The Twin Paradox on a Hypertorus: a case study on the value of reflecting on first principles

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Abstract

In this talk, I highlight what I see as a major benefit that teaching has on research: reminding the researcher of first principles. The exercise of packaging physics concepts into concise, digestible form forces you to think more about ideas often taken for granted, and can lead to interesting new directions when applied to more advanced topics. As an example, I'll discuss the Twin Paradox (the puzzle taught in undergraduate special relativity to illustrate the concept of inertial reference frames) and what it can tell us about the topology of the universe.

1 Intro

So, somewhat inspired, or perhaps emboldened by Dylan's talk last week, I decided to do something a bit experimental as well. I'm going to talk a bit about what I think is a valuable practice: going back to first principles to test our thinking when we encounter new ideas.

Last year, I was TAing the first-year physics lab course. As first year courses go, UofT's is fairly ambitious in scope, giving the students an introduction not just to mechanics and electromagnetism, but also to quantum mechanics and special relativity. So, I was explaining the twin paradox to my students. You'll remember from your first special relativity course, the basic premise: Alice hops in a rocket, zips of to Alpha Centauri at a speed v close to the speed of light, takes some pictures, talks to E.T., then comes back to Earth. Her twin Bob was chillin' on Earth the whole time, meaning to him, he's stationary and she's moving, so he'll say time ticks by slower for her and she ought to get back the younger twin.

But, relativity says that there are no preferential reference frames, so shouldn't Alice, stationary in her frame, see Bob as moving and then argue that he's gonna end up the younger one? Do they really disagree? Why can't we all just get along? Is this a true paradox? Well, no, because Alice has to turn around. She isn't in the same reference frame for her outbound journey as the return, so in that period of acceleration between two separate inertial frames, Bob will overtake her in age, and though time will tick by slower for him on her inbound journey, she'll still get back younger than him. Alice and Bob agree. Hurray!

I was TAing this course not long after meeting Glenn Stakrman at a conference in Madrid. Starkman, a Canadian and former CITAzen, is a professor at Case Western Reserve University and an expert on the CMB with a particular interest in possible non-trivial topologies of the universe. So having just met an interesting perosn and come across ideas I hadn't thought about much before, I found myself explaining the Twin pardox to a bunch of first years and thinking as I did so, ***"but what if Alice doesn't have to turn around?"***

I confess, I thought at first I'd sunk the whole field of topology, but of course others had thought of this same problem before me. Still, I think it's a valuable case of using intuition to challenge theories. And it's a fun exercise, so we'll see what the twin paradox tells us about topology and think about whether we think this raises issues.

I'll be talking about [arXiv:gr-qc/0101014].

2 The Twin Paradox

The "paradox" arises because each twin sees the other as moving, so your first impression might be that each twin thinks the other should have experienced less time passing while Alice was travelling. This is resolved by recognizing that Alice is in one reference frame for her outbound journey and another for the return. If **she travels a distance ℓ at constant speed v along the $\hat{\mathbf{x}}$ axis, then returns at $\mathbf{v} = -v\hat{\mathbf{x}}$**, she must accelerate for a period. If we **assume a constant acceleration** $\frac{dv}{dt} = -a$, then the time Bob measures is

$$\tau_B = \int_0^\ell \frac{dx}{v} + \int_v^{-v} \frac{dv}{-a} + \int_\ell^0 \frac{dx}{-v} = \frac{2\ell}{v} + \frac{2v}{a}$$

And for Alice, we can use the invariance of the spacetime interval ds, equating the space-like trajectory of her journey as seen by Bob to the time-like one she experiences and integrate to find her proper time

$$\tau_A = \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = \int \sqrt{dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2}}$$

so for a trajectory with $dy = dz = 0$, then plugging in $\frac{dx}{dt} = v$, we can express $d\tau_A = \sqrt{\frac{dx^2}{v^2} - \frac{dx^2}{c^2}} = \gamma^{-1} v^{-1} dx$ or $d\tau_A = \sqrt{dt^2 - \beta^2 dt^2} = \gamma^{-1} dt$

$$\tau_A = \frac{1}{v\gamma} \int_0^\ell dx + \int \frac{dt}{\gamma(t)} - \frac{1}{v\gamma} \int_\ell^0 dx$$

Now with $dt = -a^{-1} dv$

$$\begin{aligned}
\tau_A &= \frac{2\ell}{v\gamma} + 2 \int_0^v \sqrt{1 - \frac{v'^2}{c^2}} \frac{dv'}{a} \\
&= \frac{2\ell}{v\gamma} + \frac{v}{a\gamma} + \frac{c}{a} \arcsin\left(\frac{v}{c}\right) \\
&= \frac{\tau_B}{\gamma} - \frac{v}{a\gamma} + \frac{c}{a} \arcsin\left(\frac{v}{c}\right)
\end{aligned}$$

Note that $\arcsin\beta - \beta/\gamma \geq 0$ and $\gamma \geq 1$, so the first term is shorter than τ_B, but then we add a bit for the time it takes to accelerate. In the limit that a is large, the second term is small, so time is dilated as we expect. And in the limit $\lim_{v\to 0}\gamma = 1$ so $\lim_{v\to 0}\tau_A = \tau_B$.

So what does this actually mean? **To Alice, Bob is younger until she turns around, then as she's accelerating, he catches up and overtakes her. Alice will see Bob age more slowly again during her constant-v return, but she won't catch up, and he'll be older when she gets back**. To Bob, from the instant Alice leaves, she is always younger.

This is demonstrates that ***simultaneity of space-like separated events is not preserved by Lorentz transformations***. If Bob is watching a really long movie while Alice is away, he would say that he's halfway through it when Alice turns around, but Alice would say he's just a quarter of the way through it when she starts to accelerate, then he watches the middle half while she's turning around, and the last quarter on her return trip.

But then, what if Alice doesn't have to turn around?

3 Compact spaces

As we know, the laws of physics allow for non-trivial geometry of spacetime. As astrophysicists, we're particularly familiar with curvature of space, which is predicted by general relativity to arise around concentrations of energy. There is an entirely independent geometrical property that is also allowed for though, topology.

General Relativity allows for some pretty delinquent solutions. Spacetimes that solve the equations of GR are 4-dimensional manifolds endowed with a Lorentzian metric (*i.e.* one with signature $+ - --$), but that allows for non-compact 4-manifolds and some compact ones. **A compact manifold has no punctures or missing end points**, so $(-\infty,\infty)\cap\mathbb{R}$, $(0,1)\cap\mathbb{R}$ and $[0,1]\cap\mathbb{Q}$ are not compact, while $[0,1]\cap\mathbb{R}$ is compact.

Physics places some additional constraints. It's generally thought that **physically meaningful manifolds must be orientable**, meaning they can be endowed with a normal vector that is unchanged on closed paths. In an analogy to 2D, a ***Möbius strip*** is not orientable because, you drag a normal vector one turn around and it's pointing the other way. Physically, **time-orientability ensures the arrow of time**, it stops you from having an event that is both in the past and the future (absent this you are gonna have weird casualty issues). **Spatial orientability ensures we don't violate the strong CPT theorem**, which says the conjugation of charge, parity and time symmetries can't be broken. Again, on the Möbius strip, we have a parity violation.

With these constraints, there are still a range of orientable compact spaces that are allowed by the equations of GR. A compact solution doesn't mean the universe has a hard edge though, it means if you "go off one end" you'll find yourself back on the other side, like if you travel around the Earth's equator, you'll eventually wind up back where you started. This can be described by **tessellation of space by a fundamental volume called the covering space**. An example of a covering space for the plane is show in figure 1.

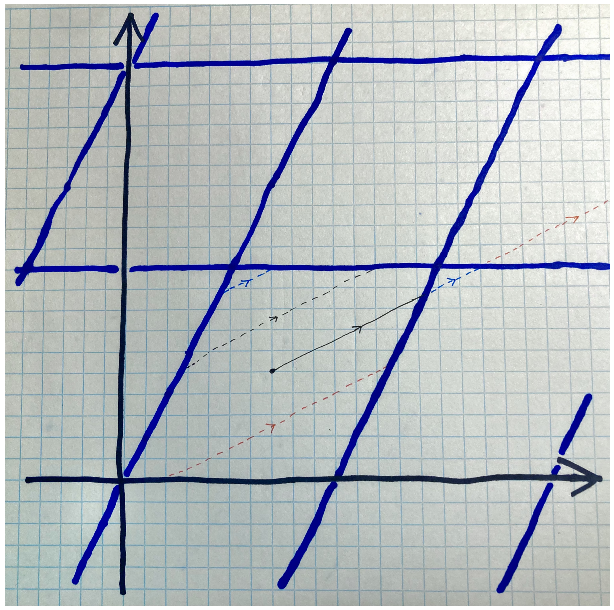

Figure 1: an example tessellation of the plane.

As I mentioned, this is all independent of curvature. The allowed tessellations of 3D space depend on the curvature:

- the 3-sphere $\mathbb{S}^3$ for closed universes $k < 0$
- Euclidean space $\mathbb{R}^3$ for flat universes $k = 0$
- hyperbolic 3-space $\mathbb{H}^3$ for open universes $k > 0$

I'll focus on Euclidean spaces because because I like to think of the cosmological constant as energy not intrinsic curvature (and also because I don't remember what the other covering spaces look like). There are six orientable covering spaces where the fundamental polyhedron is either a parallelepiped:

- E_1 - opposite faces by translations (*e.g.* the hypertorus)
- E_2 - opposite faces, one pair rotated by π
- E_3 - opposite faces, one pair rotated by $\pi/2$
- E_4 - opposite faces, all three pairs rotated by π

or a hexagonal prism:

- E_5 - opposite faces, hexagonal rotated by $2\pi/3$
- E_6 - opposite faces, hexagonal rotated by $\pi/3$

The peculiar thing here is that we can get ***periodic orbits***, where you can fly off in one direction and end up back where you started. Which brings us back to the Twin Paradox.

4 The Twin Paradox Revisited

If we think of figure 1 as a slice of the tessellation, we can draw a periodic orbit.

For the flat torus with fundamental axes $\mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ and where the parallelepiped has corresponding lengths L_x, L_y, L_z, it is useful mathematically to embed the (3+1)-dimensional spacetime in a $(4+1)$-dimensional Minkowski spacetime with the fourth spatial coordinate fixed at 1. Then the spacetime is invariant under translation T_x, T_y, T_z given by

$$x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \\ 1 \end{pmatrix}, \quad T_x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & L_x \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & L_y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & L_z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For example

$$(T_x)^\mu{}_\nu x^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & L_x \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ x + L_x \\ y \\ z \\ 1 \end{pmatrix} \equiv \begin{pmatrix} t \\ x \\ y \\ z \\ 1 \end{pmatrix}$$

And similarly for the rotation matrix $R(\theta)$ for the faces of the polyhedra. In general, any periodic orbit of compactified space is invariant under a composite transform φ which is a matrix product of T_x, T_y, T_z (and rotations for topologies other than E_1)

$$\varphi^\mu{}_\nu x^\nu \equiv x^\mu$$

For instance, in figure 2, we get identity $x \equiv T_y T_x^2 x$.

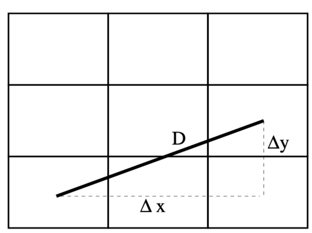

Figure 2: an example periodic orbit with composite transform $\varphi = T_y T_x^2$ identifying x.

If Alice goes off on a periodic orbit about the compactified space with a speed $\beta = v/c$ close to 1, the coordinate system in her rest frame is $x' = \Lambda x$ where the Lorentz transformation embedded in $(4+1)$ is

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z & 0 \\ -\gamma\beta_x & 1 + \frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & 0 \\ -\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1 + \frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 0 \\ -\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1 + \frac{(\gamma-1)\beta_z^2}{\beta^2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_x = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

If Bob sees Alice travel a total distance ℓ in his frame during one complete orbit, then the time that passes on his clock is

$$\tau_B = \ell/v$$

and Bob will tell you that Alice's clock should run slower by a factor of γ^{-1}

$$\tau_A = \ell/v\gamma$$

Alice's orbit is described by some composite matrix φ. For the sake of simplicity, I'll stick with the hypertorus where she's moving along the $\hat{\mathbf{x}}$ axis, so for a point in Alice's frame

$$x'^\mu = (\Lambda_x)^\mu{}_\nu x^\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma(t-\beta x) \\ \gamma(x-\beta t) \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

this should then identify with

$$(T_x)^\mu{}_\nu x'^\nu = (\Lambda_x)^\mu{}_\nu (T_x)^\nu{}_\sigma x^\sigma = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x+L_x \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma(t-\beta(x+L_x)) \\ \gamma(x+L_x-\beta t) \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

As before, this identifies points in x with $x \equiv x + L_x$, but it also identifies points in t. The statement

$$\delta t' = (\gamma t - \gamma\beta x - \gamma\beta L_x) - (\gamma t - \gamma\beta x) = -\gamma\beta L_x$$

Similar to how each time you go around the Möbius strip, you find yourself reflected, or how each time you travel a distance $\Delta z = L_z$ in E_2, you find yourself rotated by π, this statement means that ***each time Alice travels a distance L_x in Bob's reference frame, she picks up a time offset of $\gamma\beta L_x$***. Which is deeply weird if you ask me. But it also means that Alice and Bob can't synchronize their watches when Alice does an orbit because each orbit she picks up a time offset. So while she will say that Bob was aging more slowly during her journey, because she picks up this offset, she still finds herself younger upon return.

We can see this mathematically by considering that Alice sees Bob travelling a time dilated distance γL_x at speed β, the time for him to reach her again is this plus the offset

$$\tau_A = \gamma L_x/\beta - \gamma\beta L_x = L_x/\gamma\beta$$

Which agrees with what we found earlier. To see this in a little more detail, let's look at a **spacetime diagram**. This very odd statement is also a reflection of what we already knew about special relativity, ***events that happen at the same time and different places in one frame are not simultaneous in another***. So another way of looking at this is that in Alice's frame, the copy of Bob from the next cell has already left the next copy of Alice when she leaves her Bob.

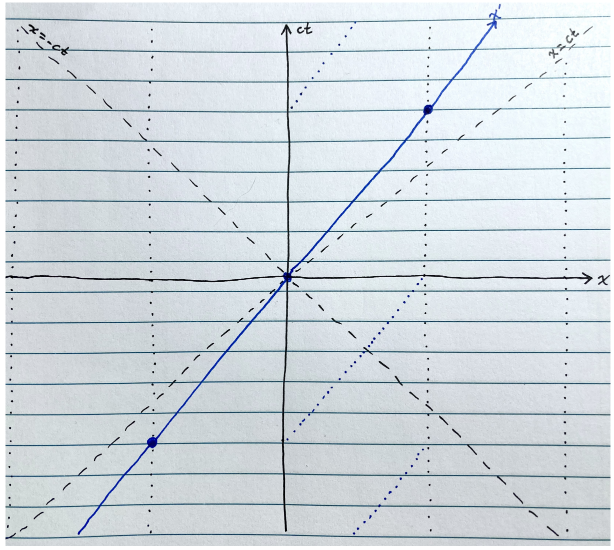

Figure 3: the spacetime diagram describing the twin paradox involving a stationary twin and a twin on a periodic orbit about the compact space.

5 What does all this mean for cosmology?

This has some pretty dramatic effects on cosmology. First of all, ***there exists one and only one inertial reference frame that is at rest with respect to the compact spatial sections***. Bob can set up an experiment where many observers in his frame exchange information by light rays to synchronize all their clocks. Alice can't do this because her observers will start getting messages from their doubles in the next tessellation cell telling them they are out of sync by factors of $\gamma\beta L_x$.

Thus, a compact topology has a **preferred frame**, the frame for which each side of the tessellation cell is shortest, in which there are no offsets.

There are other complications as well, all your particles are going to be interacting with infinitely many copies of themselves out to infinity, which sounds like it would make QFT even less pleasant to work with.

6 Should you care?

Probably not.

I spoke with Glenn about this and he said that **there is nothing pointing physically toward a given topology** or a given one of the allowed Euclidean tessellation schema, for instance. It's more that there's nothing pointing to there being only trivial topology. For me, from a purely aesthetic perspective, I tend to prefer more motivation. In GR, energy curves space, there's a cause and effect there that has some meat to it because we know that energy is a specifically meaningful thing.

And more importantly, we have looked for the universe's topology, and we haven't found anything. We looked for ***matched circles* in the CMB** and didn't find any, which means that if the universe does have non-trivial topology, the covering space is really big, too big

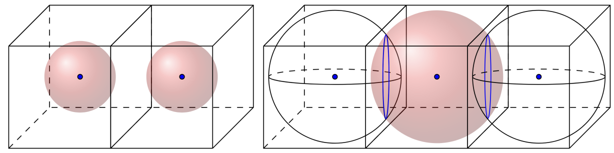

Figure 4: if the universe is a flat hypertorus, but the side length of the torus is much greater than the Hubble diameter, then we simply won't see any evidence the topology (a). But, if a side is less than the Hubble diameter, we will see matched circles in the CMB (b).

to see in the CMB. They may be visible through other means, the boundary conditions will result in discretization of energies, so maybe there is some quantum gravity phenomena to probe there, but this is not something that we'll test any time soon.

What I do think is worth your time though is asking these sorts of questions. Whether you're asking if your simulation is obeying important conservation laws or recasting an old question to apply it to a new theory, it can help you get at the core of the idea. By following this lead, we've understood something fundamental about what the breakdown of simultaneity means for these sorts of cosmologies, and I would say that in doing so we've understood cosmic topology on a deeper level. And also if that idea is nagging at you, you can use it as an excuse to write a blackboard talk and then force yourself to actually get to the bottom of the original puzzle.