Exercises for Cosmology (WS2013/14)

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Exercise sheet 7

Due: Dec. 3, 2013 16:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; groups of ≤ 3 are allowed.

1. **Power Spectrum and Correlation Function** (10 points)

   The power spectrum is conveniently expressed in dimensionless form as the variance per $\ln k$,

   $$\Delta^2(k) \equiv \frac{\mathrm{d}\langle\delta^2\rangle}{\mathrm{d}\ln k} = \frac{4\pi}{(2\pi)^3} k^3 P(k). \tag{1}$$

   Consider a dimensionless power spectrum with small-scale truncation:

   $$\Delta^2(k) = \left(\frac{k}{k_0}\right)^{n+3} \exp(-k/k_\mathrm{c}). \tag{2}$$

   Show that the corresponding correlation function is

   $$\xi(r) = \frac{(k_\mathrm{c}/k_0)^{n+3}}{y(1+y^2)^{1+n/2}} \Gamma(2+n) \sin[(2+n)\arctan y], \tag{3}$$

   where $y = k_\mathrm{c} r$. For this model, explain why ξ is negative at large r for $n > 0$. For what values of n does ξ stay positive at large r?

   Hint: to solve the final integral analytically, use the following identity

   $$\exp(-x)\sin(xy) \equiv \mathcal{I}m\{\exp[-(1+iy)x]\}, \tag{4}$$

   which suggests the change of variables $z = (1+iy)x$ in your integral.

2. **The Zel'dovich Approximation** (10 points)

   Consider a one-dimensional plane parallel density perturbation. Show that in this case, the Zel'dovich approximation provides an exact solution at all times before particle trajectories intersect.

   Hint: one way to go about this is to substitute the Zel'dovich approximation trajectories into the true equation of motion, and then to demonstrate that the gravitational potential $\vec{\nabla}\delta\Phi$ implied by the resulting equation is agrees with the solution for $\vec{\nabla}\delta\Phi$ that one gets from the Poisson equation.

3. **Spherical Collapse with Cosmological Constant** (10 points)

We consider a spherical overdensity, which expands, slows down, turns around, and collapses in a spatially flat universe with a cosmological constant. We normalize the scale factor a and the radius of the overdensity R by their values at turn-around,

$$x \equiv \frac{a}{a_{\rm ta}}, \qquad y \equiv \frac{R}{R_{\rm ta}}, \tag{5}$$

introduce the dimensionless time, $\tau \equiv H_{\rm ta} t$ (where $H_{\rm ta}$ is the Hubble function at turn-around), and define ζ as the relative overdensity inside the sphere at turn-around.

(a) Use both(!) Friedmann equations to show that the dimensionless scale factor x and the radius y obey the differential equations

$$x' = \sqrt{\frac{\omega}{x} + (1-\omega)x^2}, \qquad y'' = -\frac{\omega\zeta}{2y^2} + (1-\omega)y, \tag{6}$$

where ω is the matter-density parameter at turn-around, $\omega \equiv \Omega_{\rm m}(a_{\rm ta})$ and the prime denotes a derivative with respect to the dimensionless time τ.
Hint: you may find it useful to follow the (simplified) derivation of such a pair of differential equations in the Einstein-de Sitter universe as it is being done in the script by Bartelmann on page 50.

(b) Use the boundary condition $y'(x=1) = 0$ to obtain the first integral of y''

$$y' = \left[\zeta\omega\left(\frac{1}{y} - 1\right) + (1-\omega)\left(y^2 - 1\right)\right]^{1/2}. \tag{7}$$