DERIVATION OF ENERGY FLUX IN A GRAVITATIONAL WAVE

$$t_{\mu\nu} = \frac{c^4}{8\pi G} R^{(2)}_{\mu\nu} \qquad [R \to 0 \text{ as we show}]$$

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} \sum_{\beta} g^{\alpha\beta}\left(\frac{\partial g_{\nu\beta}}{\partial x^{\mu}} + \frac{\partial g_{\mu\beta}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right)$$

(CRISTOFFEL SYMBOL)

$$R_{\mu\nu} = \sum_{\alpha} \frac{\partial}{\partial x^{\alpha}} \Gamma^{\alpha}_{\mu\nu} - \frac{\partial}{\partial x^{\mu}} \sum_{\alpha} \Gamma^{\alpha}_{\alpha\nu} + \sum_{\alpha,\gamma} \left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\gamma}_{\alpha\gamma} - \Gamma^{\alpha}_{\gamma\nu}\Gamma^{\gamma}_{\alpha\mu}\right)$$

(RICCI TENSOR)

NOW DROP SUM ON REPEATED INDEX...

TO FIRST ORDER IN $h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik_\alpha x^\alpha} + \varepsilon^*_{\mu\nu} e^{-ik_\alpha x^\alpha}$

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\left(ik_\mu \varepsilon_{\nu\beta} + ik_\nu \varepsilon_{\mu\beta} - ik_\beta \varepsilon_{\mu\nu}\right)\eta^{\alpha\beta} e^{ik_\rho x^\rho} + (*)$$

$$\overline{\Gamma^{\alpha}_{\mu\nu}\Gamma^{\gamma}_{\alpha\gamma} - \Gamma^{\alpha}_{\gamma\nu}\Gamma^{\gamma}_{\alpha\mu}} =$$

AVERAGE OVER TIME

$$\frac{1}{2}(k_\mu\varepsilon_{\nu\beta} + k_\nu\varepsilon_{\mu\beta} - k_\beta\varepsilon_{\mu\nu})\eta^{\alpha\beta}\frac{1}{2}(k_\alpha\varepsilon^*_{\gamma\delta} + k_\gamma\varepsilon^*_{\alpha\delta} - k_\delta\varepsilon^*_{\alpha\gamma})\eta^{\gamma\delta} \times 2$$

$$-\frac{1}{2}(k_\gamma\varepsilon_{\nu\beta} + k_\nu\varepsilon_{\gamma\beta} - k_\beta\varepsilon_{\gamma\nu})\eta^{\alpha\beta}\frac{1}{2}(k_\alpha\varepsilon^*_{\mu\delta} + k_\mu\varepsilon^*_{\alpha\delta} - k_\delta\varepsilon^*_{\alpha\mu})\eta^{\gamma\delta} \times 2$$

$$= \frac{1}{2}(k_\mu\varepsilon_{\nu\beta} + k_\nu\varepsilon_{\mu\beta} - k_\beta\varepsilon_{\mu\nu})\left(k^\beta\varepsilon^\delta{}_\delta + k^\delta\varepsilon^\beta{}_\delta - k_\delta\varepsilon^{\beta\delta}\right)^*$$

$$-\frac{1}{2}(k_\gamma\varepsilon_{\nu\beta} + k_\nu\varepsilon_{\gamma\beta} - k_\beta\varepsilon_{\gamma\nu})\left(k^\beta\varepsilon_\mu{}^\gamma + k_\mu\varepsilon^{\beta\gamma} - k^\gamma\varepsilon^\beta{}_\mu\right)^*$$

Work in TT gauge. Then $k^\alpha\varepsilon_{\alpha\beta} = 0$, $\varepsilon^\delta{}_\delta = 0$

$k^2 = 0$ and $\boxed{\overline{\Gamma^{\alpha}_{\mu\nu}\Gamma^{\gamma}_{\alpha\gamma} - \Gamma^{\alpha}_{\gamma\nu}\Gamma^{\gamma}_{\alpha\mu}} = -\frac{1}{2}k_\mu k_\nu \varepsilon^*_{TT,\beta\gamma}\varepsilon^{\beta\gamma}_{TT}}$

To second order in $h_{\mu\nu}$,

$$\overline{\Gamma^{\alpha}_{\mu\nu}} = \frac{i}{2} \varepsilon^{\alpha\beta}\left(k_\mu \varepsilon^*_{\nu\beta} + k_\nu \varepsilon^*_{\mu\beta} - k_\beta \varepsilon^*_{\mu\nu}\right) + (*)$$

$$\frac{\partial}{\partial x^\alpha} \overline{\Gamma^{\alpha}_{\mu\nu}} = 0 \quad \text{since} \quad \varepsilon^{\alpha\beta} k_\alpha = 0 \text{ for TT}$$

(Terms in $\Gamma^{\alpha}_{\mu\nu} \propto e^{\pm 2 i k_\alpha x^\alpha}$ average to 0.)

$\boxed{\text{Also term } + k_\mu k_\nu \varepsilon^{\alpha\beta}_{TT} \varepsilon^*_{TT,\alpha\beta} \text{ from } -\frac{\partial}{\partial x^\mu} \Gamma^{\alpha}_{\alpha\nu}.}$

$$\text{So} \quad \overline{R_{\mu\nu}} = +\frac{1}{2} k_\mu k_\nu \varepsilon^{\alpha\beta}_{TT} \varepsilon^*_{TT,\alpha\beta} \qquad (R \propto k^2 = 0)$$

$$\boxed{\overline{t_{\mu\nu}} = \frac{c^4}{8\pi G} \times \overline{R_{\mu\nu}}^{(2)} = \frac{c^4 k_\mu k_\nu}{16\pi G} \varepsilon^{\alpha\beta}_{TT} \varepsilon^*_{TT,\alpha\beta}}$$

More generally $\varepsilon^{\alpha}_{\alpha} \neq 0$ and

THIS RESULT DOES <u>NOT</u> DEPEND ON HARMONIC GAUGE CONDITION

$$\boxed{\overline{t_{\mu\nu}} = \frac{c^4 k_\mu k_\nu}{16\pi G}\left[\varepsilon^{\alpha\beta} \varepsilon^*_{\alpha\beta} - \frac{1}{2}\left|\varepsilon^{\alpha}_{\alpha}\right|^2\right]}$$

Writing directly in terms of x^μ-derivatives,

$$t_{\mu\nu} = \frac{c^4}{32\pi G} \frac{\partial}{\partial x^\mu}\left(h^{\alpha\beta}_{TT}\right) \frac{\partial}{\partial x^\nu}\left(h^*_{TT,\alpha\beta}\right) \quad \Big\} \text{ TRANSVERSE TRACELESS GAUGE}$$

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\left[\frac{\partial h^{\alpha\beta}}{\partial x^\mu} \frac{\partial h^*_{\alpha\beta}}{\partial x^\nu} - \frac{1}{2} \frac{\partial h^{\alpha}_{\alpha}}{\partial x^\mu} \frac{\partial h^{\alpha *}_{\alpha}}{\partial x^\nu}\right]$$