Hyperbolicity of force-free electrodynamics

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Introduction

We are interested in a magnetized fluid,

$$\rho_{matter}\frac{D\vec{v}}{Dt} = -\nabla P + \rho\vec{E} + \vec{\jmath}\times\vec{B},$$

when the magnetic energy density dominates $\frac{B^2}{8\pi} \gg \rho_{matter}, P$

$$\boxed{\Rightarrow \text{force-free condition} \qquad \rho\vec{E} + \vec{\jmath}\times\vec{B} = 0}$$

Using Maxwell's equations,

$$\partial_t\vec{E} = \nabla\times\vec{B} - \vec{\jmath}, \qquad \nabla\cdot\vec{E} = \rho,$$

$$\partial_t\vec{B} = -\nabla\times\vec{E}, \qquad \nabla\cdot\vec{B} = 0,$$

one can solve for $\vec{\jmath}$:

$$\boxed{\vec{\jmath} = \frac{\vec{B}}{B^2}\left[\vec{B}\cdot(\nabla\times\vec{B}) - \vec{E}\cdot(\nabla\times\vec{E})\right] + \frac{\vec{E}\times\vec{B}}{B^2}\nabla\cdot\vec{E}}$$

substitute into Maxwell's equations $\Rightarrow$ matter degrees of freedom eliminated; closed set of evolutionary equations for $\vec{B}$ and $\vec{E}$ only.

FFDE in the E-B-formulation (current formulation)

$$\partial_t \vec{B} = -\nabla \times \vec{E}$$

$$\partial_t \vec{E} = \nabla \times \vec{B} - \frac{\vec{B}}{B^2}\left[\vec{B}\cdot(\nabla\times\vec{B}) - \vec{E}\cdot(\nabla\times\vec{E})\right] + \frac{\vec{E}\times\vec{B}}{B^2}\nabla\cdot\vec{E}$$

$$\nabla\cdot\vec{B} = 0$$

$$\vec{E}\cdot\vec{B} = 0$$

This talk examines the equations above:

1. Show that they are **not** hyperbolic as written above
2. Augment eqns. to restore hyperbolicity
3. Present a pseudo-spectral evolution code

Hyperbolicity

Hyperbolicity $\Rightarrow$ well-posedness – existence and uniqueness of solution
– solution depends continuously on initial data
– growth bounded independent of initial data

$\Rightarrow$ Characteristic speeds – causal properties
– boundary conditions

$\Rightarrow$ Characteristic fields – boundary conditions

First order system $\boldsymbol{\partial_t u + A^i \partial_i u = F}$ (A^i and F may depend on u)

- strictly hyperbolic: $A^i n_i$ has all real and distinct eigenvalues
- stronlgy hyperbolic: $A^i n_i$ has all real eigenvalues, and a complete set of eigenvectors
- symmetric hyperbolic: $\exists$ symmetric positive definite S s.t. SA^i is symmetric for $i = 1, 2, 3$

... at *all* points x
... for *all* directions $\hat{n}$
... for *all* allowed field-values u

FFDE in E-B formulation

$$u = \begin{pmatrix} \vec{B} \\ \vec{E} \end{pmatrix} \qquad\qquad A^i n_i = \begin{pmatrix} 0 & -\epsilon^{ij}{}_k n_i \\ \epsilon^{ij}{}_k n_i - \frac{(\hat{n}\times\vec{B})_k B^j}{B^2} & \frac{(\hat{n}\times\vec{E})_k B^j}{B^2} + \frac{(\vec{E}\times\vec{B})^j n_k}{B^2} \end{pmatrix}$$

- Consider $\vec{E} = 0$ and $\hat{n}$ such that $\hat{n}\cdot\vec{B} = 0$
- Eigenvalues $+1, -1, 0, 0, 0, 0$
- Zero-speed eigenvector equation:

$$A^i n_i \begin{pmatrix} \vec{B}' \\ \vec{E}' \end{pmatrix} = \begin{pmatrix} \hat{n}\times\vec{E}' \\ \hat{n}\times\vec{B}' - \frac{\vec{B}}{B}\cdot(\hat{n}\times\vec{B}')\frac{\vec{B}}{B} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{array}{l} \leftarrow \vec{E}' = C_1\hat{n} \\ \leftarrow \vec{B}' = C_2\hat{n} + C_3\hat{n}\times\vec{B} \end{array}$$

— Only *three*-dimensional eigenspace for *four* eigenvalues —

- $\Rightarrow$ **Not hyperbolic**

N.B. No complete set of eigenvectors whenever $(\hat{n}\cdot\vec{B})^2 = (\hat{n}\times\vec{E})^2$, otherwise ok.

Augmented E-B formulation: constraint addition

Add terms proportional to $\nabla \cdot \vec{B}$ and $\vec{E} \cdot \vec{B}$ to the evolution equations:

$$\partial_t \vec{B} = -\nabla \times \vec{E} \;\; -\gamma_1 \frac{\vec{E} \times \vec{B}}{B^2} \nabla \cdot \vec{B} - \gamma_2 \frac{\vec{B}}{B^2} \times \nabla (\vec{E} \cdot \vec{B}),$$

$$\partial_t \vec{E} = \nabla \times \vec{B} - \vec{\jmath} \;\; -\gamma_3 \frac{\vec{E}}{B^2} \times \nabla (\vec{E} \cdot \vec{B}),$$

$$\vec{\jmath} = \frac{\vec{B}}{B^2} \left[\vec{B} \cdot (\nabla \times \vec{B}) - \vec{E} \cdot (\nabla \times \vec{E}) \right] - \frac{\vec{E} \times \vec{B}}{B^2} \nabla \cdot \vec{E}$$

- *Physical* solutions unchanged
- Different behavior of constraint-violating solutions
- Derivatives in new terms change A^i-matrices and thus influence hyperbolicity

N.B. Form of new terms restricted by parity

Fixing parameters

- Revisit counter-example: $\vec{E}=0$, $\hat{n}\cdot\vec{B}=0$:

- Still four zero eigenvalues. Eigenvector equation:

$$A^i n_i \begin{pmatrix}\vec{B}'\\ \vec{E}'\end{pmatrix} = \begin{pmatrix}\hat{n}\times\left[\vec{E}'-\gamma_1\left(\frac{\vec{B}}{B}\cdot\vec{E}'\right)\frac{\vec{B}}{B}\right]\\ \hat{n}\times\vec{B}'-\frac{\vec{B}}{B}\cdot(\hat{n}\times\vec{B}')\,\frac{\vec{B}}{B}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

- If and only if $\gamma_1=1$ $\Rightarrow$ square bracket is projection $\perp\ \vec{B}$
  $\Rightarrow$ $\vec{E}'=C_1\hat{n}+C_2\vec{B}$
  $\Rightarrow$ fourth eigenvector exists
  $\Rightarrow$ complete set

- More general case $(\hat{n}\cdot\vec{B})^2=(\hat{n}\times\vec{E})^2$ requires $\gamma_1=\gamma_2=1$.

Symmetric hyperbolicity of augmented system

For $\gamma_1 = \gamma_2 = 1, \gamma_3 = 0$, the augmented system is symmetric hyperbolic:

$$S = \frac{1}{B^2}\begin{pmatrix} \Delta B^2\delta_{ij} + zE_iE_j & -\Delta B_iE_j + zE_iB_j \\ -\Delta E_iB_j + zB_iE_j & \Delta B^2\delta_{ij} - (z-2\Delta)B_iB_j \end{pmatrix}$$

$\Delta = 1 - E^2/B^2$; $z > 1$ arbitrary

Symmetrizer only valid for $\vec{E}\cdot\vec{B} = 0$

(generalization in progress...)

Eigenvalues (Characteristic speeds)	Eigenvectors (Characteristic fields)	
$v_{1,2} = \pm 1$	$\vec{B}' = P\vec{B} \mp \hat{n}\times\vec{E}$ $\vec{E}' = -P\vec{E} \mp \hat{n}\times\vec{B}$	$\vec{B} \perp \vec{E} \perp \vec{n}$ don't carry current
$v_{3,4} = \frac{\hat{n}\cdot(\vec{E}\times\vec{B})}{B^2} \pm \frac{\lvert\hat{n}\vec{B}\rvert}{B^2}\sqrt{B^2 - E^2}$	$\vec{B}' = -P\vec{E} - v_{(3,4)}\hat{n}\times\vec{B}$ $\vec{E}' = -P\vec{B} + v_{(3,4)}\hat{n}\times\vec{E}$	Alfven-like modes carry current
$v_{5,6} = \frac{\hat{n}\cdot(\vec{E}\times\vec{B})}{B^2}$	nasty	unphysical, $\vec{E}\cdot\vec{B} \neq 0$

- Always $\lvert v_\alpha \rvert \leq 1$ (inside light-cone $\Rightarrow$ inside event horizon all modes *ingoing*)
- Modes 1-4 are identical to those found by Komissarov, MNRAS (2002)

Numerical simulations of augmented system

Pseudospectral method (from BH evolution code)

- Expand solution in basis functions

$$u(x,t) = \sum_{k=0}^{N-1} \tilde{u}_k(t)\Phi_k(x)$$

(Φ_k Fourier-series, Chebyshev polynomials, spherical harmonics)

- Rewrite PDEs as set of ODEs for the spectral coefficients,

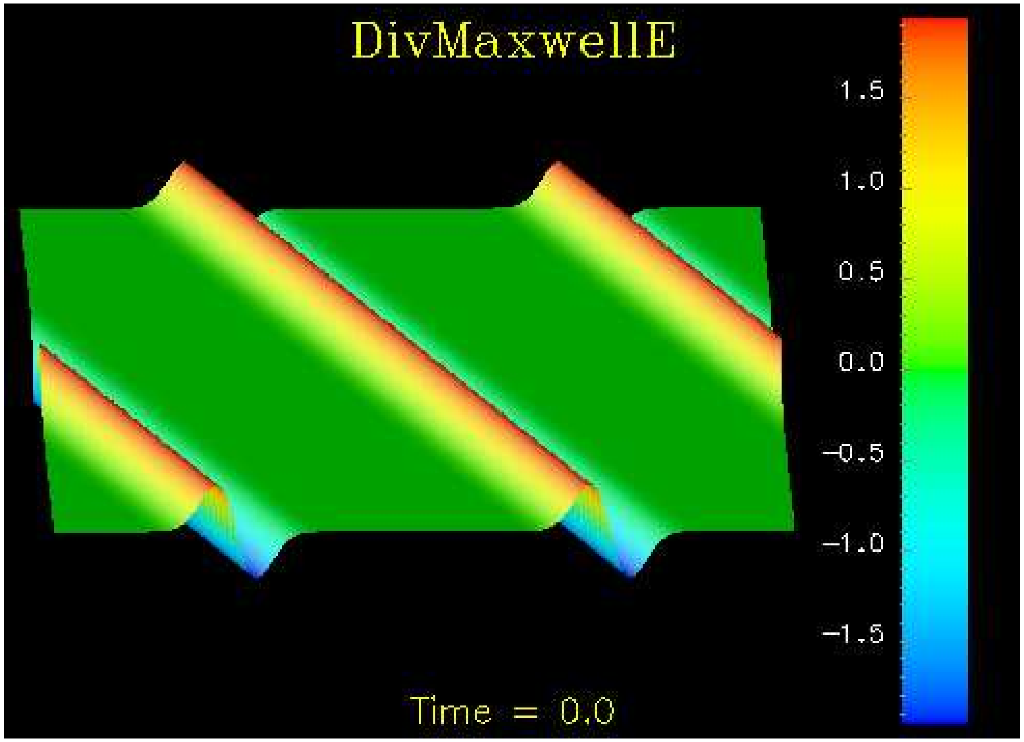
$$\partial_t \tilde{u}_k(t) = \mathcal{R}\left[\{\tilde{u}_k\}\right]$$

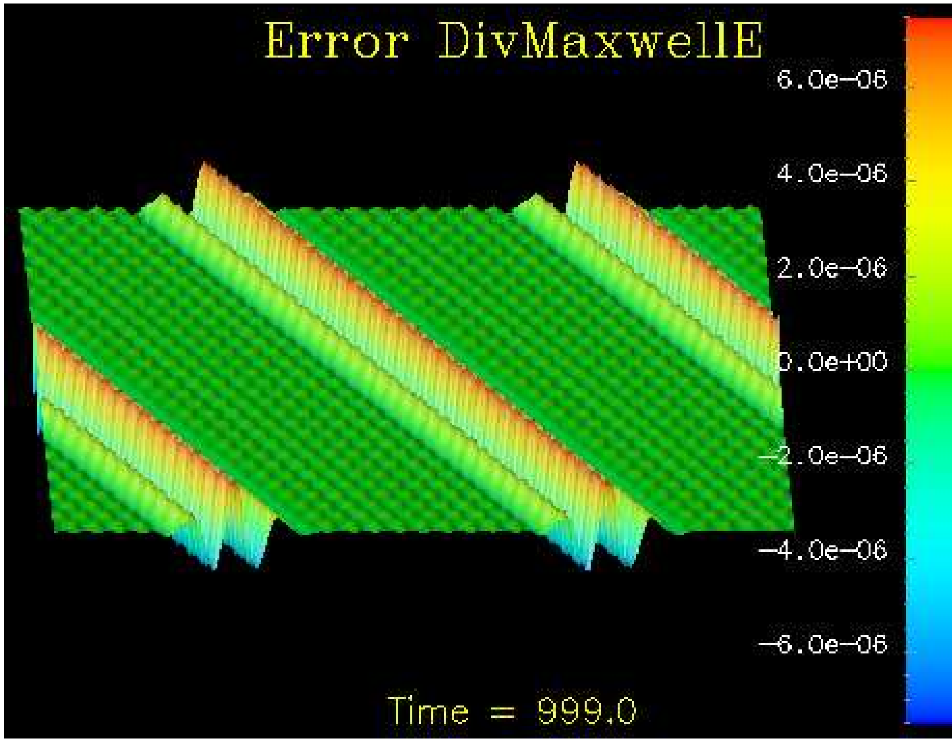
- Solve with ODE solver (Runge-Kutta 4)

So far only tests, no physics

Travelling eigenmode in 2-D

- Background field: $\vec{B}_0 = 6/5\hat{x} - 1/5\hat{y}$, $\vec{E}_0 = 1/10\hat{x} - 3/5\hat{y}$
- Gaussian pulse profile, Alfven mode

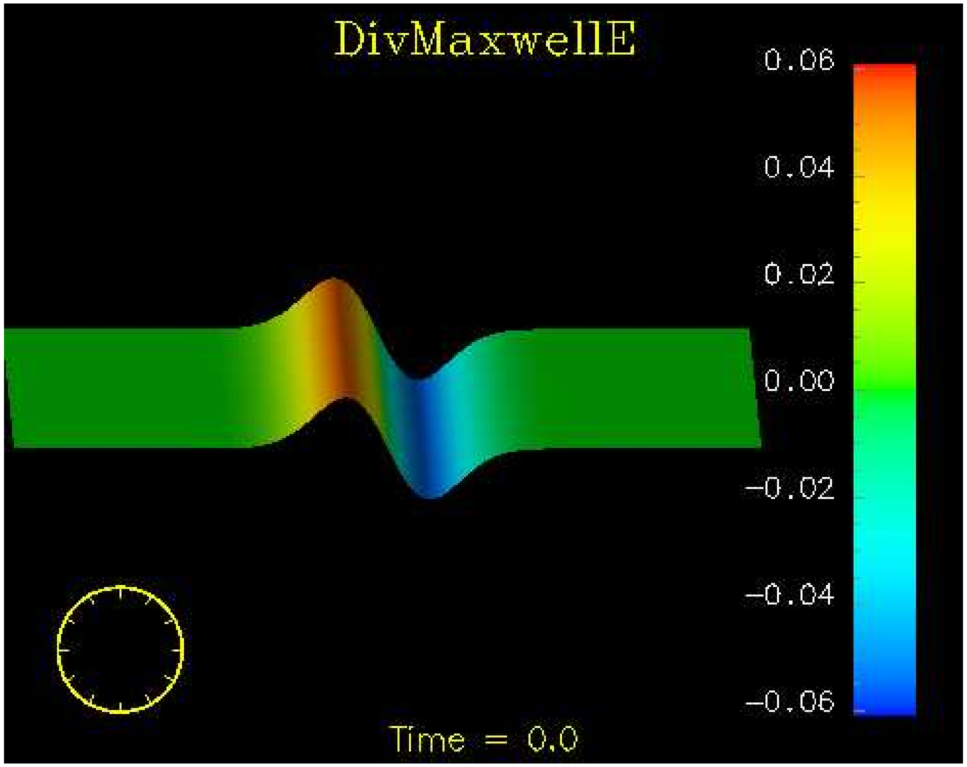


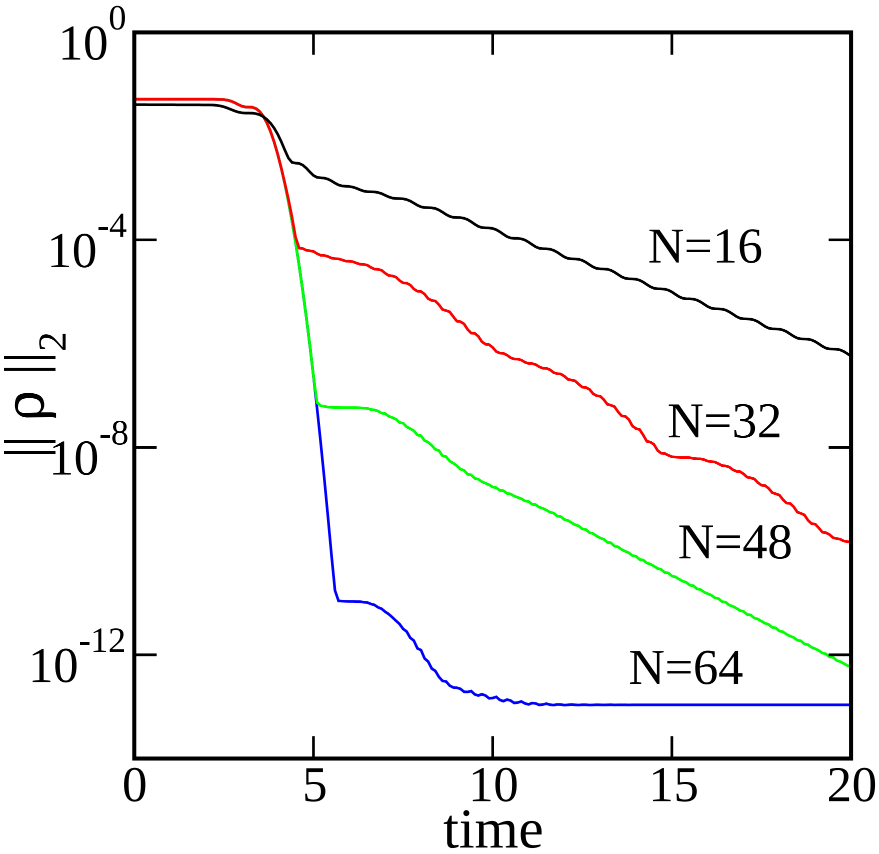


- Oblique wavefront, non-linear terms are important
- **Excellent accuracy, essentially no dispersion**

Open Boundaries

At boundary: – decompose into characteristic fields
– Apply BCs only to *incoming* fields





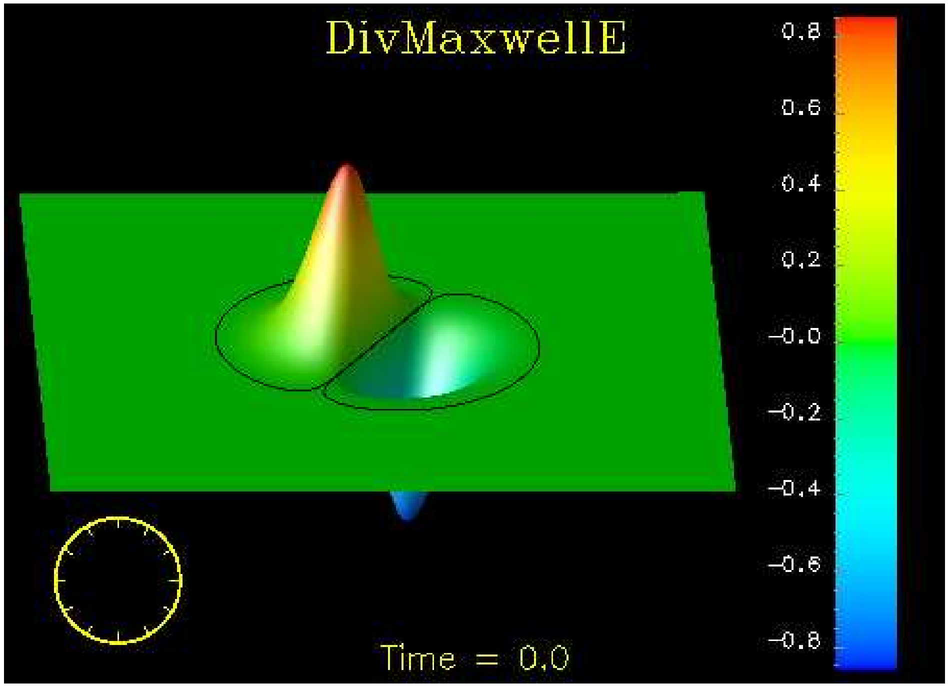
Black hole excision:
At excision boundary inside BH, **all** causal modes are outgoing ⇒ **no BC required at all**

Oblique incidence on boundaries: 2-D, all boundaries open

Background field: $\vec{B}_0 = \begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \\ 0 \end{pmatrix}$, $\vec{E}_0 = 0$

$\delta\vec{B} = f(\vec{x}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\delta\vec{E} = f(\vec{x}) \begin{pmatrix} \sin 60^\circ \\ -\cos 60^\circ \\ 0 \end{pmatrix}$

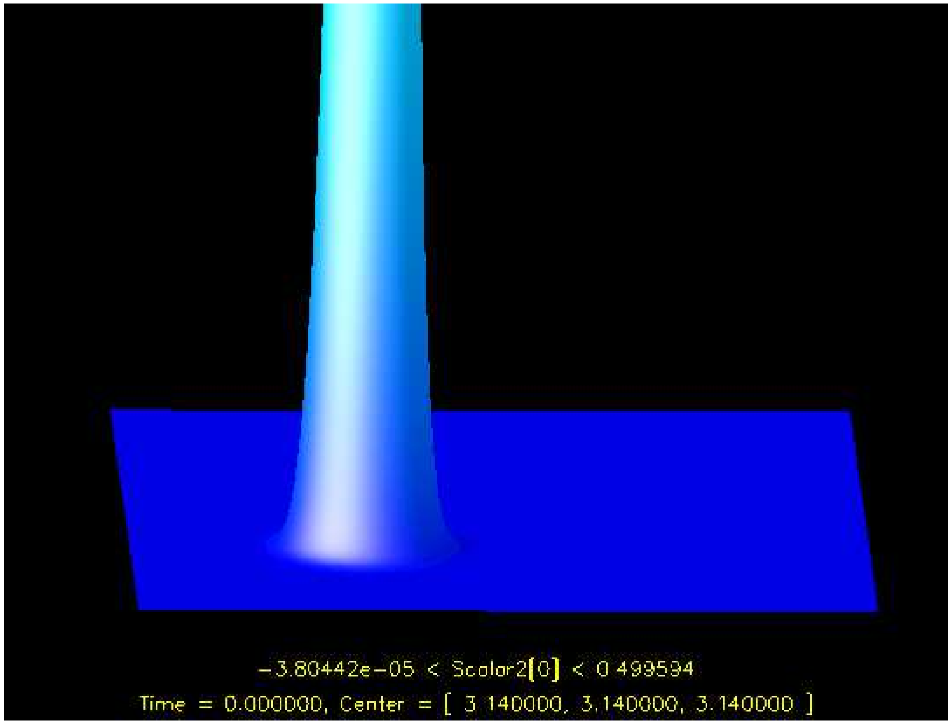
Alfven-pulse



reflections $< 10^{-6}$

$\delta\vec{B} = 0$, $\delta\vec{E} = f(\vec{x}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

fast pulse



reflections of a few per cent

Summary

- Hyperbolicity depends on **how** the equations are written
  $\longrightarrow$ *Analyze the same set of equations one implements numerically**

- This form of FFDE is symmetric hyperbolic:

$$\partial_t \vec{B} = -\nabla \times \vec{E} \; - \frac{\vec{E}\times\vec{B}}{B^2}\nabla\cdot\vec{B} - \frac{\vec{B}}{B^2}\times\nabla(\vec{E}\cdot\vec{B}),$$

$$\partial_t \vec{E} = \nabla \times \vec{B} - \frac{\vec{B}}{B^2}\left[\vec{B}\cdot(\nabla\times\vec{B}) - \vec{E}\cdot(\nabla\times\vec{E})\right] + \frac{\vec{E}\times\vec{B}}{B^2}\nabla\cdot\vec{E}$$

- Pseudo-spectral evolution code performs very well so far:
  – high accuracy $(10^{-4}\ldots 10^{-10})$
  – low dispersion
  – low reflectivity at boundaries