Ch.1: Function Characteristics and Properties

Skills and concepts you need: 1, 2(a,c)

1.1 Functions

Functions are used to represent problems in our everyday. Unlike arithmetic, functions use variables, such as "x" or "y", to represent a problem that may be solved. **Function notation** is used to represent the dependent variable in a function, for example $y = f(x)$, where y is the dependent variable, x is the independent variable, and $f(x)$ is the function of the independent variable x.

- Function notation: $y = 2\sin(3\mathrm{x}) + 4$ or $f(x) = 2\sin(3x) + 4$

The functions that we will consider are **injective**, meaning that there is a unique input for every output, or that the independent variable corresponds to one and only one value of the dependent variable. The values of the independent variable that a function can be used for is called the **domain**, and the corresponding dependent variable values are the **range**. The domain and range are dependent on the function itself, and may need to be restricted depending on what the function is intended to represent.

- When would the domain and/or range need to be restricted?
- How do we write the domain and range for a function?

- Ex. 1: Jonathan and Tina are building an outdoor skating rink. They have enough materials to make a rectangular rink with an area of about $1800\ m^2$ and they do not want to purchase any additional materials. They know, from past experience, that a good rink must be approximately 30 m longer than it is wide. What dimensions should they use to make their rink?
    - Can we make a function to describe this problem and its solution?
    - What is the independent variable? What is the dependent variable?
    - What would be the domain? What would be the range? Do either of these need to be restricted?

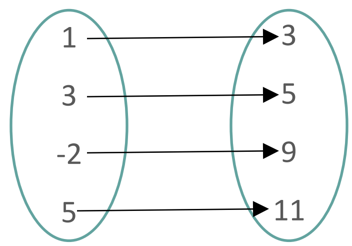
Functions can be represented **graphically, numerically, or algebraically**. Graphically implies the function is drawn according to its domain and range, algebraically implies the function is written in function notation and may be evaluated or solved, and numerically implies several or many of the dependent values evaluated by a function for a set of independent values are written out, for example, in a table, set of ordered pairs, or mapping diagram.

- For numerically representing a function, there are at least 3 ways you can use:

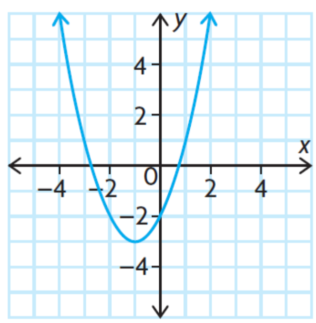
- Set of ordered pairs: {(1,3), (3,5), (-2, 9), (5, 11)}, where the first variable in each pair is an element of the domain, and the second variable in each pair is an element of the range.
- Table of values:

x	y
1	3
3	5
-2	9
5	11

- Mapping Diagram:



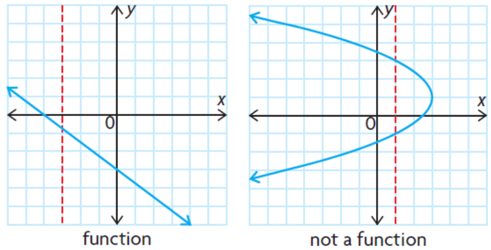
- For graphically representing a function, graph paper is the preferred medium. Every graph should have the axes clearly labelled (ex. x,y ; time, distance; etc.) with appropriate tick marks indicating quantities on both axes:



- Ex.1 continued:
  - How can we represent this function graphically, numerically, and algebraically? Which one is easier to do first?
  - Would the function change if we switched the independent and dependent variable?

One can test whether a graph represents a function by using the **vertical line test**. If a graph intersects any vertical lines within its domain more than once, then it is not a function as more than one range element corresponds to only one domain element. If a graph only intersects all vertical lines with its domain once, then it is a function as there is only one range element for each domain element.

Video on vetical line test, domain, range, and function definitions (ONLY TO 7:17!): https://www.youtube.com/watch?v=DrEXTC6mIO8



The domain and range of a function can be found out by looking for **asymptotes** in the function. Asymptotes are limits that a function tends towards but never reaches in either the horizontal or vertical axis, and imply that the domain or range, respectively, continue on to infinity above or below this line. **Vertical asymptotes** of a function can be represented as a line that lies in one value on the horizontal axis (ie. $x = 1$) and likewise, **horizontal asymptotes** of a function can be represented as a line that lies in one value on the vertical axis (ie. $y = -5$).

Video on asymptotes: https://www.youtube.com/watch?v=oBmnt5tXRws

Another useful tool is the **finite difference method**, which is particularly applicable to polynomial functions. For a series of values for a function with equally spaced independent variables, one can take the differences between each successive value to get the 1st finite differences, and then repeat the process with these new values to get the 2nd finite differences, and so on.

- Ex. What do you notice about this pattern?

x	**f(x)= x^2**	**1st finite difference**	**2nd finite difference**
-2	4	4-1 = 3	3-1=2
-1	1	1-0 = 1	1-(-1)=2
0	0	0-1=-1	-1-(-3)=2
1	1	1-4=-3	-3-(-5)=2
2	4	4-9=-5	
3	9		

- Video on finite differences (ONLY TO 3:40!), using a cubic: https://www.youtube.com/watch?v=etqJfCVC1yc

Ex. 2 in text as group work.

Ex. 3 in text as class discussion.

Check your understanding in text as individual/partner work.

1.2 Exploring Absolute Value

The **absolute value** of a number is its **magnitude**, which is always positive or non-negative. One can find the absolute value of an input by applying the **absolute value function**, which describes the distance of any input away from the origin.

The absolute value function is a **piecewise function**, meaning that it cannot be defined by only one function expression for all of its domain. It is typically written as $f(x) = |x|$. For the absolute value function:

- There is one zero located at the origin
- The graph is symmetric about the vertical axis
- As the input approaches large positive values, the output approaches large positive values
- As the input approaches large negative values, the output approaches large positive values
- Is comprised of two linear functions: $f(x) = \begin{cases} x, if\ x \geq 0 \\ -x\ if\ x < 0 \end{cases}$
    - What is the domain and range of the absolute value function?

One can use **number lines** to represent piecewise functions and their domains. (what do solid and hollow dots mean on a number line?)

Video on number lines and absolute values: https://www.youtube.com/watch?v=dQCNC97Y-mw

Video on absolute value inequalities and number lines: https://www.youtube.com/watch?v=MsDDmSMaC5Q

Group work: 1, 2

Class discussion: 3, 4

Individual/pairs work: 5 a), b), 6, 7 a), b)

1.3 Properties of Graphs of Functions

Functions can be categorized based on their **graphical characteristics**. Function types can be evaluated by determining which graphical characteristics a function does have, and eliminating those which it does not have. Note that some characteristics are more helpful than others when determining the type of function.

Graphical characteristics include:

- **X-intercept**: where a function intercepts the x axis, or for what input does $f(x) = 0$.
- **Y-intercept:** where a function intercepts the y axis, or the output of $f(0)$.
    - What are the intercepts for f, g, p, and q?
- **Interval of increase:** the interval(s) within a function's domain where the output of the function gets larger with increasing inputs, or from left to right the function increases in value.
- **Interval of decrease:** the interval(s) within a function's domain where the output of the function gets smaller with increasing inputs, or from left to right the function decreases in value.

- For intervals if increase or decrease, or the end behavior of a function, it is helpful to use the notation "$As\ x \to \infty$, y …", which reads "As x goes to infinity, y…".
  - What are the intervals of increase/decrease for k, h, m, q?
- **Odd function:** any function that has rotational symmetry about the origin. Algebraically, any function that satisfies the property $f(-x) = -f(x)$.
- **Even function:** any function that is symmetric about the vertical axis. Algebraically, any function that satisfies the property $f(-x) = f(x)$.
  - Which of f,g,h,k,m,,q are odd and even?
- **Continuous function:** any function that does not contain any holes or breaks over its entire domain.
- **Discontinuities:** a break in the graph of a function is a point of discontinuity. Graphically, if you were to draw the function, you would need to lift your pen from the paper to draw the function over its entire domain.
  - Are any of f,g,h,k,m,p,q discontinuous? Where are the points of discontinuiuty?

It may be possible to determine the equation of a function by knowing some or all of its characteristics.

Video on intervals of increase and decrease (from 2:30 onwards): https://www.youtube.com/watch?v=KxOp3s9ottg

Video on finding if a function is even, odd, or neither: https://www.youtube.com/watch?v=fKyBOLsqRlo

Video on continuity and discontinuity: https://www.youtube.com/watch?v=BIQrnx-vvx8

Class discussion: Ex.1

Group discussion: Ex. 3

Individual/partner work: Check your understanding

Assignment : OR mid-chapter review questions

- Ch. 1.1: 3. (a,d,e), 4. (b,d,f), 5. (a,c), 7
- Ch 1.2: 5 (c,d), 7 (c,d), 8
- Ch. 1.3: 4, 5(a, c, f), 8(a), 13

1.4 Sketching Graphs of Functions

Graphing functions is fairly straightforward. However, there are a number of **transformations** that can be applied to the graph of a function. Transformations may include **reflections, horizontal and vertical stretches or compressions, and translations**.

The general application of transformations to a function $f(x)$ can be written algebraically as $y = af\big(k(x-d)\big) + c$, where:

- a determines whether there is a **vertical stretch or compression, and/or a reflection in the x-axis**.
    - $|a| > 1$ implies the graph is stretched vertically by a factor of $|a|$.
    - $0 < |a| < 1$ implies the graph is compressed vertically by a factor of $|a|$.
    - $a < 0$ implies the graph is ALSO reflected in the x-axis.
- k determines the whether there is a **horizontal stretch or compression, and/or a reflection in the y-axis.**
    - $|k| > 1$ implies the graph is compressed horizontally by a factor of $\frac{1}{|k|}$.
    - $0 < |k| < 1$ implies the graph is stretched horizontally by a factor of $\frac{1}{|k|}$.
    - $k < 0$ implies the graph is ALSO reflected in the y-axis.
        - When would there ONLY be a reflection in the x or y axis?
- d determines whether there is a **horizontal translation**.
    - $d > 0$ implies the graph is shifted to the right.
    - $d < 0$ implies the graph is shifted to the left.
- c determines whether there is a **vertical translation**.
    - $c > 0$ implies the graph is shifted up.
    - $c < 0$ implies the graph is shifted down.
        - How much is the graph translated in each case?

Video on graphical transformations: https://www.youtube.com/watch?v=MkP1LJR2PyM

Transformation must be performed in a particular order. Stretches/compressions/reflections must be applied first, and translations must be applied last. a, k may be applied at the same time, and likewise c, d can be applied together. (Why?)

All points on a graph of the function $f(x)$ that has been transformed as $y = af\big(k(x-d)\big) + c$ are changed as: $(x, y) \rightarrow (\frac{x}{k} + d, ay + c)$.

Note that transformations may change the **turning points** on the graph, or the point on a curve where the function changes from increasing to decreasing or vice versa.

Group discussion: Ex.1

Class discussion: Ex. 2

Group discussion Ex. 3, 4

Individual/partner practice: check your understanding

1.5 Inverse Relations

Every function has the capability to create an **Inverse function.** Inverse functions are found by switching the independent and dependent variables in a function’s table of values or in the equation of the relation. Plotting a function and its inverse on the same graph shows that the inverse is a reflection of the original function in the line $y = x$.

- Ex. 1: The owners of a candy company are creating a spherical container to hold their small chocolates. They are trying to decide what size to make the sphere and how much volume the sphere will hold, based on its radius. The volume of a sphere is given by the relationship $V = \frac{4}{3}\pi r^3$.
  - How can you use this relationship to find the radius of any sphere for a given volume? Try graphically, numerically, and algebraically.
- Ex.3 (Individual/partner)

The inverse function of $f(x)$ is denoted by $f^{-1}(x)$, but only when the inverse is a function. Not all inverse relations are functions, and the domain and/or range of the original function may need to be restricted to ensure that the inverse of a function is also a function. Note that the domain of the function is the range of the inverse of the function, and the range of a function is the domain of the inverse.

To find the inverse algebraically, interchange the independent and dependent variables and solve. If (a, b) is a point on the graph for $f(x)$, then (b, a) is a point on the graph for $f^{-1}(x)$.

Video on algebraic, numerical, and graphical representations of inverse relations:
https://www.youtube.com/watch?v=on0E9e7P2sQ

Group discussion: Check your understanding 1,2
Class discussion: Check your understanding 3

1.6 Piecewise Functions

As we saw when discussing the absolute value function, some functions may be represented by 2 or more "pieces" or expressions, and needs to be written as more than one function expression for the entirety of its domain. Each piece of a piecewise function is defined for a specific interval in the domain.

To graph a piecewise function, graph each piece over the given interval. The pieces don't need to connect, as a piecewise function can be continuous or not. If all the pieces of a piecewise function connect, then the function is continuous (ie. You can draw the function without lifting your pen from the paper). Otherwise, the function is discontinuous at the endpoints of the pieces in the domain.

Intro to piecewise functions: https://www.youtube.com/watch?v=OYOXMyFKotc

Graphing piecewise functions: https://www.youtube.com/watch?v=Uzw9tsGq2Pw

Class discussion: Ex. 1, 2

Individual/partner: Ex. 3, 4

Group discussion: Ex. 5

Assignment 2:

- Ch 1.4: 4 (a,c,e), 5 (a, d, f), 6, 8
  Ch. 1.5: 4, 6(b,d), 7, 9, 16
- Ch. 1.6: 1,2 (a,c,e), 5 (c,d), 6, 8

1.7 Exploring Operations with Functions

Like the numbers that constitute a function domain and range, functions themselves can be added, subtracted, or multiplied to create a new function. Two functions can be added, subtracted, or multiplied by each other IF they have domains that overlap. The new function will exist on the shared domain.

- Ex 1., part class discussion (a,c,d,f,g,I,j) and part group discussion (b, e,g,h,j,k)
- Group discussion - reflecting

Graphically, functions can be added, subtracted, or multiplied by adding, subtracting, or multiplying the values of the dependent variable for identical values of the independent variable.

Algebraically, functions can be added, subtracted, or multiplied by adding, subtracting, or multiplying the expressions for the dependent variable and then simplifying.

Properties of each original function have an impact on the properties of the new function.

Individual/partner: 1, 2a, 3a, 4a, 5(a,c),6

Video on adding, subtracting, and multiplying functions: https://www.youtube.com/watch?v=3gaxVHVI4cI

Chapter 1 test prep questions:

Assignment questions, chapter practice questions, chapter self-test