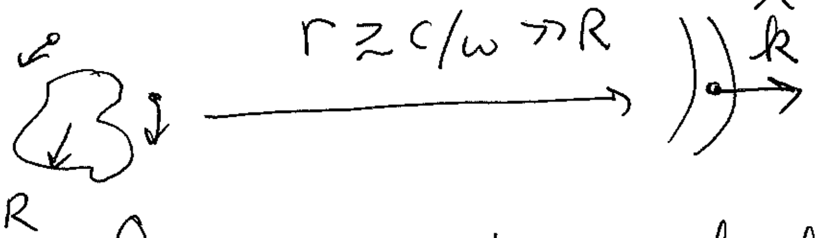
EMISSION OF GRAVITATIONAL WAVES

BY { SLOWLY MOVING $v \ll c$ / WEAKLY GRAVITATING $\phi \ll c^2$ } MASSES

$T^M_{\mu\nu}$ MATTER



Q_{ij}

$\omega \sim v/R$

$$t_{\mu\nu} = \frac{k_\mu k_\nu c^4}{32\pi G}\left(h^{\alpha\beta} h^*_{\alpha\beta} - \frac{1}{2}|h^\alpha_\alpha|^2\right)$$

Generation of waves:

We found $R^{(1)}_{\mu\nu} = \frac{1}{2}\Box h_{\mu\nu}$ $\quad \partial_\mu \bar{h}^{\mu\nu} = 0$

So
$$\left(R_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}R\right)^{(1)} = \tfrac{1}{2}\Box\left(h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h\right)$$
$$= \boxed{\tfrac{1}{2}\Box \bar{h}_{\mu\nu} = -\frac{8\pi G}{c^4} T^M_{\mu\nu}}$$

Solution at $r \gg R$ RECALL E+M $\Box A^\mu = -\frac{4\pi}{c} J^\mu$

$$A^\mu(\vec{x}, t) = \frac{1}{cr}\int d^3r'\, J^\mu\left(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c}\right)$$

Similarly
$$\boxed{\bar{h}^{\mu\nu}(\vec{x}, t) = \frac{16\pi G}{c^4 r} \times \frac{1}{4\pi}\int d^3r'\, T^{\mu\nu}\left(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c}\right)}$$

Recall procedure with $\mathcal{E}+\mathcal{M}$

① Express $\vec{A}(\vec{x},t)$ in terms of

$$\frac{\partial \vec{d}}{\partial t}(t-r/c) \qquad \vec{d} \equiv \sum e_i \vec{x}_i$$

dipole moment

$$\boxed{\vec{A}(\vec{x},t) = \frac{1}{cr}\frac{\partial \vec{d}}{\partial t}(t-r/c)}$$

[TO LEADING ORDER $\frac{c}{\omega r}$]

② Obtain components of $\vec{E}$, $\vec{B}$ $\perp$

$$\hat{k} = \hat{n} \equiv \vec{r}/r$$

$$\vec{B}_{wave} = i\vec{k}\times\vec{A} = i\frac{\omega}{c}\,\hat{n}\times\vec{A}$$

$$\vec{E}_{wave} = -\hat{n}\times\vec{B}_{wave} \quad \left[\frac{\partial \vec{E}}{\partial t} = c\vec{\nabla}\times\vec{B}\right]$$

③ Integrate energy flux

$$\frac{1}{r^2}\frac{dP}{d\Omega} = \hat{n}\cdot\vec{S} = \hat{n}\cdot\left(\frac{\vec{E}_{wave}\times\vec{B}_{wave}}{4\pi}\right)c$$

over sphere $\quad P = r^2\int d\Omega\,\hat{n}\cdot\left(\frac{\vec{E}_{wave}\times\vec{B}_{wave}}{4\pi}\right)c$

Our plan:

① Focus on spatial components h^{ij}, express these in terms of $\partial^2 Q^{ij}/\partial t^2$.

② Project h^{ij} onto transverse directions

③ Integrate energy flux

$$\frac{1}{r^2}\frac{dP_{GW}}{d\Omega} = n_i t^{0i} c$$

over sphere,

$$\boxed{P_{GW} = r^2 \int d\Omega \; n_i t^{0i} c}$$

First evaluate

$$\int T^{ij} d^3r' = -\int (x^i)' \frac{\partial}{\partial (x^k)'} T^{kj} d^3r'$$

$$= +\frac{1}{2}\int (x^i)'(x^j)' \frac{\partial^2 T^{kl}}{\partial (x^k)'(x^l)'} d^3r'$$

$$= \frac{1}{c^2}\frac{\partial^2 T^{00}}{\partial t^2}$$

$$\frac{\partial}{\partial x^j}\left(\frac{1}{c}\frac{\partial}{\partial t}T^{0j} + \frac{\partial}{\partial x^i}T^{ij}\right) = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial}{\partial t}T^{00} + \frac{\partial}{\partial x^j}T^{j0}\right) = 0$$

$$\Rightarrow \quad \frac{1}{c^2}\frac{\partial^2}{\partial t^2}T^{00} = \frac{\partial^2}{\partial x^i \partial x^j}T^{ij}$$

$$\int T^{ij}\, d^3r' = \frac{1}{2}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int (x^i)'(x^j)'\, T_{00}\, d^3r'$$

MASS QUADRUPOLE TENSOR:

$$Q^{ij} = 3\int \rho (x^i)'(x^j)'\, d^3r' = 3\int (T_{00}/c^2)(x^i)'(x^j)'\, d^3r'$$

So
$$\boxed{\bar{h}^{ij}(\vec{r}, t) = \frac{2G}{3c^4 r}\frac{\partial^2 Q^{ij}}{\partial t^2}\left(t - \frac{r}{c}\right)}$$

NOTE: $\bar{h}^{ij}$ HERE IS NOT TRACELESS OR TRANSVERSE – WE HAVE DROPPED h^{00}.

$$P = \int r^2 d\Omega\, (t^{0i} c)\, n_i \qquad k^i n_i = \omega/c$$

$$(t^{0i} c)\, n_i = \frac{\omega^2 c^3}{32\pi G}\left(\bar{h}^{\alpha\beta}\bar{h}^*_{\alpha\beta} - \frac{1}{2}|\bar{h}^\alpha_\alpha|^2\right)$$

PROJECTION ONTO TRANSVERSE COORD:

$$\boxed{\bar{h}_{ik}^{(T)} = \bar{h}^{jl}(\eta_{ij} - n_i n_j)(\eta_{kl} - n_k n_l)}$$

$$\eta_{ij} = \delta_{ij} \qquad n_i = r_i/r$$

TWO PIECES: INDEPENDENT OF θ, ϕ !

$$\int \bar{h}_{\textcircled{T}}^{ik} \bar{h}^*_{\textcircled{T}ik} \, d\Omega = \int \bar{h}^{ik} \bar{h}^{jl*} (\eta_{ij} - n_i n_j)(\eta_{kl} - n_k n_l)$$

$$-\frac{1}{2} \int |\bar{h}_{\textcircled{T}}{}^{\alpha}{}_{\alpha}|^2 \, d\Omega = -\frac{1}{2} \int |\bar{h}^{ij} (\eta_{ij} - n_i n_j)|^2 \, d\Omega$$

WE WILL NEED: $\vec{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$

$$\int d\Omega \, \eta_{ij} \eta_{kl} = \eta_{ij} \eta_{kl} \int d\Omega = 4\pi \eta_{ij} \eta_{kl}$$

$$\int d\Omega \, n_i n_j = 4\pi \left(\frac{1}{3} \delta_{ij}\right)$$

$$\int d\Omega \, n_i n_j n_k n_l = \frac{4\pi}{15} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

E.g. $0 = \int n_x n_y \, d\Omega = \int n_x n_z \, d\Omega$

BUT $\int n_x^2 \, d\Omega = \int n_y^2 \, d\Omega = \int n_z^2 \, d\Omega = \frac{1}{3} \times 4\pi$

So $$\boxed{\int \bar{h}_{\textcircled{T}}^{ik} \bar{h}^*_{\textcircled{T}ik} \, d\Omega = 4\pi \left[\frac{1}{15} |\bar{h}^i{}_i|^2 + \frac{7}{15} \bar{h}^{ik} \bar{h}^*_{ik} \right]}$$

$$\left\{ = 4\pi \left[\bar{h}^{ik} \bar{h}^*_{ik} \left(1 - \frac{2}{3}\right) + \frac{1}{15} \left(2\bar{h}^{ik}\bar{h}^*_{ik} + \bar{h}^i{}_i \bar{h}^{i*}_i\right) \right] \right\}$$

$$\boxed{-\frac{1}{2} \int |\bar{h}_{\textcircled{T}}{}^i{}_i|^2 \, d\Omega = -4\pi \left[\frac{1}{5} |\bar{h}^i{}_i|^2 + \frac{1}{15} \bar{h}^{ik} \bar{h}^*_{ik} \right]}$$

$$\left\{ = -\frac{1}{2} 4\pi \left[\bar{h}^i{}_i \bar{h}^{*i}{}_i \left(1 - \frac{2}{3}\right) + \frac{1}{15} \left(\bar{h}^i{}_i \bar{h}^{i*}_i + 2\bar{h}^{ik} \bar{h}^*_{ik}\right) \right] \right\}$$

Combining:

$$\int \left(\bar{h}^{ik}_{T} \bar{h}^{*}_{T\,ik} - \frac{1}{2} |\bar{h}^{i}_{T\,i}|^2 \right) d\Omega$$

$$= \frac{2}{5} \cdot 4\pi \left(\bar{h}^{ik} \bar{h}^{*}_{ik} - \frac{1}{3} |\bar{h}^{i}_{i}|^2 \right)$$

$$P_{GW} = \frac{\omega^2 c^3 r^2}{32\pi G} \int (\cdots)\, d\Omega \qquad (\omega \rightarrow \partial/\partial t)$$

$$= \frac{c^3 r^2}{20 G} \left(\frac{\partial \bar{h}^{ik}}{\partial t} \frac{\partial \bar{h}^{*}_{ik}}{\partial t} - \frac{1}{3} \left| \frac{\partial \bar{h}^{i}_{i}}{\partial t} \right|^2 \right)$$

Substitute $\quad \bar{h}^{ij} = \dfrac{2G}{3c^4 r} \dfrac{\partial^2 Q^{ij}}{\partial t^2}(t - r/c)$

gives

$$\boxed{P_{GW} = \frac{G}{45 c^5} \left(\frac{\partial^3 Q^{ij}}{\partial t^3} \frac{\partial^3 Q^{*}_{ij}}{\partial t^3} - \frac{1}{3} \left| \frac{\partial^3 Q^{i}_{i}}{\partial t^3} \right|^2 \right)}$$

MULTIPOLE ELECTROMAGNETIC VS GRAVITATIONAL RADIATION

POWER $\quad P = P(\vec{d}, \omega, c)$ $\quad$ EM

$\quad\quad\quad P = P(Q_{ij}, \omega, G, c)$ $\quad$ GR

$$\vec{d} = \sum_i e_i \vec{r}_i \qquad Q_{ij} = 3\sum_i m_i r_i r_j$$

$$[P] = \frac{\text{ENERGY}}{\text{TIME}}$$

$$[d^2] = e^2 \cdot (\text{LENGTH})^2 = \text{ENERGY} \cdot (\text{LENGTH})^3$$

$$[\omega] = T^{-1} \qquad [c] = L/T$$

$$[P/d^2] = \frac{1}{L^3 \cdot T} \qquad [\omega^4/c^3] = T^{-4+3}/L^3$$

$$[P] = [d^2\omega^4/c^3]$$

$$\boxed{P = \frac{2}{3}(\ddot{d})^2/c^3}$$

ELECTRIC DIPOLE RADIATION

FOR ℓ-pole, $\ell > 1$: $\quad P \sim d_\ell^2\, \omega_\ell^{2(1+\ell)} / c^{1+2\ell}$

$$[GM/L] = [v^2] = L^2/T^2$$

$$[G] = L^3/MT^2$$

$$[Q] = ML^2$$

DIMENSIONLESS QUANTITY:

$$\left[\frac{GQ\omega^3}{c^5}\right] = \frac{L^3}{MT^2} \cdot ML^2 \cdot T^{-3} \left(\frac{L}{T}\right)^{-5}$$

$$= 1$$

POWER: $[P] = \dfrac{ML^2/T^2}{T} = ML^2/T^3$

$$[P] = [c^5/G] = L^5/T^5 \cdot MT^2/L^3$$

$$P = \frac{c^5}{G} \times \underbrace{f\left(\frac{GQ\omega^3}{c^5}\right)}_{\text{QUADRATIC}}$$