Binary black hole simulations and the validity of post-Newtonian theory

Harald Pfeiffer

California Institute of Technology

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Gravitational wave detectors

LIGO (Hanford)

GEO 600

Prime target: Binary black hole systems

LISA (201x)

VIRGO
Stages of binary inspiral

- **Inspiral**
  - post-Newtonian (PN) expansions

- **“Late” inspiral**
  - PN $\rightarrow$ Numerical relativity

- **Merger**
  - Numerical relativity

- **Ringdown**
  - Perturbation theory
  - Numerical relativity
Role of numerical relativity

- Support for **GW detectors**
  - Waveforms for templates and vetos
  - Parameter estimation
  - Test general relativity

- **Explore** properties of general relativity
  - Toroidal black holes (Shaprio, Teukolsky)
  - Critical behavior in BH formation (Choptuik)
  - Verify analytic approximations

- Solve the **two-body problem**

- **Binary black hole simulations** began in earnest two years ago
Examples of Merger simulations

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Examples of Merger simulations
Black hole kicks: Gonzalez et. al., 2006

![Graph showing v (km/s) vs \( \eta \) with data points for Baker, et al, Campanelli, Damour and Gopakumar, Herrmann, et al, and Sopuerta, et al.](image)
Examples of Merger simulations
Spinning black holes: Campanelli et. al., 2006
**Mergers vs. Inspirals**

**Mergers**
- highly dynamic
- short duration: several $100M$
- large energy flux: $\Psi_4 r M \sim 0.01$

**Inspirals**
- mildly dynamic
- long duration: several $1000M$
- small energy flux: $\Psi_4 r M \sim 0.0002$
- absolute phase-error relevant for template mismatch
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  - Tests of post-Newtonian expansion
  - Matching to post-Newtonian waveforms

- Kip’s challenge:
Mergers vs. Inspirals

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- **Kip’s challenge:**
  Simulate last 15 orbits with phase error 0.01rad.
Who are we?

Simulation of eXtreme Spacetimes

- Collaboration between Cornell & Caltech
- TAPIR
  - Mike Boyle, Mike Cohen, Tony Chu,
    Lee Lindblom, Geoffrey Lovelace, Rob Owen,
    Harald Pfeiffer, Oliver Rinne, Mark Scheel.
- Cornell
  - Matt Duez, Francois Foucart, Larry Kidder,
    Francois Limousin, Abdul Mroue, Nick Taylor,
    Saul Teukolsky.
- Further collaborators: Duncan Brown, Luisa Buchman,
  Gregory Cook, Jan Hesthaven, Stephen Lau
How to simulate black holes: Essential ingredients.
Generalized Harmonic evolution system

Task: Find space-time metric \( g_{ab} \) such that \( R_{ab}[g_{ab}] = 0 \)

\[
0 = R_{ab} = -\frac{1}{2} \Box g_{ab} + \nabla(\Gamma_a) + \text{lower order terms} \quad \Gamma_a = -g_{ab} \Box x^b
\]
Generalized Harmonic evolution system

- Task: Find space-time metric $g_{ab}$ such that $R_{ab}[g_{ab}] = 0$

  $$0 = R_{ab} = -\frac{1}{2} \Box g_{ab} + \nabla(a \Gamma_b) + \text{lower order terms} \quad \Gamma_a = -g_{ab} \Box x^b$$

- The gauge condition $g_{ab} \Box x^b \equiv H_a$ (with $H_a$ given) eliminates some principal terms. One obtains wave-equations

  $$\Box g_{ab} = \text{lower order terms}$$
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\[ \Box g_{ab} = \text{lower order terms} \]

- Constraint $C_a \equiv H_a + \Gamma_a = 0$ must remain satisfied. Constraint damping (Gundlach, et al, Pretorius, 2005)

\[ 0 = -\frac{1}{2} \Box g_{ab} + \nabla (a C_b) + \gamma \left[ t(a C_b) - \frac{1}{2} g_{ab} t^c C_c \right] + \text{l. o.} \]

\[ \partial_t C_a \sim -\gamma C_a \]
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Spectral Methods

PDE $\mathcal{L}(u) = 0$

- Approximate solution by truncated series

$$u(x, t) \approx u_N(x, t) \equiv \sum_{k=1}^{N} \tilde{u}_k(t) \Phi_k(x).$$

- Analytic differentiation

$$\frac{d u_N(x, t)}{dx} = \sum_{k=1}^{N} \tilde{u}_k(t) \frac{d \Phi_k(x)}{dx}.$$  

- $\mathcal{L}(u_N)$ can be evaluated exactly.
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- Recast hyperbolic PDE as grid-based “method of lines”:

$$\partial_t u(x_i) = \text{RHS}_i[u].$$
### Spectral Methods

#### Accuracy

- **Accuracy determined by how well** \( u(x, t) \) **can be approximated by** \( u_N(x, t) \).  

- **Sinusoidal linearized gravitational wave**  
  - “Mexico City tests” (Mike Boyle et al. 2006)  
  - 3 grid points!

<table>
<thead>
<tr>
<th>Time Step ( \Delta t ) (crossing times)</th>
<th>Phase Error (radians)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-4}</td>
<td>10^0</td>
</tr>
<tr>
<td>10^{-3}</td>
<td>10^{-1}</td>
</tr>
<tr>
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</table>

- 2nd order FD
- 100s of points
- Spectral, RK4
- 3 points

---

**General features**

- **exponential convergence for smooth solutions**
  \[ \text{err} \propto e^{-\lambda N} \]

---

**Eccentricity**

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Spectral Methods

Accuracy

- Accuracy determined by how well $u(x, t)$ can be approximated by $u_N(x, t)$.
- Sinusoidal linearized gravitational wave
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General features
- exponential convergence for smooth solutions $\text{err} \propto e^{-\lambda N}$
- Low phase-errors

![Graph showing phase error vs. time step, comparing 2nd order FD, 100s of points, and Spectral, RK4, 3 points.]
Spectral methods

Domain-decomposition for binary black hole

- Expansions easiest in simple topologies: Blocks, shells, cylinders, ...
- Cover complicated domain by multiple subdomains.
Spectral methods

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Einstein Code: mostly C++, 2,000 files, 250,000 lines, principal developers L. Kidder, M. Scheel, H.P. Harald Pfeiffer (Caltech)
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Cost of high-resolution BBH simulations
- Nov ’05: 4 days for 1/4 orbit (100 CPU-h/M)
- Nov ’06: 4 weeks for 5 orbits (30 CPU-h/M)
- Feb ’07: 3 weeks for 15 orbits (4 CPU-h/M)

Improvements:
- Cubes -> Cylinders
- balancing of resolution
- understanding of constraint damping parameters
- Code optimization
- Outer boundary conditions
Boundary conditions & BH excision

- Generalized harmonic evolution system is symmetric hyperbolic

\[ u^\alpha + A^k\alpha_\beta \partial_k u^\beta = F^\beta \]

- Boundary conditions
  - Decompose into characteristic fields
    \[ e^{\hat{\alpha}}_{\alpha} n_k A^k\alpha_\beta = v(\hat{\alpha}) e^{\hat{\alpha}}_\beta \]
  - Impose BCs on incoming fields

- All modes propagate inside light cone
  \Rightarrow \text{Excision boundaries inside horizon do not require any BC}
Changing domain-decomposition is difficult
– *localize* horizons in coordinate space
(Scheel *et al.*, 2006):
Dual coordinate method

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1. Evolve **inertial frame** components of tensors
2. Represent solution at grid-points which **move** relative to inertial coordinates:

\[ x^\bar{i} = a(t, |x|) R(t)^{\bar{i}}_j x^j \]

- \( R(t)^{\bar{i}}_j \) rotation matrix, \( a(t, |x|) \) radial scaling
Dual coordinate method

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  1. Evolve **inertial frame** components of tensors
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\[ \bar{x}^i = a(t, |x|) R(t)^{\tilde{i}}_j x^j \]

- \( R(t)^{\tilde{i}}_j \) rotation matrix, \( a(t, |x|) \) radial scaling
  3. \( R(t) \) and \( a(t) \) determined by dynamic control based on current AH location
Outer boundary conditions

All physics lies in the boundary conditions!

- Must prevent influx of constraint violations
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- Must allow gravitational waves to exit without reflection.
  - Consider Newman-Penrose scalars
    - $\Psi_4$ is represented by outgoing characteristic fields (good!)
    - $\Psi_0 \equiv 0$ implies conditions on some incoming char. fields

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  - Sommerfeld BC on gauge modes
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Initial data

Zero radial velocity

- Quasiequilibrium data (Cook, HP, et al, 02-06)
- Equal mass, no spin
- Black holes in *circular* orbits
- Sequences of initial data sets at different separation

![Graph showing initial data parameters](image)

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Initial data

Non-zero radial velocities

(Pfeiffer et al., 2007)

- Allow for nonzero initial radial velocities of BHs.
- Tune radial velocities (and $\Omega_0$) to reduce orbital eccentricity.
Initial data
Non-zero radial velocities

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![Graphs showing proper separation and change in radial velocity over time.

Proper separation $s/M_{ADM}$

- Quasi-circular ($e \approx 0.01$)
- 1st iteration ($e \approx 7 \times 10^{-4}$)
- 2nd iteration ($e \approx 5 \times 10^{-5}$)

$ds/dt$

- Quasi-circular
- 1st iteration
- 2nd iteration

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Orbital trajectory

Quasi-circular

Eccentricity

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Orbital trajectory

Quasi-circular

Eccentricity reduced

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Results
Irreducible Mass

\[ M_{\text{irr}}(t) / M_{\text{irr}}(t=0) \]

Initial increase of \( M_{\text{irr}} \) by \( 6 \cdot 10^{-6} \).

\[ |\delta M_{\text{irr}}| < 10^{-7} \] in the next 3.5 orbits.
Irreducible Mass

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Waveforms

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Eccentricity

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Comparison against post-Newtonian
What to compare?

- Orbital phase?
  - Is gauge dependent

- Waveform $h$
  - For NR, $h \sim \int \int \psi_4 + A + Bt$.
  - Integration constants are problematic.

- Newman-Penrose scalar $\psi_4$
  - Choose optimal orientation (direction normal to orbital plane)
  - Decompose and compare magnitude and phase: $\psi_4 = |\psi_4| \exp(-i\varphi)$.

- Procedure
  - Set $M_1 = M_2 = M_{\text{irr,NR}}$.
  - Two-parameter fit in $t_c$ and $\phi_0$ over the 2nd quarter of the evolution.
  - Examine how closely PN and NR track each other at other times.
Equation of motion (EoM):

\[ a_i^1 = \frac{Gm_2 n_{12}^i}{r_{12}^2} + \frac{1}{c^2} (\ldots) + \ldots + \frac{1}{c^7} (\ldots) + \mathcal{O}\left(\frac{1}{c^8}\right) \]

Radiation reaction enters at \(1/c^{5/2}\); thus, corrections to radiation reaction only known to \(\mathcal{O}(1/c^2)\) beyond leading order.
Digging through post-Newtonian

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- EoM imply energy

\[
E = -\frac{\mu c^2 x}{2} \left\{ 1 + (\ldots) x + \ldots + (\ldots) x^3 + (\ldots) x^{7/2} \right\} + \mathcal{O}\left(\frac{1}{c^8}\right)
\]

where \(x = (Gm\omega/c^3)^{2/3} = \mathcal{O}(1/c^2)\)

- For circular orbits, multipole expansions give energy flux

\[
\mathcal{F} = \frac{32 c^5}{5G} \nu^2 x^5 \left\{ 1 + (\ldots) x + \ldots + (\ldots) x^{7/2} \right\} + \mathcal{O}\left(\frac{1}{c^8}\right)
\]
**Digging through post-Newtonian**

**Assume** energy balance \( E/dt = -F \)

\[
\Rightarrow \quad \frac{dx}{dt} = -\frac{F}{dE/dx} = \frac{32c^3\nu}{5Gm}x^4 \left\{ 1 + (\ldots)x + \ldots + (\ldots)x^{7/2} \right\}
\]

**Integration yields orbital phase**

\[
\phi = \frac{1}{4} \left( \frac{5Gm}{c^3\nu} \right)^{1/4} (t_c - t)^{-1/4} \left\{ 1 + \ldots + (\ldots)(t_c - t)^{-7/8} + \mathcal{O}(c^{-8}) \right\}
\]

**Alternative:** use \( F/(dE/dx) \) as is without series expansion.
Digging through post-Newtonian

- **Waveforms**

\[ h_{+/x} = \frac{2G\mu x}{c^2 R} \left\{ H_{+/x}^{(0)}(\nu, \psi) + \ldots + x^{5/2} H_{+/x}(\nu, \psi)^{(5/2)} + \mathcal{O}(c^{-6}) \right\} \]

with \( \psi = \phi - \frac{2GM_{\text{ADM}}\omega}{c^3} \ln \frac{\omega}{\omega_0} \)

- **Newman-Penrose scalar**

\[ \psi_4 = \ddot{h}_+ - i\dot{h}_x = A_{\text{PN}} \exp(i\varphi_{\text{PN}}) \]

- For “n-th order comparison”, use \( \phi \) up to order \( n \), and use \( H_{+/x} \) up to order \( \min(n, 5/2) \).
Comparison of Phase

![Graph showing comparison of phase](image)

- **Lev2**
- **Lev3**
- **Lev4**
- **Lev5**
- **Lev6**
- **2PN**
- **2.5PN**
- **3PN**
- **3.5PN**
- **Lev5 R=200**
- **Lev 5 R=300**
- **Lev5 R=400**

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Eccentricity

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PN vs NR – Phase

Error budget in simulation

- Spatial truncation error $\sim 0.02\text{rad}$
**Spatial truncation error** $\sim 0.02\text{rad}$

**Finite extraction radius** $\sim 0.2\text{rad}$

(Extrapolation $R_{\text{extr}} \to \infty$ underway)
PN vs NR – Phase

Error budget in simulation

- Spatial truncation error $\sim 0.02\text{rad}$
- Finite extraction radius $\sim 0.2\text{rad}$
  (Extrapolation $R_{\text{extr}} \to \infty$ underway)
- Outer boundary $\sim 0.1\text{rad}$
2PN errors significant throughout run
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- Good convergence of PN toward NR for $t \lesssim 2500M$
  - Verification of NR
PN vs NR – Phase

Differences PN-NR

• 2PN errors significant throughout run
• Good convergence of PN toward NR for $t \lesssim 2500M$
  ➤ Verification of NR
• half-PN orders diverge a few 100M before coalescence
• 3PN–NR $\lesssim 0.5\text{rad}$
Pn vs NR – Amplitude

Error budget in simulation

- Spatial truncation error $\sim 0.1\%$
- Extraction radius $\sim 5\%$
- Outer boundary $\sim 1\%$
Pn vs NR – Amplitude

Differences PN - NR

Normalized $\Delta |R_{\Psi_4}|$

-0.08 to 0.08

0 to 4000 t/M

R$_{bdry} = 470$ vs R$_{bdry} = 210$

R = 300 vs R = 400

R = 200 vs R = 400

No significant differences for $t \lesssim 3000 M$

$\Delta PN - NR \lesssim 10\%$ throughout run

other PN-orders abysmal at late times

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No significant differences for $t \lesssim 3000M$
No significant differences for $t \lesssim 3000M$

- 3PN–NR $\lesssim 10\%$ throughout run
- other PN-orders abysmal at late times
Making contact with data-analysis
Overlap with detection templates

- In collaboration with D. Brown
- Template too short, causes loss in overlap (3.5PN turnover?)
Masses in simulation: 10.07Msun. Recovered 10.05Msun.
Summary

- Numerical relativity can finally do science
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- Accuracy sufficient to investigate breakdown of PN (With extrapolation $R_{\text{extr}} \to \infty$ even for 3PN.)
- For equal-mass non-spinning BHs, and with the caveat of large errors due to $R_{\text{extr}}$:
  - Agreement between NR and PN at early times
  - Taylor-3PN agrees very well throughout simulation
    - $\delta \phi \lesssim 0.5\text{rad}$
    - $\delta A/A \lesssim 0.1$
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