Exercises for
Cosmology & Galaxies (Summer Term 2018)

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Exercise sheet 1

Due: Apr. 17, 2018, 16:00

To be handed in after the lectures or emailed as a pdf or scanned hand-written document (via e-mail to kehlert@aip.de). Remember to put your names on the document; you can work in groups of ≤ 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it.

1. **Newtonian Friedmann Equations** (10 points)
   Consider a spherical sub-volume of radius $r(t)$, within an (infinite) expanding, homogeneous mass density distribution $\rho(t)$ of a idealized pressureless fluid. The radius $r(t) \equiv a(t) \cdot R$ (where $R = \text{const.}$) is defined to enclose constant mass as a function of time. At $t = t_0$, we have $r(t_0) = R$ and $\rho(t_0) = \rho_0$ and some expansion rate $\dot{a}(t_0)$.

   (a) Use the equation of motion for a test mass on the sphere's surface to derive an equation of motion for $a(t)$? How does this compare to the 2nd Friedmann equation?

   (b) Give an expression for the change in total energy $\mathrm{d}E_{\text{tot}}$ of a test mass on the sphere's surface. What is the solution to the resulting equation of motion? Explain why the integration constant can be identified as the total energy per unit mass of the test mass U. Which cosmological parameter is represented by U?

   (c) If $a(t = 0) = 0$ and $E_{\text{tot}} = 0$, give an expression for the time t_0 since the 'big bang' (i.e. $a = 0$). Compare to the canonical $1/\dot{a}(t_0)$. Does that t_0 depend on the choice of R?

2. **Loitering Universes and the Big Bang** (10 points)
   Consider a curved Friedmann model with non-relativistic matter ($\rho_{\text{m}} \propto a^{-3}$) and a cosmological constant ($\rho_\Lambda \propto a^0$) with densities relative to critical of Ω_{m} and Ω_Λ, respectively.

   (a) By considering the 'second Friedmann equation' for $\ddot{a}$ show that there is an inflexion point in the scale factor evolution when
   $$(1+z)^3 = \frac{2\Omega_{\Lambda,0}}{\Omega_{\text{m},0}}. \tag{1}$$

   (b) Show that the following relation holds at this inflexion point (i.e. $\dot{a} \to 0$):
   $$27\,\Omega_{\text{m},0}^2 \Omega_{\Lambda,0} = 4(\Omega_{\text{m},0} + \Omega_{\Lambda,0} - 1)^3. \tag{2}$$

(to be continued on the back)

(c) Show that the age of the Universe becomes infinite at the inflexion point. Models near this one, with a long period where a is nearly constant, are known as loitering models. The extreme loitering model is Einstein's static universe, also known as Eddington-Lemaitre universes, and are the reason Einstein introduced Λ in the first place. Models with long loitering phases are heavily disfavored by the classical tests (and eventually by Olber's paradox!).

(d) What region of the $\Omega_{\rm m,0} - \Omega_{\Lambda,0}$ plane leads to models with no 'big bang' (i.e., a does not tend to zero at early times)?

(e) Show that such universes have a maximum redshift $z_{\rm max}$ with

$$\Omega_{\rm m,0} \leq \frac{2}{z_{\rm max}^2 (z_{\rm max} + 3)}. \tag{3}$$

Use this relation to test if current observations of high-redshift objects and measurements of $\Omega_{\rm m,0}$ are in agreement with this prediction.

3. **Cosmological Epochs** (10 points)
We have seen in the lecture that during the course of the evolution of the universe different components have dominated the expansion history. Here we assume a canonical cosmological model of $(\Omega_{\rm m,0}, \Omega_{\Lambda,0}, h) = (0.3, 0.7, 0.7)$.

(a) **Matter-Radiation Equality.** The energy density of the CMB can be written in terms of the CMB temperature, $T_{\rm CMB}$, as

$$u_{\rm CMB} = \frac{\pi^2}{15} \frac{(kT_{\rm CMB})^4}{(\hbar c)^3}. \tag{4}$$

With the help of this equation, write down an expression for $\Omega_{\rm r}(a)$ in terms of $T_{\rm CMB}$. (You can neglect the contribution from neutrinos that are subdominant.) If $T_{\rm CMB} = 2.726\,{\rm K}$ at $z = 0$, what is the value of $\Omega_{\rm r,0}$? Calculate the redshift $z_{\rm eq}$ at which the energy densities of matter and radiation are equal.

(b) **Dominance of the Cosmological Constant.** Calculate the redshift $z_{\rm eq,\Lambda}$ at which the energy densities of matter and the cosmological constant are equal.

(c) **Accelerated Expansion.** Rewrite the 'second Friedmann equation' for $\ddot{a}$ and show that this defines a deceleration parameter

$$q(t) \equiv \frac{a\ddot{a}}{\dot{a}^2} = -\frac{\Omega_{\rm m}(a)}{2} + \Omega_\Lambda(a), \tag{5}$$

which is valid for times well after the radiation-dominated epoch. Use this equation to determine the redshift at which the universe started its accelerated expansion phase. Compare and discuss your result in comparison to $z_{\rm eq,\Lambda}$.