{5}, \frac{-27}{5}, -5\right)$
5. a. $\left[\begin{array}{ccc|c} 5 & -2 & 1 & 5 \\ 3 & 1 & -5 & 12 \\ 1 & -5 & 2 & -3 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} -2 & 1 & -3 & 0 \\ 1 & 5 & 0 & 8 \\ 0 & 3 & 2 & -6 \end{array}\right]$
c. $\left[\begin{array}{ccc|c} 0 & 4 & -3 & 12 \\ 2 & 5 & 0 & 15 \\ 4 & 0 & 6 & 10 \end{array}\right]$ **6. a.** $x = 8$, $y = -6$, $z = 3$
b. $x - 6z = 4$, $y + 5z = -5$, $0z = 0$ **c.** $x = 0$, $y = 0$, $0z = 1$
7. $\left(\frac{27}{4}, \frac{-15}{4}, \frac{-25}{4}\right)$ **8. a.** unique solution point $(x, y, z) = (2, 3, 4)$
b. no solution, 3 distinct parallel planes **c.** infinite number of solutions, the planes intersect in the line with equation $(x, y, z) = (7 - t, t, 2)$ **d.** Infinite number of solutions, the 3 planes are coincident, $x - 2y - 3z = 1$ **e.** no solution, 2 of the planes are parallel and distinct **f.** no solution, 2 planes are coincident, and the third is parallel and distinct **g.** infinite number of solutions intersecting in the line $(x, y, z) = (6 + t, -1 - t, 2t)$ **h.** no solution, planes form a triangular prism **i.** unique solution, the origin (0, 0, 0) **9.** $\frac{-7}{19}$

Review Exercise

2. a. $\vec{r} = (-1, -1, 2) + s(5, 4, 2) + t(0, 0, 1)$; $x = -1 + 5s$, $y = -1 + 4s$, $z = 2 + 2s + t$
b. $\vec{r} = (1, 1, 0) + s(0, 1, 0) + t(3, 1, -3)$; $x = 1 + 3t$, $y = 1 + s + t$, $z = -3t$
c. $\vec{r} = (0, 0, 4) + s(2, -3, 0) + t(1, 0, 2)$; $x = 2s + t$, $y = -3s$, $z = 4 + 2t$
d. $\vec{r} = s(1, 1, 1) + t(3, 4, 5)$; $x = s + 3t$, $y = s + 4t$, $z = s + 5t$
e. $\vec{r} = (3, -1, 2) + s(4, 0, 1) + t(4, 0, 2)$ $x = 3 + 4s + 4t$, $y = -1$, $z = 2 + s + 2t$ **3. a.** $x + 3y + 5z - 67 = 0$
b. $2x - 3y - 11z + 33 = 0$ **c.** $y + z - 6 = 0$
d. $8x + 2y + z - 18 = 0$ **e.** $z - 7 = 0$ **f.** $x - 3y - 3 = 0$
4. a. $\frac{1}{3}$ **b.** $k = 5$ or $k = -4$ **5.** $7x + 2y - 4z - 13 = 0$
6. $\vec{r} = s(1, 2, -1) + t(2, -3, 2)$ **7.** $2x - y = 0$
8. $x - 3y + 2z - 14 = 0$ **9.** $17x - 7y + 13z - 23 = 0$
11. a. $\frac{54}{\sqrt{37}}$ **b.** $\frac{4}{\sqrt{14}}$ **c.** $\frac{2}{\sqrt{5}}$ **d.** $\frac{3}{2\sqrt{30}}$ **12.** $\frac{22}{7}$ **13.** $\left(\frac{4}{5}, \frac{-2}{5}, 1\right)$
14. (−5, 0, 0), (0, −4, 0), (0, 0, 20) **17. a.** $k \neq \frac{9}{2}$
b. will never intersect in a line **c.** $k = \frac{9}{2}$
18. a. $3x + 4y - z - 1 = 0$ **b.** $\vec{r} = (0, 3, 3) + t(3, 4, -1)$
c. $\left(\frac{-12}{13}, \frac{23}{13}, \frac{43}{13}\right)$ **19.** $27x + 11y + 7z - 53 = 0$
20. a. $4x - y + z = 0$ **21.** coincident **22. a.** in R^2 − 2 lines intersect in the point $\left(\frac{1}{2}, \frac{3}{2}\right)$, in R^3 − 2 planes intersecting in the line $\vec{r} = \left(\frac{1}{2}, \frac{3}{2}, t\right)$ **b.** no solution, 2 parallel planes
c. line $\vec{r} = (-1, 1, 0) + t(6, 5, 7)$ **d.** point (2, 3, −1)
e. no solution, triangular prism
f. line $\vec{r} = (5, 1, 0) + t(-3, 0, 1)$
g. line $\vec{r} = (1, -2, 0) + t(-1, 3, -5)$
h. planes coincident with $x - z = 4$ **i.** no solution, 2 planes are parallel and distinct

Chapter 8 Test

1. a. planes are perpendicular and intersect in a line **b.** planes are parallel **c.** planes are parallel **2. a.** line is parallel to the plane, no solution **b.** intersects the plane at (2, 2, 0) **4. a.** (−5, 0, 0)
b. $\vec{r} = (0, 0, 5) + t(1, 0, 1)$ **5.** $4x - 4y + 7z = 0$

6. planes intersect in a line with equation $\vec{r} = (0, 1, -5) + t(1, -3, 5)$

7. a. $\sqrt{14}$ **b.** $\frac{54}{\sqrt{14}}$ **c.** opposite side

Cumulative Review Chapters 4–8

3. 0 **4.** $(0, 8) = \frac{8}{5}(2, 4) + \frac{8}{5}(-2, 1)$ **6.** $\frac{5}{9}$
7. a. $-\vec{x} + \vec{y}$ **b.** $(1 + \sqrt{3})\vec{y} - (1 + \sqrt{3})\vec{x}$
c. $(-6 - 4\sqrt{3})\vec{x} + (3 + 2\sqrt{3})\vec{y}$ **8.** $\frac{19}{2}$ **12. b.** $(-5, 5, 5)$
13. yes **14.** $2x - 3y - 3z - 12 = 0$ **15.** $A\left(\frac{7}{11}, \frac{4}{11}, 3\right)$, $B\left(\frac{3}{11}, \frac{6}{11}, \frac{27}{11}\right)$
16. $x + 2y + 2z - 20 = 0$ **17. a.** no intersection
b. $\vec{r} = (-1, 3, 0) + t(1, 2, -1)$ **c.** $\left(\frac{-5}{2}, 0, \frac{3}{2}\right)$ **18.** $(2, \frac{1}{2}, 0)$
19. $\left(2, \frac{5}{2}, \frac{5}{2}\right)$ **20. a.** $x = -1 - t, y = 3 + t, z = t$
b. xy plane at $(-1, 3, 0)$, xz plane at $(2, 0, -3)$, yz plane at $(0, 2, -1)$ **c.** $3\sqrt{3}$ **21.** $\left(\frac{103}{11}, \frac{-93}{11}, \frac{60}{11}\right)$
22. $(24, 36, 8)$ **23.** $\pm\frac{3}{2\sqrt{109}}$ **24.** $a = \frac{1}{2}b, b \neq -2$
25. $(x, y, z) = (3, -1, 0)$ **26. a. (i)** $k = 2$ **(ii)** $k \neq 2, k \neq -1$
(iii) $k = -1$ **b.** planes intersect in the line $(x, y, z) = (t, t, t)$

CHAPTER 9

Review of Prerequisite Skills

2 a. isosceles **b.** isosceles **c.** isosceles **d.** right-angled
3 a. $\left(\frac{7}{2}, -1\right)$ **b.** $\left(\frac{7}{5}, \frac{9}{5}\right)$ **4.** $(0, -7)$ **5.** $(6, 0)$ **6. a.** $x^2 + y^2 = 9$
b. $(x + 1)^2 + (y - 4)^2 = 16$ **c.** $(x - 3)^2 + (y - 2)^2 = 20$
7. $x = 9, -3$ **8.** $(4, 4), \left(-1, \frac{9}{2}\right), \left(2, \frac{11}{2}\right)$ **9.** $y = 3$
10. $E\left(4, \frac{11}{2}\right), F\left(\frac{9}{2}, \frac{7}{2}\right), G\left(\frac{17}{4}, \frac{9}{2}\right)$

Exercise 9.1

1. a. $C(2, -1), r = 5$ **b.** $C(-1, -2), r = 3$
c. $C(-1, -3), r = \frac{7}{2}\sqrt{2}$ **2. a.** point **b.** $C\left(0, \frac{4}{3}\right), r = \frac{4}{3}$, circle
c. imaginary circle **3.** 2 **4.** 12 **5.** $\sqrt{10}$ **6.** 8
7. $(0, 0), (-6, 0), (0, 2)$ **8.** 6 **9.** $(-5, -3)$ **10.** $x^2 + (y - 4)^2 = 68$
11. $(x + 5)^2 + (y + 5)^2 = 25$ **12.** $\frac{125}{12}$ **13.** $3\sqrt{5} - 5$ **14.** $2\sqrt{3}$

Exercise 9.3

3. 21 km or 11 km **5. a.** $h = \frac{2}{3}$ **b.** 1:4

Exercise 9.4

1. $61x^2 - 78xy + 25y^2 = 16$ **2.** part of a circular arc
3. $x^2 + y^2 - x - y + 1 - \frac{k}{2} = 0$

Review Exercise

1. a. $y = x, y = -x$ **b.** $y = 5$ **c.** $x^2 + y^2 = 1$ **2.** $y = x + 3$
3. $\left(6, \frac{53}{4}\right)$ **4. a.** 6:5 **b.** 5:6 **5.** $x = \frac{(y-1)^2}{12}$ **6.** $PB = 6, PD = 7$; $PB = 7, PD = 6$; $PB = 9, PD = 2$; $PB = 2, PD = 9$
7. $x^2 - 3y^2 + 48y - 144 = 0$ **8.** $7x - 9y = -21$ or $7x - 9y = -17$ **9.** 52 m^2 **10.** $2x^2 - 12x - 3y + 12 = 0$
11. the locus is a circle with equation $(x - 5)^2 + (y - 8)^2 = 13$
12. the required point is $\left(\frac{-5}{2}, 6\right)$ **14.** $2x - 5y - 15 = 0$
15. $x^2 + (y - \frac{5}{2})^2 = \frac{25}{4}$ **16. b.** $\left(\frac{a}{2}, 0\right), \frac{a}{2}$ **17.** Circle

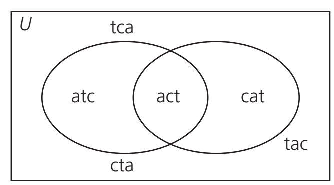
Chapter 9 Test

1. a. $x = 4$ **b.** $y = 2x + 6$ **c.** $(x + 3)^2 + (y - 2)^2 = 25$
2. a. a circle with $C(1, -3)$ and $r = \sqrt{13}$
b. a sphere with $C(1, -2, 3)$ and $r = 3$ **3. a.** $(-2, 9)$ **b.** $x = 2$
4. the circle has equation $(x - 3)^2 + (y - 6)^2 = 8$
5. the locus has equation $x^2 + 16y^2 = 64$
6. the locus is a circle with its centre at $\left(1, \frac{\sqrt{3}}{3}\right)$ with a radius of $\sqrt{\frac{7}{3}}$ **7.** equation of locus is $y^2 + 2xc - c^2 - k^2 = 0$

CHAPTER 10

Exercise 10.1

1. $U = \{10, 11, 12, 13, ..., 97, 98, 99\}$
$A = \{17, 27, 37, ..., 70, 71, 72, ..., 79, 87, 97\}$
2. $U = \{M_1, D_1, M_2D_2, M_1D_3, M_2D_1, M_2D_2, M_2D_3, M_3D_1, M_3D_2, M_3D_4, M_4D_1, M_4D_2, M_4D_3\}$
3. $\{C_1C_2, C_1C_3, C_1C_4, C_1C_5, C_2C_3, C_2C_4, C_2C_5, C_3C_4, C_3C_5, C_4C_5\}$
4. $V = \{a, e, i, o, u\}$
$\overline{V} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$
$n(V) = 5, n(\overline{V}) = 21$
5. a. $U = \{cat, cta, tac, tca, act, atc\}$ **b.** $n(U) = 6$
c. $A = \{act, atc\}, B = \{cat, act\}$
d.



e. $\overline{A}$: words not beginning with a
$\overline{B}$: words that do not end in t
6. a. $U = \{000, 001, 010, 011, 100, 101, 110, 111\}$
b. $n(u) = 8$ **c.** $E = \{001, 010, 100\}$
$F = \{001, 010, 011, 100, 101, 110, 111\}$
d. $\overline{F}$: subsets that do not have a 1
7. a. R: multiples of 5
S: integers from 53 to 76 inclusive
$\overline{R}$: integers not divisible by 5
b. $P = \{71, 72, 73, ..., 100\}$
$Q = \{7, 17, 27, ..., 97\}$
c. $\overline{P}$: integers less than 71
$\overline{Q}$: integers that do not end in 7
8. S: multiples of 10
$\overline{S}$: integers not divisible by 10
$n(S) = 100, n(\overline{S}) = 900$
9. a. no, a square is not prime **b.** no, there are composite numbers that are not squares
10. a. U = {1b, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf, de, df, ef}
b. $n(u) = 15$ **c.** W = {ab, ac, ad, ae, af},
V = {ab, ac, ad, ae, af, bf, cf, df, ef}
11. a. U: $100a + 10b + c$; $a, b, c \in \{1, 2, 3, ..., 9\}$; $a \neq b \neq c$
b. $n(A) = 56$ $n(B) = 168$
12. U: $100a + 10b + c$; $a, b, c \in \{1, 2, 3, ..., 9\}$
$n(A) = 81, n(B) = 217$
13. a. $n(E) = 142$ **b.** $n(F) = 31$ **c.** $n(G) = 667$ **d.** $n(H) = 900$
14. (A, B, C):(x, y, z); A in envelope x, B in envelope y, C in envelope z

(A, B, C):(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, b, a), (c, a, b), (b, c, a), (c, a, b)

Exercise 10.2

1. U = {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}
R = {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, CABD, CADB, CDAB, DABC, DACB, DCAB}
S = {ACDB, ADCB, CADB, CDAB, DACB, DCAB}
$R \cup S$ = {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, CABD, CADB, CDAB, DABC, DACB, DCAB}
$R \cap S$ = {ACDB, ADCB, CDAB, DACB, DCAB}
2. $n(A) = 3$, $n(\overline{A}) = 6$ **3.** $n(E \cap F) = 7$ **4.** no **5.** no
6. b. {12, 15} **c.** no **d.** {3, 6, 9, 11, 12, 13, 14, 15, 16}; integers divisible by 3 or greater than 10 **e.** $n(A) = 5$, $n(B) = 6$, $n(A \cap B) = 2$, $n(A \cup B) = 9$
7. a. U = {pppqq, ppqpq, ppqqp, pqppq, pqpqp, pqqpp, qpppq, qppqp, qpqpp, qqppp}
C = {pppqq, ppqpq, ppqqp, pqppq, pqpqp, pqqpp}
D = {ppqqp, pqpqp, pqqpp, qppqp, qpqpp, qqppp}
b. $C \cup D$ = {pppqq, ppqpq, ppqqp, pqppq, pqpqp, pqqpp, qppqp, qpqpp, qqppp}; sequences starting with p or ending with p **c.** $C \cap D$ = {ppqqp, pqpqp, pqqpp}; sequences starting and ending with p
d. {qqppp, qpqpp, qppqp, qpppq}
9. a. U = {ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE} **b.** F = {ABC, ABD, ABE}
G = {ABC, ACD, ACE, BCD, BCE, CDE}
c. $F \cup G$ = {CBA, CBE, CBD, CAE, CAD, CED, ABD, ABE}
$F \cap G$ = {ABC} **10. a.** no, all four sets have a common element
b. no **c.** {1234} **12.** $n(E) = 6$, $n(F) = 6$
$n(E \cup F) = 9$, $n(E \cap F) = 3$ **16.** $u = (A \cap B) \cup (\overline{A \cup B}) \cup (\overline{A} \cap B) \cup (A \cap \overline{B})$

Exercise 10.3

1. $n(A) = 10$, $n(B) = 25$
$n(A \cap B) = 5$, $n(A \cup B) = 30$ **2.** 18
3. a. A = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}
B = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}
b. $A \cap B = \varnothing$ **c.** 12
4. A = {abcd, abdc, acbd, acbd, adbc, adcb}
B = {abcd, abdc, cbad, cbda, dbac, dbca}
$A \cap B$ = {abcd, abdc} **6.** $n(V \cup W) = 34$
7. 20 **8.** 14 **9. a.** $n(A) = 8$, $n(B) = 11$, $n(A \cup B) = 16$
b. $n(U) = 33$ **12.** 105 **13. a.** 200 **b.** 142 **c.** 28 **d.** 686
e. 172 **14.** 208 **16. a.** $n(A) = 8$, $n(B) = 8$, $n(C) = 6$, $n(A \cap B) = 4$, $n(A \cap C) = 3$, $n(B \cap C) = 3$, $n(A \cap B \cap C) = 1$ **17.** 734

Exercise 10.4

1. a. 2828, 4136, 1333 **b.** $n(U) = 6561$ **3.** 36 **4.** 6859000
5. 3×10^6 **6.** 48 **7.** 720 **10. b.** 1024 **11.** yes
12. b. $n(U) = 17576$, $n(A) = 3380$, $n(B) = 125$, $n(C) = 15600$
c. 0 **e.** 3505 **f.** 1976 **13.** $n(U) = 720$, 600 **14.** $n(U) = 360$, 300 **15. b.** 1800 **c.** 4536 **d.** 4464 **16. a.** 62^8
b. $62^6 \times 10^2$ **c.** $\frac{62!}{54!}$ **d.** $62^8 - 61^8$ **17. b.** 6 **c.** 6 **d.** 15 **e.** 3
18. a. 90 **b.** $\frac{5}{6}$ **19. a.** $(a + 1)(b + 1)(c + 1)$ **b.** $\frac{a}{a + 1}$
20. a. $m(m - 1)(m - 2)$ **b.** m^3
21. a. $m(m - 1)(m - 2) \dots (m - r + 1)$ **b.** m^r **22. a.** 365^5
b. 97.29% **c.** 2.71%
23. a. $-\frac{365 \cdot 364 \cdot 363 \cdot \dots (366 - n)}{365^n}$ **b.** $n = 22$

Exercise 10.5

1. 18 **2.** 4717440 **3.** 62 **4.** 224 **5. a.** 64 **b.** 16 **c.** 49 **6. a.** 340
b. 85 **c.** 220 **7.** 14515200 **8.** 8987928 **9.** 249 **10.** 216
11. 1242, $\frac{589}{1242}$ **12. a.** 402 **b.** 54 **c.** 122 **d.** 162
13. a. 11111111110 **b.** 9864100 **c.** 7188478660
14. a. $3r^2 - 3r + 1$ **b.** $3r^2 - 9r + 6$ **15.** 32

Review Exercise

2. a. $n(U) = 9!$, $n(G) = 5!$, $n(F) = 4!$ **d.** $n(G \cup F) = 143$, $n(G \cap F) = 1$ **3. b.** $n(U) = 5000211$, $n(A) = 263169$, $n(C) = 748683$ $n(D) = 708588$ **c.** 789507
4. b. $n(U) = 720$, $n(A) = 360$, $n(B) = 360$, $n(C) = 660$
c. $n(A \cap B) = 160$ **5.** $\frac{3}{4}$ **6.** 8×10^6, $\frac{1}{100}$ **7.** 271
8. a. 11000 **b.** 1100 **c.** 3710

Chapter 10 Test

1. b. 800 **3. b.** 48 **4.** 60 **5.** 16 **6. b.** 30 **7. a.** 456976 **b.** 358800
c. 66351 **8.** 16 **9.** 756 **10.** 727

CHAPTER 11

Exercise 11.1

1. a. 5040 **b.** 35280 **c.** 8 **d.** 56 **e.** 20160 **f.** 110 **g.** 26970
h. 463474 **2.** $\frac{n - 1}{n + 1}$ **3. a.** $P(6, 6)$ **b.** $P(6, 4)$ **c.** $P(5, 4)$
d. $3 \times P(5, 5)$ **e.** $P(3, 2) \times P(4, 2)$ **4.** $P(10, 6)$ **5. a.** $\frac{1}{9}$
b. $\frac{4}{9}$ **c.** $\frac{5}{9}$ **d.** $\frac{5}{6}$ **e.** $\frac{2}{3}$ **f.** $\frac{5}{12}$ **g.** $\frac{11}{12}$ **h.** $\frac{7}{18}$ **6.** $8! \times 16!$
7. $P(9, 1) \times P(98, 98) \times P(10, 1)$ **8.** 28560 **9. a.** $6! \times 6!$
b. $2 \times (6!)^2$ **c.** $6 \times 11!$ **d.** $36 \times 10!$ **e.** $96 \times 10!$
10. $n(U) = P(n, 4)$, $n(A) = P(n - 1, 3)$
$n(B) = 3 \times P(n - 2, 2)$, $n(C) = 6 \times P(n - 2, 2)$
11. $n(U) = P(n, r)$, $n(A) = P(n - 1, r - 1)$,
$n(B) = (r - 1) \times P(n - 2, r - 2)$
$n(C) = 2(r - 1) \times P(n - 2, r - 2)$,
13. $n(U) = P(40, 5)$, $n(A) = 4 \times P(10, 5)$,
$n(C) = P(40, 5) - 4^5 \times P(10, 5)$, $n(B) = 16P(38, 3)$
14. $\frac{132!}{144!}$ **15.** $8 \times 9!$ **16.** 10^4 **20.** 24

Exercise 11.2

1. 64×10^8 **2. a.** $\frac{1}{10}$ **b.** $\frac{1}{100}$ **c.** $\frac{1}{128}$ **d.** $\frac{1}{4}$ **e.** $\left(\frac{9}{10}\right)^7$
3. a. n^{11} **b.** n^{10} **c.** $n^{12} - (n - 1)^{12}$ **4.** 3066 **5. a.** 2^{r-1}
b. 2^{r-2} **c.** $3 \times 2^{r-2}$ **6.** 3×2^{10} **7.** 16 **8. a.** $26^3 \times 10^3$
b. 1123200 **c.** 6344000 **d.** 553280 **9.** 3276 **10.** 1324 hours
11. 7 **12.** 29701 **13.** $\frac{1}{2}$ **14.** 81 **15. a.** 63 **b.** odd numbers less than 64 **c.** odd numbers from 33 to 63 inclusive **16.** 7
17. a. 512 **b.** 256 **c.** 384 **18.** 72.7% **19.** $\frac{P(n, r)}{n^r}$
21. r even $n^{\frac{r}{2}}$, r odd $n^{\frac{r+1}{2}}$ **23.** $(n(T))^{n(s)}$ **24. a.** yes **b.** 64

Exercise 11.3

1. a. 20 **b.** 34220 **c.** 35820200 **2.** $\frac{4}{11}$ **3. a.** 252 **b.** 21 **c.** 100
d. 226 **e.** 126 **f.** 196 **4. a.** $\frac{3}{7}$ **b.** $\frac{4}{7}$ **c.** $\frac{2}{7}$ **5. a.** 5040 **b.** 4!
6. 2970 **7.** 10626 **9. a.** $n(\mu) = 66$, $n(U) = 3$,

$n(B) = 12, n(C) = 19$ **b.** 47 **10. a.** 462 **b.** 200 **c.** 455
d. 378 **11. a.** $\binom{49}{6}$ **b.** $\binom{48}{5}$ **c.** $\binom{24}{6}$ **d.** $\binom{24}{3} \times \binom{25}{3}$

12. a. $\binom{1200}{60}$ **b.** $\left[\binom{300}{15}\right]^4$ **c.** $\binom{300}{60}$ **13.** $\frac{4657}{32340}$ **14.** 1440
15. a. (R1, R2, A), (R1, R2, B), (R1, R2, C), (R1, R3, A), (R1, R3, B), (R1, R3, C), (R2, R3, A), (R2, R3, B), (R2, R3, C), (R1, R2, R3)
16. a. $\binom{100}{5}$ **b.** $\binom{97}{5}$ **c.** $\binom{3}{1}\binom{97}{4}$ **d.** $\binom{3}{2}\binom{97}{3}$ **e.** $\binom{3}{3}\binom{97}{2}$
17. a. $\frac{n(n-3)}{2}$ **b.** none if n is odd, $\frac{n}{2}$ if n is even **18. a.** 45
b. 45 **19. a.** 1140 **b.** 460 **c.** 90 **20. b.** 78 **21.** 2600

Exercise 11.4

1. a. 00011, 01111, 10001 **b.** 32 **2.** 30 **3. a.** 415800 **b.** 151200 **c.** 45360 **d.** 75600 **e.** 189000 **4.** $\frac{12!}{2^6}$ **5.** $\frac{35}{128}$ **6.** 26 **7.** 225
8. 20160 **9. a.** 128 **b.** 35 **c.** 120 **d.** 20 **10.** $\frac{10}{21}$ **11. a.** 1024 **b.** 128 **c.** 768 **12. a.** $\frac{16!}{(4!)^4}$ **b.** $(4!)^4$ **13. a.** 0 **b.** 30
14. a. $\frac{319}{324}$ **b.** $\frac{5}{2592}$ **c.** $\frac{5}{16}$ **15. a.** 2^{10} **b.** $\binom{10}{r}, 0 \le r \le 10$
16. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n$ **18.** 36 **19.** 201 **20.** $\frac{8}{15}$
21. 56 **22. b.** $\frac{32!}{20!12!}$ **c.** $\frac{20!}{10!10!} \times \frac{12!}{10!2!}$ **d.** $\frac{12!}{8!4!} \times \frac{8!}{4!4!} \times \frac{12!}{8!4!}$
23. a. 286 **b.** 36036 **c.** $\binom{12}{2} \cdot \binom{9}{5}$ **24. a.** $(12, \pm 2k)$, k is an integer, $-6 \le k \le 6$ **c.** 924

Exercise 11.5

1. $26^k - 25^k$ **2.** $101 - k$ for $2 \le k \le 100$; 0 for $k = 1$ or 2
3. $\binom{L-1}{4}$ **4.** $\binom{n+1}{o} \cdot \binom{n}{e}$ **5. a.** $rP(n-1, r-1)$
b. $n^r - (n-1)^r$ **6.** $\frac{(n-1)!}{(r-1)!}$ **7.** $\binom{n}{k} - \binom{n-2}{k}$
8. $2 \times (n-2) \times (n-2)!$ **9.** $\frac{r!}{2} \times \binom{998}{r-2}$ **10.** $\frac{(2n)!}{2^n}$
11. $\frac{(b+c)!}{b!c!} \times \binom{b+c+1}{a}$

Review Exercise

1. a. 151200 **b.** 15120 **c.** 105840 **d.** 10080 **e.** 70560 **f.** 13440 **g.** 141120 **2. a.** 10^5 **b.** 9×10^4 **c.** 10^3 **d.** 19×10^3 **e.** 10^4 **f.** 30240 **g.** 99990 **h.** 40951 **3. a.** 495 **b.** 81 **c.** 15 **d.** 369 **e.** 117 **f.** 90 **g.** 324 **4. a.** 256 **b.** 64 **c.** 192 **d.** 28 **e.** 6 **f.** 7 **g.** 16 **h.** 247 **5. a.** 2520 **b.** 6 **c.** 56 **d.** 784 **e.** 2016 **f.** 952 **g.** 60
8. a. 67 **b.** 59 **c.** cdabe **9. a.** $\binom{n-2}{r-2}$ **b.** $2\binom{n-1}{r-1} - \binom{n-2}{r-2}$
c. $\binom{n-2}{r}$ **10. a.** $\frac{1}{2}\binom{n-2}{r-2}r!$ **b.** $\binom{n-3}{r-3} \times \frac{r!}{3!}$
c. $2 \times \binom{n-3}{r-3} \times \frac{r!}{3!}$

Chapter 11 Test

1. 600 **2. a.** 6720 **b.** 2520 **c.** 4800 **4. a.** 495 **b.** 45 **c.** 450
5. 7776 **6.** 300 **7.** $3 \cdot 2^{n-2}$
8. $1333 \times 36^6 - 703 \times 26^6 - 111 \times 10^6$

CHAPTER 12

Review of Prerequisite Skills

1. a, e are Arithmetic; b, f are Geometric c, d are something else
2. $t_{10} = 31, S_{10} = 175$ **3.** -180 **4.** $b_n = a + 2(n-1)d$ **5.** -682
7. a. 3, 6, 10, 15, 21; no **b.** yes

Exercise 12.1

1. a. 1, −1, 5, −13, 41 **b.** 1, 1, 2, 3, 5
c. $1, x+1, x^2+x+1, x^3+x^2+x+1$
$x^4+x^3+x^2+x+1$
d. $x, 1+2x, 3+4x, 7+8x, 15+16x$ **2. a.** $t_n = 27 - 5n$
b. 5 **3. b.** \$22053.13 **6. a.** $g_n = 9.4^{n-1}$ **b.** $T_n = 3.4^{n-1}$
c. $G_1 = 3_1, G_n = 9 \cdot 2^{2n-3}, n \ge 2$ **7.** 4 **9. c.** 63 months
10. $r_1 = 3_1, r_{3k-1} = 3, r_{3k} = 3, r_{3k+1} = 4$ **11.** a if n is odd, $s - a$ if n is even **13.** $x_n = 0.5, n \ge 1$ **14. a.** 2, 4, 7, 11
15. a. $f_n(x) = \frac{x}{1+nx}$ **c.** 0 **d.** $\frac{-2}{3}$ **16. a.** 1, 3, 9, 27
d. $b_n = \frac{1}{2}b_{n-1}$ **e.** $b_n = \left(\frac{1}{2}\right)^{n-1}$ **f.** $A_n = \frac{3^{n-1}\sqrt{3}}{4^n}, A_n \to 0$
17. $A \to \frac{2\sqrt{3}}{5}$, perimeter is infinite

Exercise 12.2

1. a. $\sum_{i=1}^{25} i$ **b.** $\sum_{i=1}^{15} 3 \times 2^{i-1}$ **c.** $\sum_{i=50}^{100} i^2$ **d.** $\sum_{i=1}^{30} t_{2i-1}$
2. a. $1 + 2 + 3 + ... + 10$ **b.** $4^2 + 5^2 + 6^2 + 7^2 + 8^2$
c. $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ **5. a.** $\frac{1}{100}\sum_{i=1}^{100} a_i$
6. a. $\frac{3^{10}-1}{2 \times 3^9}$ **b.** $\frac{3^n - 1}{2 \times 3^{n-1}}$ **c.** $\frac{9(3^{16}-1)}{8 \times 3^{16}}$ **7.** $\frac{1}{r^{60}}$
8. $\frac{10(10^n-1)-9n}{9}, \frac{10(10^n-1)-9n}{81}, \left[\frac{10(10^n-1)-9n}{81}\right]k$
9. 2^n **10.** $\frac{n}{2}d$ for n even, $-a - \frac{n-1}{2}d$ for n odd; $\frac{(-a)[1-(-r)^n]}{1+r}$
11. a. 2046, 145 **b.** 6118 **12.** $\frac{n(n+1)}{2}$ **13.** $\frac{n}{n+1}$
14. $\frac{n(4n^2-12n+11)}{3}$ **16.** $\frac{a(1-r^n)}{(1-r)^2} - \frac{nar^n}{1-r}$

Exercise 12.3

20. a. 2^{k-1}

Exercise 12.4

1. a. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5a^4b - b^5$
b. $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$
c. $x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$
d. $1 - 5s^2 + 10s^4 - 10s^6 + 5s^8 - s^{10}$
e. $x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$
f. $z^{15} - 5z^{12}b^2 + 10z^9b^4 - 10z^6b^6 + 5z^3b^8 - b^{10}$
2. a. 26 **b.** $\binom{25}{4}, \binom{25}{23}$ **c.** 1081575 **d.** 2300, 3268760
e. 5200300 **f.** 2^{25} **3. a.** $(-1)^k\binom{15}{k}3^kx^k$
b. $(-1)^k\binom{20}{k}5^{20}2^kx^k$ **c.** $\binom{13}{k}a^{13-k}x^{2k}$ **d.** $\binom{10}{k}a^{30-5k}$
4. a. $(-1)^k\binom{12}{k}2^{12-k}x^k$ **b.** 264 **5.** 90720 **6. a.** 16 **b.** −32
8. 1 **9.** $1 + x - 7x^2 - 8x^3$ **10.** 528 **11.** 728 **12. a.** 810 **b.** 196
13. a. $t_4 = -54648a^{-15}$ **b.** no term containing a^{10} **14.** 6
15. a. $(-1)^k\binom{10}{k}2^kz^{20+3k}$ **b.** $t_6 = -8064z^{35}$ **c.** −15360
d. z^{36} does not occur in the expansion **e.** no **16.** 10 **17.** 3

19. $(1 + 3x)^7$ **28. a.** 2^n **b.** $\binom{n}{k}$ **29.** $\frac{(n+1)(n)}{2}$ **31.** 7

33. a. $\binom{n+r-k}{r-k}$ **36.** $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots$
$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$
$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots$
40. a. $1 + x + x^2 + x^3$; x^k
b. $1 - 2x + 3x^2 - 4x^3$; $(-1)^k(k+1)x^k$
c. $1 + 3x + 6x^2 + 10x^3$; $\frac{(k+2)(k+1)x^k}{2}$
d. $1 - 4x + 10x^2 - 20x^3$; $(-1)^k\frac{(k+3)(k+2)(k+1)}{3!}x^k$
e. $1 + 8x + 40x^2 + 160x^3$; $\frac{(k+3)(k+2)(k+1)}{3!}2^kx^k$
f. $1 - 15x + 135x^2 - 945x^3$;
$(-1)^k\frac{(k+4)(k+3)(k+2)(k+1)}{4!}3^kx^k$

Review Exercise

1. -15 **3. a.** $-\frac{2^{99}+1}{2^{99}}$ **b.** $\frac{-6(4^{50}-1)}{5 \times 4^{50}}$ **c.** $-\frac{2}{9}(1 + 2^{98})$
4. a. $A_n = a + \frac{n-1}{2} \bullet d$; arithmetic
$G_n = \sqrt[n]{a(a+d)(a+2d) \ldots (a+(n-1)d)}$; neither
b. $A_n = a\left(\frac{1-r^n}{1-r}\right)$; neither, $G_n = ar^{\frac{n-1}{2}}$; geometric
5. a. 1, 5, 17, 53, 161, 485 **6 a.** $U_n = .999U_{n-1} + 100000$
b. grow **c.** 30000 **7.** escapes on the 9th move
8. a. $h_n = h_{n-1} + \left(\frac{1}{3}\right)^{n-1}$, $A_n = A_{n-1} + \left(\frac{1}{3}\right)^{2n-2}$ **b.** $h \to \frac{3}{2}$,
$A \to \frac{9}{8}$ **16.** 924, 0 **17.** 4 **18.** 924 **19.** 1

Chapter 12 Test

2. $\frac{a^2(r^{2n}-1)}{r^2-1}$ **3. a.** \$3840 **b.** $t_n = 1.01t_{n-1} - 200$, $t_1 = 4000$
6. a. 495 **b.** 924 **7.** 3^n **8.** -576

Cumulative Review Chapters 10–12

1. a. 1413720 **c.** 684420 **2.** 969 **3. a.** 64 **b.** 20 **c.** 32
4. $-\binom{20}{5} \bullet 2^5$ **6.** 420 **9. a.** 10, 28, 82 **10. b.** 155 **11.** 25560
12. 0, $1 \le r \le 4$; $5!\binom{r}{5}$ for $r \ge 5$.