Exercises for
Cosmology (WS2013/14)

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Exercise sheet 12

Due: Jan. 21, 2014 16:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; groups of ≤ 3 are allowed. While this is an **optional exercise sheet**, it is highly advisable to solve those problems as you may encounter similar ones in your exam!

1. **Angular Momenta of Galaxies** (10 bonus points)

   The angular momentum of a galaxy (L) can be expressed in terms of the dimensionless spin parameter
   $$\lambda \equiv \frac{L|E|^{1/2}}{GM^{5/2}}, \tag{1}$$
   where E is the binding energy and M is the mass of the galaxy. A system with appreciable rotational support has $\lambda \sim 1$. Observations suggest that disk galaxies have $\lambda \approx 0.4 - 0.5$.

   One possible way of understanding the angular momenta of galaxies is through the effects of tidal torques during gravitational collapse. Simulations indicate that this process can provide an initial λ of $\lambda_{\mathrm{i}} \approx 0.05$, which is only about 10% of the observed value. During the collapse of gas the binding energy increases due to cooling, while mass and angular momentum remain the same. This will allow λ to increase as $\lambda \propto |E|^{1/2}$ and (possibly) reach observed values.

   (a) Consider the collapse of a homogeneous overdense sphere of mass M and initial radius R_0 from its initial state that is expanding with the Hubble flow. Show that the collapse time is given by
   $$t_{\mathrm{coll}} \sim \pi \left(\frac{R_0^3}{2GM} \right)^{1/2}. \tag{2}$$

   (b) Examine the idea of the increase of the spin parameter λ during the collapse in the absence of dark matter halos and show that it is not viable by estimating the initial size that the gas cloud would need and the timescale for it to collapse to the scales associated with galaxies ($M_{\mathrm{disk}} \sim 10^{11}\mathrm{M}_\odot$ and $R_{\mathrm{disk}} \sim 10\,\mathrm{kpc}$).

   (c) Next, consider the same process in the presence of a dark matter halo. Show that the difficulties in part (b) can be circumvented and a sufficiently high value of λ can result if $M_{\mathrm{disk}} \approx 0.1 M_{\mathrm{halo}}$.

2. **Gravitational Lensing at Galaxies** (10 bonus points)

Suppose that a galaxy has an effective radius $r_\mathrm{g} \simeq 10\,h^{-1}$kpc, and an effective cross section of πr_g^2. Furthermore assume a constant comoving number density of galaxies $n_\mathrm{g} \simeq 0.02\,h^3$Mpc^{-3}.

(a) Show that the optical depth due to galaxies for an object at redshift z in a flat, matter dominated model is

$$\tau \simeq 10^{-2}\left[(1+z)^{3/2} - 1\right]. \tag{3}$$

(b) Keep the model flat by lowering Ω_m and introducing Λ. At what value of Ω_Λ does the optical depth become unity? Evaluate your answer analytically and express it as a function of redshift. This sensitivity to path length is the principle behind limits on Λ from e.g., gravitational lensing.

3. **Imprints of Primordial Non-Gaussianity on the Distribution of Halos** (10 bonus points)

Here, we derive an analytic expression for the peak height and clustering of dark matter halos in the presence of local non-Gaussianity in the primordial density field,

$$\Phi_\mathrm{NG}(\boldsymbol{x}) = \Phi(\boldsymbol{x}) - f_\mathrm{NL}\left(\Phi^2(\boldsymbol{x}) - \langle\Phi^2\rangle\right), \tag{4}$$

where Φ denotes the Newtonian potential, which is connected to the overdensity δ through Poisson's equation on subhorizon scales. With this choice of convention, positive f_NL corresponds to a positive skewness of the density probability distribution, and hence an increased number of massive objects.

(a) Since we are interested in the formation of halos, we focus on high peaks in the density field, where the derivative of Φ vanishes. Thus, we can neglect $|\nabla\Phi|^2$ in comparison to the curvature term $\Phi\nabla^2\Phi$ in the vicinity of rare, high peaks of the density distribution. Use Poisson's equation in combination with equation (4) to show that *in the vicinity of high peaks*, the density contrast in a model with non-Gaussianity (δ_NG) is connected to the matter density contrast of a purely Gaussian model (δ) via the equation

$$\delta_\mathrm{NG} \approx \delta\left(1 - 2f_\mathrm{NL}\Phi\right). \tag{5}$$

This shows that non-Gaussianity *enhances* the peak height by a factor that is proportional to the primordial potential $|\Phi| = |\Phi_\mathrm{late}|a/D_+(a)$, where Φ_late is the evolved potential at late times and $D_+(a)$ is the linear growth factor.

(b) The halo correlation function is parametrized in terms of the halo bias b, which is the rate of change of the halo abundance as the background density is varied. Writing the halo overdensity as δ_h and the matter overdensity (in either model) as δ, we can define the halo bias as

$$\delta_\mathrm{h} = b\delta. \tag{6}$$

Consider a long-wavelength mode that provides a background density perturbation δ and corresponding potential fluctuation Φ. In the absence of non-Gaussianity, this perturbation raises (small-scale) subthreshold perturbations above the threshold, and thereby enhances the abundance of superthreshold peaks by $b_{\rm L}\delta$, where $b_{\rm L}$ is the (Gaussian) Lagrangian bias. As shown in part (a), for non-zero $f_{\rm NL}$, the long-wavelength mode also enhances the peak height by $2f_{\rm NL}|\Phi|\delta_{\rm pk}$, and we will focus on peaks near the threshold such that $\delta_{\rm pk} \simeq \delta_{\rm c}$. Show that in this case, the Lagrangian bias acquires the *scale-dependent* correction

$$\Delta b_{\rm L}(k) = 3 b_{\rm L} f_{\rm NL} \frac{\delta_{\rm c}\Omega_{\rm m} H^2}{D_+(a) k^2}, \tag{7}$$

where the total Lagrangian bias is $b_{\rm L}(k) = b_{\rm L} + \Delta b_{\rm L}(k)$. If you want to maximize your constraining power on the local non-Gaussianity parameter $f_{\rm NL}$, which type of objects would you target for a survey?

(c) Calculate the corresponding Eulerian bias in the presence of local non-Gaussianity.