Chapter 3: Polynomial Functions (Ch. 1, McGraw-Hill)

Skills and concepts you need: 1 (a, d), 2a, 3(a, d), 5a, 6a – what is the quadratic formula and how/when do you use it? What are the rules associated with the quadratic formula?

Graphing possibilities: online mathematica, python/jupyter

3.1 Exploring Polynomial Functions

Class discussion: explore the math A,B,C

A **polynomial function** in one variable is a function of the form:

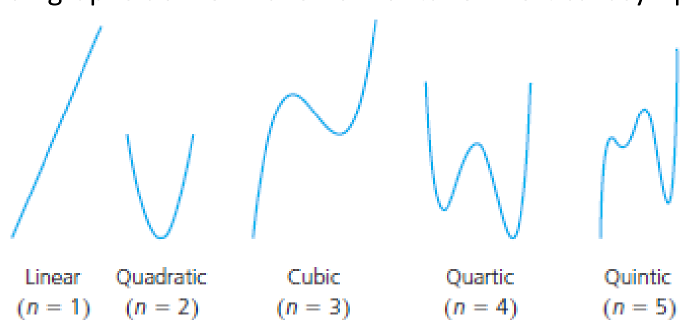
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are real numbers and n is a positive integer. Polynomial functions are named according to their **degree**, which is the number n, or the largest exponent value in the expression. Polynomial functions of degree 1,2,3,4 and 5 are commonly called **linear, quadratic, cubic, quartic,** and **quantic**, respectively. Note that polynomial functions and expressions are typically written with the powers arranged in descending order (as we have written it above).

- Group discussion: D (paper), E, F, J,K,L

Polynomial functions have a few properties that we need to keep in mind:

- The n^{th} **finite differences** of a polynomial function of degree n are constant. The finite difference method entails finding the differences in the dependent variable values that correspond to equally spaced independent variable values.
- The domain of a polynomial function is the set of real numbers.
- The range of a polynomial function may be all real numbers, or it may have a lower bound OR an upper bound – but not both! (why?)
- Polynomial function graphs do NOT have horizontal OR vertical asymptotes (why?)



Individual/Partner: 1. (a,b,c), 2(a,d,e), 3(a, c), 5

Intro to polynomials (UNTIL 5:30): https://www.youtube.com/watch?v=m7mIkplcCmk

3.2 Characteristics of Polynomial Functions

Class discussion: A, B

Polynomial functions of the same degree have similar characteristics, such as shape, turning points, and zeros. In general, a polynomial function of degree n has at most $n-1$ turning points and up to n distinct zeros. However, an odd-degree polynomial must have at least one zero while an even-degree polynomial may have no zeros.

The degree and the **leading coefficient** of a polynomial function indicate the **end behaviours** of the graph. The leading coefficient is the coefficient of the term with the highest degree in the polynomial (ie. a_n), and end behaviours are the behaviours of the polynomial at the "end", or towards +/- infinity.

End behaviours are different for odd- and even-degree polynomial functions.

For odd-degree polynomial functions, if the leading coefficient is –ve, then the function extends from the second quadrant to the fourth quadrant, or $x \to -\infty, y \to \infty$ and $x \to \infty, y \to \infty$. If the leading coefficient is +ve, then the function extends from the third quadrant to the first quadrant, or $x \to -\infty, y \to -\infty$ and $x \to \infty, y \to \infty$. In general, odd-degree polynomial function have opposite end behaviours.

For even-degree polynomial functions, if the leading coefficient is –ve, the function extends from the third quadrant to the fourth quadrant, or $x \to \pm\infty, y \to -\infty$. If the leading coefficient is +ve, then the function extends from the second quadrant to the first quadrant, or $x \to \pm\infty, y \to \infty$. Even-degree polynomials must also have an **absolute maximum** or **absolute minimum**. A **local min/max** is a min/max within a subset of the function, and a global or absolute min/max is the smallest/largest possible min/max for the entire function.

- Group discussion: Reflecting (J,K,L,M)

Polynomials may also be even or odd functions, and may have symmetrical properties. Polynomials that are even functions (NOT necessarily even degree!), ie. $f(-x) = f(x)$, are symmetrical in the y-axis. Polynomials that are odd functions, ie. $f(-x) = -f(x)$, have rotational symmetry about the origin. However, most polynomial functions are neither even or odd functions, and have no symmetrical properties.

- Group discussion: Ex1, Ex2

Individual/Partner: 1(a,c), 2, 3(ii, v)

End behaviours (FROM 5:30 on): https://www.youtube.com/watch?v=m7mIkplcCmk

Local max and mins (or relative max and mins): https://www.youtube.com/watch?v=3LfEPjgmWWM

3.3 Characteristics of Polynomial Functions in Factored Form

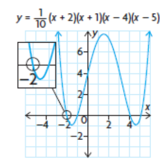
In general, the zeros of a polynomial function $y = f(x)$ are the same as the roots of the related polynomial equation $f(x) = 0$. To find these roots or zeros, it is most helpful to determine the equations of the polynomial function in factored form, which can be done by:

1. Substituting the zeros $(x_1, x_2, \dots, x_n)$ into the general equation of the appropriate **family of polynomial functions** of the form $y = a(x - x_1)(x - x_2) \dots (x - x_n)$
2. Substituting the coordinate of an additional point for x, y and solving for a.

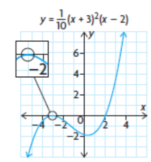
A family of polynomial functions is a set of polynomial functions whose equations have the same degree and whose graphs share characteristics, and have the generic form $y = a(x - x_1)(x - x_2) \dots (x - x_n)$ where n is the degree. Additionally, each factor in a factored polynomial expression has an **order**, or the exponent to which each factor is raised.

- Group discussion A,B,D

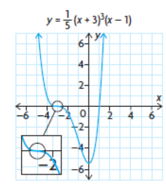
If any factors of a polynomial function are linear (ie. order 1) then the corresponding x-intercept is a point where the curve passes through the x-axis, and the graph has a linear shape near this x-intercept.



If any factors of a polynomial function are quartic (ie, order 2, or squared) then the corresponding x-intercepts are turning points of the curve and the x-axis is the tangent line to the curve at these points, with the graph having a parabolic shape near these points.



If any factors of a polynomial function are cubed, then the corresponding x-intercepts are points where the x-axis is tangent to the curve and passes through the x-axis, with the graph having a cubic shape near these points or “sitting” on the x-axis.



- Group discussion: Ex 1,2,5

Individual/partner: 1, 2, 3, 4

Review of sections 3.1-3.3: https://www.youtube.com/watch?v=bbPjssv2n94

Assignment:

- Ch. 3.1: 1(e,d,f), 2(b,c,e), 3(b,d), 8
- Ch. 3.2: 4(a,c,e), 5, 8
- Ch. 3.3: 5, 6(a,d), 15

3.4 Transformations of Cubic and Quartic Functions

What are the family forms of cubic and quartic functions? What are some examples? What would be the simplest form of cubic and quartic functions?

The parent function for cubic polynomials is $y = f(x) = x^3$, and the parent function for quartic polynomials is $y = f(x) = x^4$. These are also the simplest forms of these functions, and thus we can investigate transformations.

- Group discussion: Reflecting I,J,K

The polynomial function $y = a\left(k(x-d)\right)^n + c$ can be graphed by applying transformations to the graph of the parent function $y = x^n$, where n is positive integer. Each point $(x, y) \rightarrow \left(\frac{x}{k} + d, ay + c\right)$ on the graph of the function.

In $y = a\left(k(x-d)\right)^n + c$:

- a represents a vertical stretch or compression, and possibly a vertical reflection
  - Which values would do which actions?
- k represents a horizontal stretch or compression, and possibly a horizontal reflection
  - Which values would do which actions?
- d represents a horizontal translation
  - What happens when +ve? –ve?
- c represents a vertical translation
  - What happens when +ve? –ve?

https://www.youtube.com/watch?v=0w5tUm8HNP4

Group discussion: Ex1, 2

Individual/partner: 1, 2(a,c,e), 3(a,d)

3.5 Dividing Polynomials

There are 4 main mathematical operations: adding, subtracting, multiplying, and dividing. Adding, subtracting, and multiplying polynomial functions is relatively straightforward, however dividing is not as obvious or intuitive.

- Class discussion: long division (quotient, dividend, divisor)

Polynomials can be divided in a similar way that numbers or divided, but a polynomial can only be divided by a polynomial of the same degree or less. (why?)

- Class discussion: Ex 1 A

For general polynomial division, focus on the first terms of the dividend and the divisor first, and then determine the quotient when these terms are divided. This becomes the first term of the quotient. Multiply this first term of the quotient by the divisor, and write the answer below the dividend, lining up like terms. Bring down the next term and repeat the process until the remainder degree is less than the degree of the divisor.

- Class discussion Ex. 1B
- Group discussion : reflecting A,B,C, Ex 2

When the divisor is a linear polynomial, synthetic division can be used. When using either polynomial or synthetic division, the terms should be arranged in descending order of degree in both the divisor and the dividend and zeros must be used as the coefficient of any missing powers of the variable in both the divisor and dividend. If the remainder of polynomial or synthetic division is zero, then the divisor and the quotient are factors of the dividend.

- Group discussion: Ex 3,4

Individual/Partner: 1,2,3,4

Polynomial division: https://www.youtube.com/watch?v=_FSXJmESFmQ

Synthetic division: https://www.youtube.com/watch?v=FxHWoUOq2iQ

3.6 Factoring Polynomials

You have covered factoring polynomials before, as well as combining factors into full expressions – especially for quadratics, or degree 2.

- Group discussions: review of factoring quadratic functions, including quadratic formula.
- What about higher degree polynomials?
- Group discussion: Investigate the math

The **remainder theorem** states that when a polynomial $f(x)$ is divided by the divisor $x - a$, the remainder is equal to $f(a)$. If the remainder is 0, then the divisor is a factor of the polynomial. The **factor theorem** is a special case of the remainder theorem, and states that $x - a$ is a factor of $f(x)$ if and only if (**iff**) $f(a) = 0$. Note that "A iff B" means that if A is true, then B is also true and vice versa.

With this in mind, to factor a polynomial of degree 3 or high:

- Use the factor theorem to determine a factor of the polynomial
- Divide the polynomial by that factor
- Factor the quotient, if possible
- Repeat, if necessary
- Group discussion: Ex 1,2,3,4,5

Even armed with these tools, not all polynomial functions are factorable (why?).

Individual/partner: 1,2,3

Remainder theorem+synthetic division: https://www.youtube.com/watch?v=p1lSRAeEMR0

Factor theorem+synthetic division: https://www.youtube.com/watch?v=zAGP46nR6-0

Assignment:

- Ch. 3.4: 4(a,e), 5a, 8
- Ch. 3.5: 5(a,c,e), 6 (a,c,e), 7(a,c), 11
- Ch. 3.6: 4(a,c,e), 5(a,c), 6(a,c), 7(a,c), 8

3.7 Factoring a Sum or Difference of Cubes

Factoring a sum or difference of squares is something you've seen before. Can we extend this to a sum or difference of cubes?

- Class discussion: sum or difference of squares, learn about the math
- Group discussion: Ex 1, any solution
- Class discussion: reflecting A,B,C

An expression that contains two perfect cubes that are summed may be factored as:

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

An expression that contains two perfect cubes that are subtracted from the other may be factored as:

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

- Group discussion: Ex. 2,3,4

Individual/Partner: 1,2,3

https://www.youtube.com/watch?v=ADj8sGSjewg

Chapter test prep:

Assignment questions, chapter practice questions, mid-chapter review, chapter self-test