Chapter 5: Rational Functions, Equations, & Inequalities

Skills and concepts you need: 1 (a,c,e), 2(a,c,e), 3(a,d), 4(a,c,e), 5(a,d)

5.1 Graphs of Reciprocal Functions

What IS a reciprocal function? What does it sound like it might be? What reciprocal functions have we run into already?

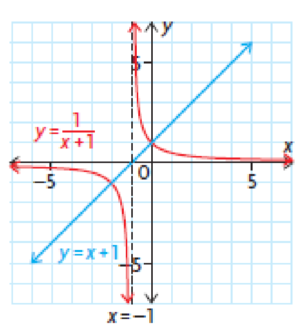
- Class discussion: Investigate the math A-E

A **reciprocal function** is the reciprocal or inverse of a function, ie. The reciprocal function of $f(x)$ is $\frac{1}{f(x)}$. A reciprocal function has:

- a range that is the reciprocal of the range of the original function
- a vertical asymptote at each zero of the original function (why?)
- a $y = 0$ asymptote if the original function is linear or quadratic
- the same +/-ve intervals as the original function
- opposite intervals of increase/decrease (ie. Intervals of increase in the original function are intervals of decrease in the reciprocal function and vice versa)
- an intersection with the original function at all points where the dependent coordinate is 1 or -1 (why?)
- opposite local min/max points (ie. Where the original function has a local min, the reciprocal function will have a local max) (why?)
- Group discussion: investigate the math F-H, Ia, J, K,L,M

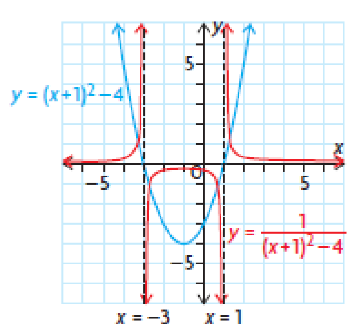
Particularly for linear and quadratic functions, one can use key characteristics of their graphs to graph the related reciprocal function

A linear function and its reciprocal



Both functions are negative when $x \in (-\infty, -1)$ and positive when $x \in (-1, \infty)$. The original function is increasing when $x \in (-\infty, \infty)$. The reciprocal function is decreasing when $x \in (-\infty, -1)$ or $(-1, \infty)$.

A quadratic function and its reciprocal



Both functions are negative when $x \in (-3, 1)$ and positive when $x \in (-\infty, -3)$ or $(1, \infty)$. The original function is decreasing when $x \in (-\infty, -1)$ and increasing when $x \in (-1, \infty)$. The reciprocal function is increasing when $x \in (-\infty, -3)$ or $(-3, -1)$ and decreasing when $x \in (-1, 1)$ or $(1, \infty)$.

Group discussion: Ex 1, 2

Individual/partner: 1,2,3

Graphing reciprocal functions: https://www.youtube.com/watch?v=k-jFNP-w9Tc

5.2 Exploring Quotients of Polynomial Functions

Quotients of polynomial functions are by definition **rational functions**. Rational functions do not necessarily always take the form of a quotient of polynomial functions, but CAN be expressed as $f(x) = \frac{p(x)}{q(x)}$, where *p*,*q* are polynomial functions and $q(x) \neq 0$. q may still have zeros, which results in the rational function having discontinuities. (why?)

- Class discussion: A, B, C, D
- Group discussion: E, F(I,v), G, H,I,M,N

Discontinuities occur where the function is undefined, which happens at values were the denominator is 0. These values must be restricted from the domain of the function. This results in key characteristics that define the shape of the graph, and can include a combination of vertical asymptotes (**infinite discontinuities**), and holes (**point discontinuities**). A hole will occur at $x = a$ if $f(a) = \frac{p(a)}{q(a)} = \frac{0}{0}$, which occurs when both the numerator and denominator contain a factor of $x - a$. A vertical asymptote will occur at $x = a$ if $f(a) = \frac{p(a)}{q(a)} = \frac{p(a)}{0}$.

The end behaviours of rational functions are typically determined either by a horizontal asymptote or an **oblique asymptote**, an asymptote that is neither vertical or horizontal, but slanted. A horizontal asymptote occurs only when the degree of $p(x)$ is less than or equal to the degree of $q(x)$, and an oblique asymptote occurs only when the degree of $p(x)$ is greater than the degree of $q(x)$ by exactly 1.

Individual/partner: 1, 2 (a,e,c,g,j,l), 3(a,c,d)

https://www.youtube.com/watch?v=Bctd33KVe4A

Assignment:

- Ch 5.1: 4, 5(a,c,e,g), 6(a,d), 9(a,c), 12
- Ch. 5.2: 2 (b,d,g,h,I,k), 3(b,e)

5.3 Graphs of Rational Functions of the Form $f(x) = \frac{ax+b}{cx+d}$

Most rational functions of the form $f(x) = \frac{b}{cx+d}$ and $f(x) = \frac{ax+b}{cx+d}$ have both a vertical asymptote and a horizontal asymptote. The vertical asymptote can be found directly by solving for the zero of the denominator. The horizontal asymptote can be found by examining the ratio of the leading coefficients in the numerator and the denominator, which gives you the end behaviours of the function.

- Class discussion: Investigate the math A-C
- Group discussion: D,E,F,G

Specifically, rational functions of the form $f(x) = \frac{b}{cx+d}$ have a vertical asymptote $x = -\frac{d}{c}$ and a horizontal asymptote $y = 0$. Most rational functions of the form $f(x) = \frac{ax+b}{cx+d}$ have a vertical asymptote $x = -\frac{d}{c}$ and a horizontal asymptote $y = \frac{a}{c}$. However, there is an exception when the numerator and denominator share a common linear factor. This results in a graph of a horizontal line with no asymptotes, with a hole where the zero of the common factor occurs.

- Group discussion: Ex 1,2,3

Individual/partner: 1,2,3

https://www.youtube.com/watch?v=aNyC1sIYWGo

5.4 Solving Rational Equations

We already know how to solve polynomial equations, and so transforming rational equations into polynomials is the simplest way of solving them. To solve a rational equation algebraically, multiply each term by the lowest common denominator and solve the resulting polynomial equation. One can verify the solution by using graphing technology, and determining the zeros of the corresponding rational function or determining the intersection of the two functions.

- Class discussion: Ex1 solution A, B
- Group discussion: reflecting A,B,C

Note that the root of the equation $\frac{ax+b}{cx+d} = 0$ is the zero, or x-intercept, of the function $f(x) = \frac{ax+b}{cx+d}$.

In general, the zeros of a rational function are the zeros of the function in the numerator as reciprocal functions, and hence the denominator of rational functions, do not have zeros (why?). All functions of the form $f(x) = \frac{1}{g(x)}$ have the x-axis as a horizontal asymptote and do not intersect the x-axis.

As with all contextual problems, one must check for inadmissible solutions that are outside of the domain determined by context. Especially when solving or verifying a solution of a rational equation using graphing technology, one can avoid inadmissible roots by specifying the window settings to the domain of the problem.

- Group discussion: Ex 2,3,4

Individual/partner: 1,2,3,4

Assignment:

- Ch. 5.3: 4(a,c), 5(a,c), 6(a,c), 9

- Ch. 5.4: 5(a,c,e), 6(a,c,e), 7(a,d), 9

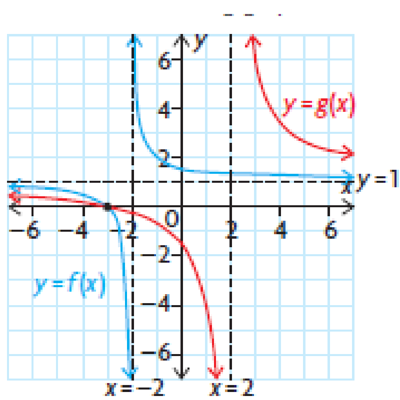
https://www.youtube.com/watch?v=1fR_9ke5-n8

5.5 Solving Rational Inequalities

A **rational inequality** is a statement that one rational expression is greater than or less than another rational expression. As we saw previously with polynomial inequalities, solving an inequality means finding all possible values of the variable that satisfy the inequality.

- Class discussion: Ex 1 solution A,B,C

Rational inequalities can always be solved graphically by graphing both sides of the inequality and identifying all the intervals created by the vertical asymptotes and points of intersection. Rational inequalities can also be solved graphically by creating an equivalent inequality with 0 on one side and identifying the intervals created by the zeros on the graph of the new function. (where would the graph(s) need to be in either case to find the solution?)



In this graph, there are four intervals to consider: $(-\infty, -3)$, $(-3, -2)$, $(-2, 2)$ and $(2, \infty)$. In these intervals, $f(x) > g(x)$ when $x \in (-\infty, -3)$ or $(-2, 2)$, and $f(x) < g(x)$ when $x \in (-3, -2)$ or $(2, \infty)$.

To solve a rational inequality algebraically, rearrange the inequality so that one side is 0. Then combine the expressions on the non-zero side using a common denominator. Examine the sign of each factor and the entire expression on the intervals created by the zeros of the numerator and denominator. If each denominator is positive, then one can multiply both sides by the lowest common denominator to make the inequality easier to solve (why?).

Multiplying or dividing both sides of an inequality by a negative causes the inequality sign to reverse to maintain equivalence.

- Group discussion: Ex 2, 3

Individual/partner: 1,2,3

Good review questions: Ch. 5.5: 4(a,c,e), 5(a,d), 6(a,d), 9

https://www.youtube.com/watch?v=gfnVHwhEe6U

Chapter review:

Assignments, mid chapter review, chapter review, self-test