Polynomials Assignment – A4 Solutions

1. [6K] a) $y = 12(x-9)^3 - 7$

$a = 12 \Rightarrow$ vertical stretch of 12.
$d = 9 \Rightarrow$ horizontal translation of 9 to the right.
$c = -7 \Rightarrow$ vertical translation of 7 down.

b) $y = -2(-3(x-4))^3 - 5$

only one needed for full marks.

option 1
$a = -2 \Rightarrow$ reflection across the x-axis, vertical stretch of 2.
$k = -3 \Rightarrow$ reflection across the y-axis, horizontal compression of 3
$d = 4 \Rightarrow$ horizontal translation of 4 to the right.
$c = -5 \Rightarrow$ vertical translation of 5 down.

option 2
$y = -2(-3)^3(x-4)^3 - 5 = 54(x-4)^3 - 5$
$a = 54 \Rightarrow$ vertical stretch of 54
$d = 4, \quad c = -5$

2. [5A] parent function : $y = x^2$. vertex has been moved 11 down, so $d = 0$, $c = -11$. (ie. $h = 0$, $k = -11$) From given formula, only 1 variable left to solve for : a. If there is no horizontal compression or stretch, then our coordinate mapping is: $(x, y) \to (x + d, ay + c) \Rightarrow (x, ay + c)$ since $d = 0$.
In our transformed function, when $x = \pm 1$, $y = -3$. In our parent function, when $x = \pm 1$, $y = 1$. So $ay + c = -3$
$\Rightarrow a(1) - 11 = -3$, $a = +8$
∴ The parent function had a vertical stretch of 8 and a vertical translation of 11 down.

3. [4A] $g(x) = x^3$; reflected in x-axis $\Rightarrow a < 0$, vertically compressed by $2/3 \Rightarrow |a| = \frac{2}{3}$ horizontally translated by 13 unit right $\Rightarrow d = 13$, vertically translated 13 units down $\Rightarrow c = -13$. ∴ $a = -\frac{2}{3}$, $d = 13$, $c = -13$.

$(x, y) \to (\frac{x}{k} + d, ay + c)$, have ① $(11, -23/3)$, ② $(13, -13)$, ③ $(15, -55/3)$

① $(11, -23/3) \to$ $x + d = 11$, $x = 11 - d = -2$; $ay + c = -23/3$, $-\frac{2}{3}y - 13 = -\frac{23}{3}$, $y = -8$ → $(-2, -8)$

② $(13, -13) \to$ $x = 13 - d$, $x = 0$; $y = \frac{1}{a}(-13 - c)$, $y = 0$ → $(0, 0)$

③ $(15, -55/3) \to$ $x = 15 - d = 2$; $y = \frac{1}{a}(-\frac{55}{3} - c) = 8$ → $(2, 8)$

4. [6K] a)

$$\begin{array}{r} x^2+4x+14 \\ x-4\overline{)\,x^3+0x^2-2x+1} \\ -(x^3-4x^2) \\ \hline 4x^2-2x \\ -(4x^2-16x) \\ \hline 14x+1 \\ -(14x-56) \\ \hline 57 \end{array}$$

$$\Rightarrow \frac{x^3-2x+1}{x-4} = (x-4)(x^2+4x+14)+57$$

b)

$$\begin{array}{r} x^2+2x-3 \\ 2x+1\overline{)\,2x^3+5x^2-4x-5} \\ -(2x^3+x^2) \\ \hline 4x^2-4x \\ -(4x^2+2x) \\ \hline -6x-5 \\ -(-6x-3) \\ \hline -2 \end{array}$$

$$\Rightarrow \frac{2x^3+5x^2-4x-5}{2x+1} = (2x+1)(x^2+2x-3)-2$$

c)

$$\begin{array}{r} x+1 \\ x^3-x^2-x+1\overline{)\,x^4+0x^3+6x^2-8x+12} \\ -(x^4-x^3-x^2+x) \\ \hline x^3+7x^2-9x+12 \\ -(x^3-x^2-x+1) \\ \hline 8x^2-8x+11 \end{array}$$

$$\Rightarrow \frac{x^4+6x^2-8x+12}{x^3-x^2-x+1} = (x^3-x^2-x+1)(x+1) + 8x^2-8x+11.$$

5. [6K] a) $x-3 \Rightarrow k=3$

$$\begin{array}{r|rrrr} 3 & 1 & 0 & -7 & -6 \\ & & 3 & 9 & 6 \\ \hline & 1 & 3 & 2 & \underline{0} \end{array} \Rightarrow \frac{x^3-7x-6}{x-3} = (x-3)(x^2+3x+2)$$

b) $x+3 \Rightarrow k=-3$

$$\begin{array}{r|rrrrr} -3 & 6 & 13 & -34 & -47 & 28 \\ & & -18 & 15 & 57 & -30 \\ \hline & 6 & -5 & -19 & 10 & \underline{-2} \end{array} \Rightarrow \frac{6x^4+13x^3-34x^2-47x+28}{x+3} = (x+3)(6x^3-5x^2-19x+10) - 2.$$

c) $2x+1 = 2(x+\frac{1}{2}) \Rightarrow k=-\frac{1}{2}$

↑ divide quotient by 2 after synthetic division!

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 12 & -56 & 59 & 9 & -18 \\ & & -6 & 31 & -45 & 18 \\ \hline & 12 & -62 & 90 & -36 & \underline{0} \end{array} \Rightarrow \frac{12x^4-56x^3+59x^2+9x-18}{2x+1} = 2(x+\tfrac{1}{2})(6x^3-31x^2+40x-18)$$

extract a factor of 2 for the actual quotient!

6. [4T] $f(x) = x^n - 1$, $n \in \mathbb{N}$. We know that for $n = 1, 2$, $f(x)$ is divisible by $x-1$. Let's see if we can find a pattern for higher n.

$n=3$: $\frac{x^3-1}{x-1}$

1	1	0	0	-1
		1	1	1
	1	1	1	0

$\therefore$ divisible, quotient is x^2+x+1.

$n=4$: $\frac{x^4-1}{x-1}$

1	1	0	0	0	-1
		1	1	1	1
	1	1	1	1	0

$\therefore$ divisible, quotient is x^3+x^2+x+1.

For every sequential n, we would add another 0 to the middle columns of our synthetic division, which implies that we would still always get a 0 remainder, just more iterations of $0+1$ in between.

$\therefore$ we can state $\frac{x^n-1}{x-1} \stackrel{!}{=} (x-1)(x^{n-1}+x^{n-2}+x^{n-3}+\dots+1)$, yes $f(x)$ is divisible by $x-1$ for all n.

(optional, mathematical induction. Mathematical induction states that if a statement is true for $k=1$ (lowest possible n in this case), some general k, then if we can prove it holds for $k+1$ it is true for all k.

Base case: $k=1$: $f(x) = x-1$ evidently divisible by $(x-1)$)

assume $f(x) = x^k - 1$ is divisible by $(x-1)$. Is $f(x) = x^{k+1} - 1$?)

* If we take n to include 0, then $f(x)$ is still divisible by all n! $f(x) = x^0 - 1 = 1 - 1 = \underline{0}$ $\quad \frac{0}{x-1} = \underline{0}$.

7. [2K] $x+2 \Rightarrow a = -2$. Using the remainder theorem:

a) remainder $= (-2)^2 + 2(-2) + 9$
$= \underline{-1}$

b) remainder $= (-2)^3 + 3(-2)^2 - 10(-2) + 6$
$= \underline{30}$

8. [4K] a) $x^3 - 3x^2 - 10x + 24 = f(x)$

step 1: factors to try: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$.

$f(2) \stackrel{!}{=} 0$ $\therefore$ $x-2$ is a factor.

step 2:

2	1	-3	-10	24
		2	-2	-24
	1	-1	-12	0

step 3 $\Rightarrow$ $f(x) = (x-2)(x^2-x-12)$
$f(x) = (x-2)(x-4)(x+3)$

8b) $x^5 - 5x^4 - 7x^3 + 29x^2 + 30x = f(x) \Rightarrow f(x) = x(x^4 - 5x^3 - 7x^2 + 29x + 30)$ — focus on factoring this!

step 1: factors to try: $\pm1, \pm2, \pm3, \pm5, \pm6, \pm15, \pm30$

$f(-1) \stackrel{!}{=} 0 \therefore x+1$ is a factor.

step 2:

$$\begin{array}{r|rrrrr} -1 & 1 & -5 & -7 & 29 & 30 \\ & & -1 & +6 & +1 & -30 \\ \hline & 1 & -6 & -1 & 30 & \underline{0} \end{array} \Rightarrow f(x) = x(x+1)(x^3 - 6x^2 - x + 30)$$

need to repeat step 1. Same potential factors to try.

(repeat!)

step 1: $f(-2) \stackrel{!}{=} 0 \therefore x+2$ is a factor. Now factor just the largest component.

$$\begin{array}{r|rrrr} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & \underline{0} \end{array} \Rightarrow f(x) = x(x+1)(x+2)(x^2 - 8x + 15)$$

can factor directly.

step 3: $f(x) = x(x+1)(x+2)(x-3)(x-5)$

9. [4A] $f(x) = ax^3 - x^2 + 2x + b$. Using the remainder theorem,

$f(1) = 10$, $f(2) = 51$. $\Rightarrow 10 = a - 1 + 2 + b$, $51 = 8a - 4 + 4 + b$

① $10 = a + b + 1$, ② $51 = 8a + b$.

Solve system of equations! From ②: $b = 51 - 8a$. Plug into ①:

$10 = a + 51 - 8a + 1 \Rightarrow -42 = -7a \Rightarrow \underline{a = 6}, \underline{b = 3}$

10. [4C]. The factor theorem states that ~~[scribbled out]~~

$f(a) = 0$ iff (if and only if) $x - a$ is a factor of $f(x)$.

This works since we know from the remainder theorem that the remainder of $\frac{f(x)}{x-a}$ is $f(a)$ <u>and</u> that if a function has a remainder of 0, then the divisor is a factor, because $f(x)$ is perfectly divided by that divisor. Additionally, we know that factors of $f(x)$ correspond to roots/zeros/x-intercepts, which by definition are the location on the x-axis where the function value is zero.