Chapter 6: Trigonometric Functions

Skills and Concepts you need: 1, 3, 5, 6

Related acute angle: The acute angle that can be found between the terminal arm and the x-axis when the terminal arm is in quadrants 2,3, or 4.

Principal angle: The CCW angle between the initial arm and the terminal arm of an angle in standard position.

Special Triangles: There are 2 special triangles:

1. 30-60-90 triangle is a right triangle with a 30deg angle and a 60deg angle. For these types of triangles, the hypotenuse is twice as long as the shortest leg, and the longer leg is sqrt(3) times the shortest leg.
2. 45-45-90 triangle is a right triangle with two 45deg angles. For these triangles, the hypotenuse is the sqrt(2) times longer than either leg, which are the same length.

Period: The interval of x-values on which the cycle of the graph that's repeated in both directions lies.

Amplitude: Half the distance from the highest point of the curve to the bottom point of the curve. Amplitude = (Max – Min)/2.

Equation of Axis: The equation of the horizontal line that is halfway between the maximum value and the minimum value of a sinusoidal function, which can be determined by y = (max+min)/2.

6.1 Radian Measure

The **radian** is an alternative way to represent the size of an angle. The radian is defined using the fact that the arc length of a circle is proportional to its radius, and the central angle it subtends by the formula $\theta = \frac{a}{r}$. One radian is defined as the angle subtended by an arc that is the same length as the radius, ie. $\theta = \frac{a}{r} = \frac{r}{r} = 1 \approx 57.3°$.

Ex 1 – Class discussion

Using radians enables you to express the size of an angle as a real number without units, often in terms of π. It is related to degree measure via π radians = 180deg. Hence, to convert from degrees to radians, multiply by $\frac{\pi}{180°}$. To convert from radians to degrees, multiply by $\frac{180°}{\pi}$.

Ex 2,3 – Class discussion for a in both, group discussion for b in both

Ex 4 – Group discussion/take home

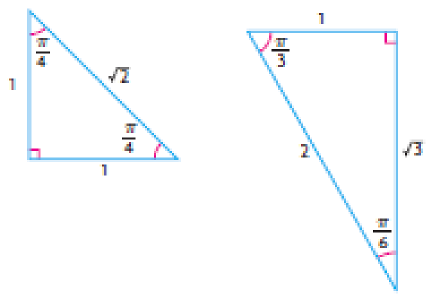
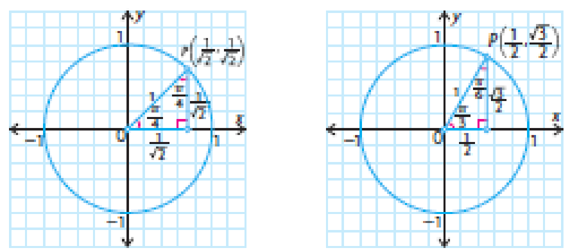
https://www.youtube.com/watch?v=EnwWxMZVBeg

6.2 Radian Measure and Angles on the Cartesian Plane

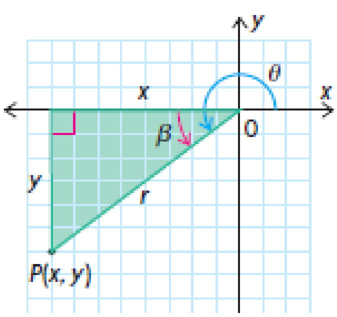
The angles in special triangles can be expressed in radians OR degrees. The radian measures can be used to determine the exact values of the trigonometric ratios for multiples of these angles between 0 and 2π.

The strategies that are used to determine the values of trigonometric ratios when an angle is expressed in degrees on the Cartesian plane can also be used when the angle is expressed in radians.

Ex 1 – Class discussion

The Special Triangles	The Special Triangles on the Cartesian Plane Using a Circle of Radius 1
	

The trigonometric ratios for any principal angle in standard position can be determined by finding the related acute angle, using coordinates of any point that lies on the terminal arm of the angle.



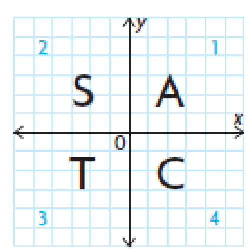
From the Pythagorean theorem, $r^2 = x^2 + y^2$, if $r > 0$.

$\sin\theta = \frac{y}{r}$ $\quad\cos\theta = \frac{x}{r}$ $\quad\tan\theta = \frac{y}{x}$

$\csc\theta = \frac{r}{y}$ $\quad\sec\theta = \frac{r}{x}$ $\quad\cot\theta = \frac{x}{y}$

Ex 2, 3 – class discussion part a, group discussion part b

The CAST rule is an easy way to remember which primary trigonometric ratios are +ve in which quadrant. Since *r*, or the length of the terminal arm, is always +ve, the sign of each primary ratio depends on the signs of the coordinates of the point.



- Q1: all ratios are +ve
- Q2: Only sine is +ve
- Q3: only tangent is +ve
- Q4: only cosine is +ve

Ex 4 – group discussion/take home

https://www.youtube.com/watch?v=nh__OEPKXnY

Assignment (6.1 – 6.2):

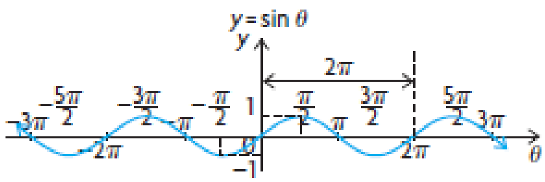
- 6.1: 6, 10, 11, 12, 13
- 6.2: 5, 6, 7, 9, 11

6.3 Exploring Graphs of the Primary Trigonometric Functions

The graphs of the primary trigonometric functions can be summarized as:

Key points when $0 \le \theta \le 2\pi$

θ	$y = \sin(\theta)$
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

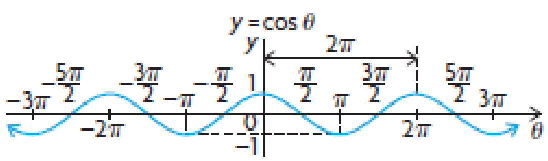


Period $= 2\pi$
Axis: $y = 0$
Amplitude $= 1$
Maximum value $= 1$
Minimum value $= -1$

$D = \{\theta \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} \mid -1 \le y \le 1\}$

Key points when $0 \le \theta \le 2\pi$

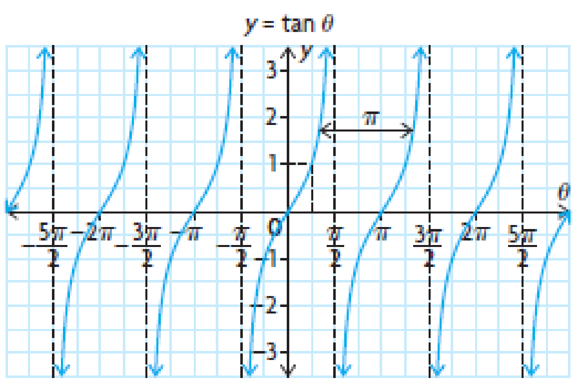
θ	$y = \cos(\theta)$
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1



Period $= 2\pi$
Axis: $y = 0$
Amplitude $= 1$
Maximum value $= 1$
Minimum value $= -1$

$D = \{\theta \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} \mid -1 \le y \le 1\}$

Key points:
- y-intercept $= 0$
- θ-intercepts $= 0, \pm\pi, \pm 2\pi, \ldots$



Period $= \pi$
Axis: $y = 0$
Amplitude: undefined
No maximum or minimum values
Vertical asymptotes:
$\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$D = \left\{\theta \in \mathbf{R} \,\middle|\, \theta \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\right\}$

$R = \{y \in \mathbf{R}\}$

FYU 1 – Class discussion

https://www.youtube.com/watch?v=QmxMPPkZpME

6.4 Transformation of Trigonometric Functions

The graphs of the functions of the form $f(x) = a\sin\big(k(x-d)\big) + c,\ \ g(x) = a\cos\big(k(x-d)\big) + c$ are transformations of the parent functions $y = \sin(x)\,, \cos(x)$ respectively.

Ex 1 – Class discussion

To sketch these functions, you can either:

1. Begin with the key points in one cycle of the parent function and apply any stretches/compressions and reflections to these points, and then apply any translations. As before, $(x, y) \to (\frac{x}{k} + d, ay + c)$. To graph more cycles, as required by the domain, add multiples of the period to the x-coordinates of the transformed points and draw a smooth curve.
2. Using the given equation, determine the equation of the axis, amplitude, and period of the function. Use this information to determine the location of the maximum and minimum points and the points that lie on the axis for one cycle. Plot these points, and then apply the horizontal translation. To graph more cycles, as required by the domain, add multiples of the period to the x-coordinates of the transformed points and draw a smooth curve

We can make connections between the transformations of the parent function and characteristics of the transformed function:

Transformations of the Parent Function	Characteristics of the Transformed Function
$\lvert a\rvert$ gives the vertical stretch/compression factor. If $a < 0$, there is also a reflection in the x-axis.	$\lvert a\rvert$ gives the amplitude.
$\left\lvert\frac{1}{k}\right\rvert$ gives the horizontal stretch/compression factor. If $k < 0$, there is also a reflection in the y-axis.	$\frac{2\pi}{\lvert k\rvert}$ gives the period.
d gives the horizontal translation.	d gives the horizontal translation.
c gives the vertical translation.	$y = c$ gives the equation of the axis.

Note that if the independent variable has a coefficient other than +1, the **argument**, or the expression on which a function operates, must be factored to separate the values of k and d. (ex. $y = 3\cos(2x + \pi) \to y = 3\cos\left(2\left(x + \frac{\pi}{2}\right)\right)$)

FYU 1 a, d – Group discussion

Ex 2 – Class discussion

Ex 3 – Group discussion/take home

https://www.youtube.com/watch?v=Vw-RwPBWS8g

6.5 Exploring Graphs of the Reciprocal Trigonometric Functions

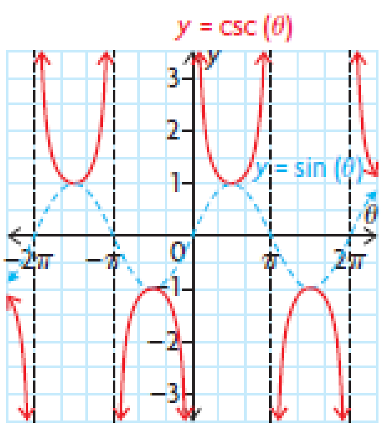
Each of the primary trigonometric graphs have a corresponding reciprocal function:

Cosecant	**Secant**	**Cotangent**
$y = \csc\theta$	$y = \sec\theta$	$y = \cot\theta$
$y = \dfrac{1}{\sin\theta}$	$y = \dfrac{1}{\cos\theta}$	$y = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

The graphs of the reciprocal trigonometric functions are related to the graphs of the corresponding primary trigonometric functions:

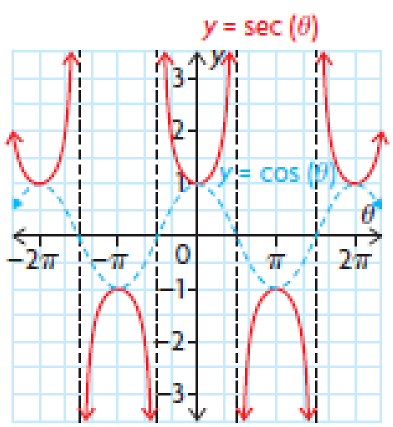
- The reciprocal graph has a V.A. at each zero of the corresponding primary
- The reciprocal graph has the same +ve/-ve intervals as the corresponding primary.
- Intervals of increase and decrease are opposite in the reciprocal graph compared to the primary.
- The reciprocal intersects with the corresponding primary function where the y-coordinate is equal to +/-1.
- The primary has a local min where the reciprocal has a local max at the same x-value, and vice versa.

Cosecant



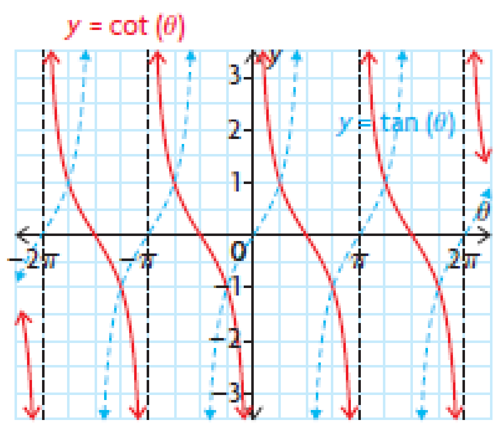
- has vertical asymptotes at the points where $\sin\theta = 0$
- has the same period (2π) as $y = \sin\theta$
- has the domain $\{x \in \mathbf{R} \mid \theta \neq n\pi, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R} \mid |y| \geq 1\}$

Secant



- has vertical asymptotes at the points where $\cos\theta = 0$
- has the same period (2π) as $y = \cos\theta$
- has the domain $\{x \in \mathbf{R} \mid \theta \neq (2n-1)\frac{\pi}{2}, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R} \mid |y| \geq 1\}$

Cotangent



- has vertical asymptotes at the points where $\tan\theta = 0$
- has zeros at the points where $y = \tan\theta$ has asymptotes
- has the same period (π) as $y = \tan\theta$
- has the domain $\{x \in \mathbf{R} \mid \theta \neq n\pi, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R}\}$

FYU 1,2,3 – Group discussion/take-home

https://www.youtube.com/watch?v=kpBNFv8HJyY

Assignment (6.3-6.5):

- 6.3: 3, 4, 5
- 6.4: 4, 5, 7, 10, 11
- 6.5: 4,5,6,7

6.6 Modelling with Trigonometric Functions

The graphs of $y = \sin(x)$, $y = \cos(x)$ can model periodic phenomena when they are transformed to fit a given situation. The transformed functions are of the form $y = a\sin\big(k(x-d)\big) + c, y = a\cos\big(k(x-d)\big) + c$, where:

- |a| is the amplitude (ie. a = (max-min)/2)
- |k| is the number of cycles in 2π radians, when the period is $\frac{2\pi}{k}$
- d gives the horizontal translation
- c is the vertical translation, and so y=c is the horizontal axis.

Tables of values, graphs, and equations of sinusoidal functions can be used as mathematical models when solving problems. Determining the equation of the appropriate sine or cosine function from the data or graph provided is the most efficient strategy, however, since accurate calculations can be made using the equation.

Ex1,2 – Class discussion

Ex 3 – Group discussion

CYU 1,2,3 – group discussion / take home

https://www.youtube.com/watch?v=4KLVZ1FQflc