Chapter 7: Trigonometric Identities and Equations

Skills and Concepts you need: 1, 2, 4, 7

7.1 Exploring Equivalent Trigonometric Functions

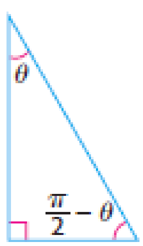
There are many equivalent trigonometric expressions due to the periodic nature of trigonometric functions. Two expressions may be equivalent if their graphs coincide. To demonstrate that two expressions are equivalent requires additional reasoning about the properties of both graphs.

Ex1 A,B- Class discussion

Horizontal translations that involve multiples of the period of a trigonometric function can be used to obtain two equivalent functions with the same graph. Horizontal translations of $\pi/2$ that involve both a sine and cosine can be used to obtain two equivalent functions with the same graph. Note that translating cosine π/2 to the right results in sine, and translating sine π/2 to the left results in cosine.

Cosine is an even function, and so reflecting the graph across the y-axis results in an equivalent function with the same graph. Sine and Tan are odd functions, and so have rotational symmetry about the origin. Reflecting these functions across the x-axis and the y-axis produces an equivalent function with the same graph.

The cofunction identities describe trigonometric relationships between the complementary angles $\theta, \frac{\pi}{2} - \theta$ in a right triangle.



$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

Ex1 L,M – Group discussion

One can identify equivalent trigonometric expressions by comparing principle angles drawn in standard position in quadrants II, III, and IV with their related acute angle θ in quadrant I.

Principal Angle in Quadrant II	Principal Angle in Quadrant III	Principal Angle in Quadrant IV
$\sin(\pi - \theta) = \sin\theta$	$\sin(\pi + \theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
$\cos(\pi - \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
$\tan(\pi - \theta) = -\tan\theta$	$\tan(\pi + \theta) = \tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$

FYU 3 – Group discussion/take home

https://www.youtube.com/watch?v=dOxKANjyvD0

7.2 Compound Angle Formulas

A **compound angle** is an angle that is created by adding or subtracting two or more angles, such as (a-b).

ITM – Class discussion

The trigonometric ratios of compound angles are related to the trigonometric ratios of their component angles by the following:

Addition Formulas

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

Subtraction Formulas

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$
$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

You can use compound angle formulas to obtain exact values for trigonometric ratios and to show that some trigonometric expressions are equivalent.

Ex1 – Class discussion

Ex2,3 – Group discussion Ex4 – Group discussion/take home

https://www.youtube.com/watch?v=sU2pyMR8GZ4

Assignment (7.1 – 7.2):

- 7.1: 1, 3, 5, 6, 7
- 7.2: 5,8,10,11,12,1

7.3 Double Angle Formulas

ITM – Class discussion

The double angle formulas show how the trigonometric ratios for a double angle, 2θ, are related to the trigonometric ratios for the original angle θ.

Double Angle Formula for Sine

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Double Angle Formulas for Cosine

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$

Double Angle Formula for Tangent

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

The double angle formulas can be derived from the compound angle formulas and can be used to develop other equivalent formulas.

Ex1 – Class discussion

Ex2,3 – Group discussion Ex4 – Group discussion/take-home

https://www.youtube.com/watch?v=-Wu6ekO2Sdw

7.4 Proving Trigonometric Identities

A trigonometric identity states the equivalence of two trigonometric expressions. It is written as an equation that involves trigonometric ratios, and the solution set is all real numbers for which the expressions on both sides of the equation are defined. Therefore, the equation has an infinite number of solutions.

Ex1 – Class discussion

Some trigonometric identities are the result of a definition, while others are derived from relationships that exist among trigonometric ratios.

Identities Based on Definitions

Reciprocal Identities

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Identities Derived from Relationships

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Addition and Subtraction Formulas

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin (x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos (x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

You can verify a trigonometric identity by graphing each side separately and showing that the two graphs are equivalent.

To prove that a given equation is an identity, the two sides of the equation must be shown to be equivalent. This can be done by:

- Simplifying the more complicated side until it is identical to the other side, or manipulating both sides to get the same expression.
- Rewriting expressions using any of the trigonometric identities.
- Using a common denominator or factoring, where applicable.

Ex2 – Class discussion

Ex 3,4 - Group discussion

Ex 5 – Group discussion/take home

https://www.youtube.com/watch?v=NK4MlZXepYQ

7.5 Solving Linear Trigonometric Equations

The same strategies can be used to solve linear trigonometric equations when the variable is measured in degrees or radians.

Ex 1 – Class discussion

A scientific or graphing calculator provide very accurate estimates of the value for an inverse trigonometric function. The inverse trigonometric function of a positive ratio yields the related acute angle. Use the related acute angle and the period of the corresponding function to determine all the solutions in the given interval.

Ex 2 – Class discussion

Ex 3 – Group discussion

Ex 4 – Group discussion/take-home

https://www.youtube.com/watch?v=Y9CQHjprLkc

Assignment (6.3-6.5):

- 7.3: 4, 6, 8,9,12
- 7.4: 5, 8, 9, 10, 11
- 7.5: 6, 7,8,9,11,13

7.6 Solving Quadratic Trigonometric Equations

A quadratic trigonometric equation is an equation that contains a square of a trigonometric ratio, and may have multiple solutions between 0 and 2pi. Some of the solutions may be inadmissible, however, in the context of the problem. Quadratic trigonometric equations can be solved graphically or algebraically.

To solve a quadratic trigonometric equation:

1. Factor the equation to create two linear trigonometric equations and then solve.
2. If the equation cannot be factored, use the quadratic formula and then solve the resulting linear trigonometric equations.

To create a quadratic equation that contains only a single trigonometric function whose arguments all match, you may need to use a Pythagorean identity, compound angle formula, or double angle formula.

https://www.youtube.com/watch?v=ZkK4ifsQoGk