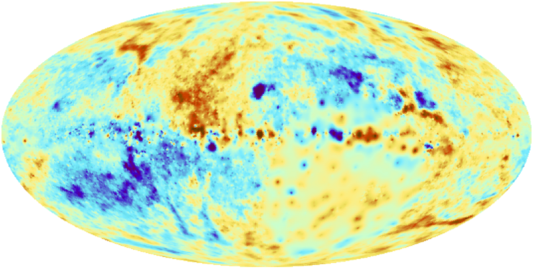
Reconstructing statistically isotropic Gaussian signals from noisy data with uncertain signal and noise covariance

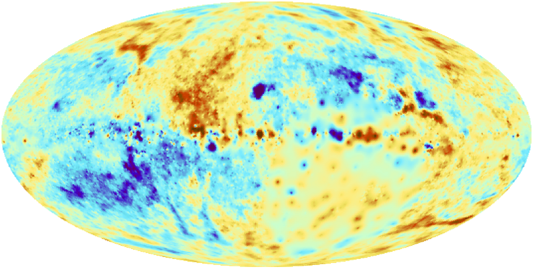
–or–

How to make neat images

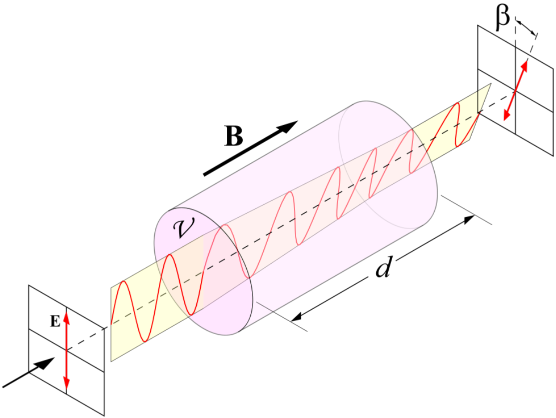


Niels Oppermann, Cosmology group meeting, 2011-11-29

A map of the Galactic Faraday depth

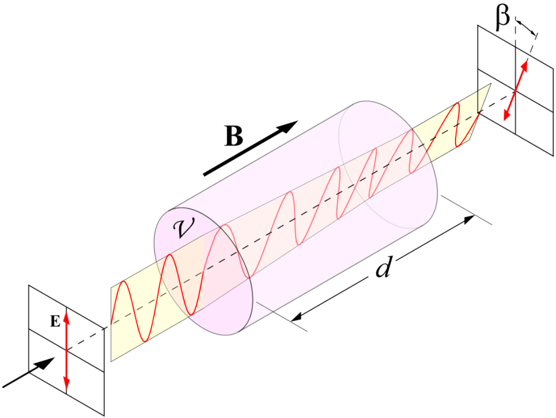


The Physics



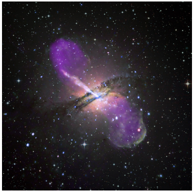
$$\mathrm{d}\beta \propto \lambda^2 n_{\mathrm{e}}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$$

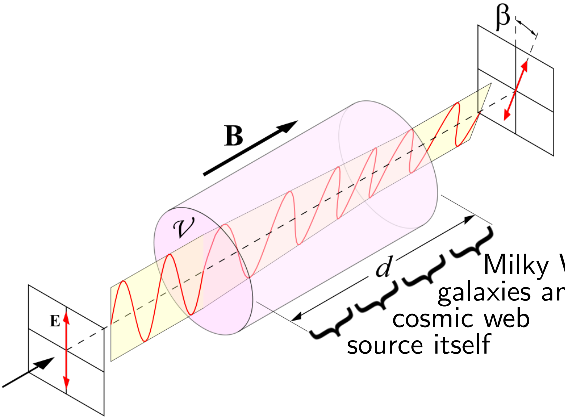
$$\Rightarrow \quad \beta \propto \lambda^2 \int_{r_{\mathrm{source}}}^{0} n_{\mathrm{e}}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$$



Faraday depth: $\phi \propto \int_{r_{\mathrm{source}}}^{0} n_{\mathrm{e}}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$

$$\beta = \phi\lambda^2$$





Faraday depth: $\phi \propto \int_{r_{\mathrm{source}}}^{0} n_{\mathrm{e}}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$

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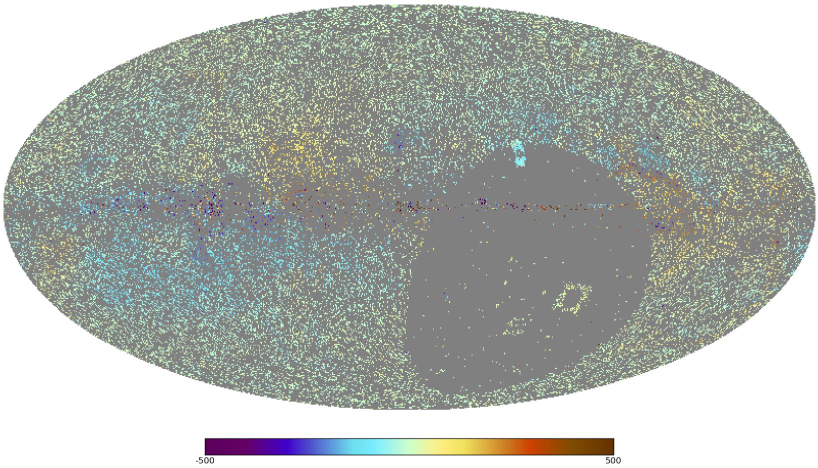
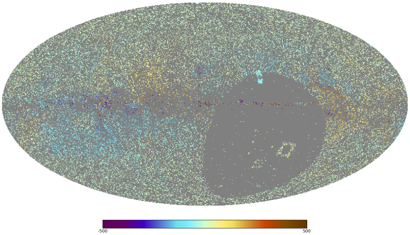


41 330 data points

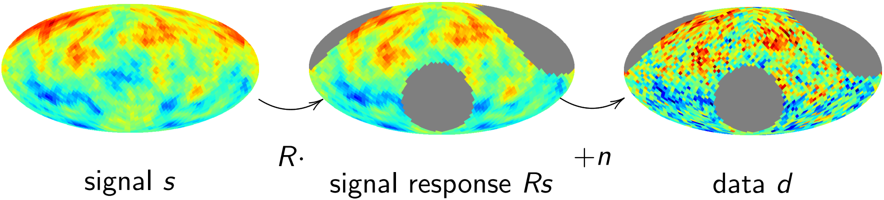


Challenges

- Regions without data
- Uncertain error bars:
    - complicated observations
    - $n\pi$-ambiguity
    - extragalactic contributions unknown

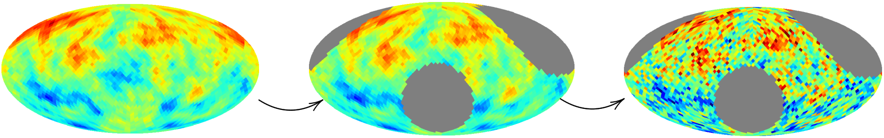
The Statistics

Reconstructing statistically isotropic Gaussian signals from noisy data with uncertain signal and noise covariances



$$\boxed{d = Rs + n}$$

Reconstructing statistically isotropic Gaussian signals from noisy data with uncertain signal and noise covariances



signal s $\quad R\cdot \quad$ signal response Rs $\quad +n \quad$ data d

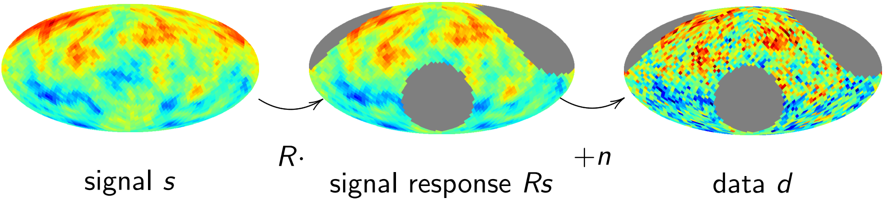
$$\boxed{d = Rs + n}$$

$$\mathcal{P}(s) = \mathcal{G}(s, S)$$

$$\mathcal{P}(n) = \mathcal{G}(n, N)$$

$$\mathcal{G}(s, S) = \frac{1}{|2\pi S|^{1/2}} \exp\left[\frac{1}{2} s^\dagger S^{-1} s\right]$$

Reconstructing statistically isotropic Gaussian signals from noisy data with uncertain signal and noise covariances

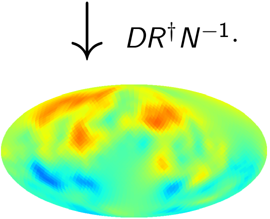


Wiener Filter

$$m = \int \mathcal{D}s \; s \; \mathcal{P}(s|d)$$

$$d = Rs + n$$

$$m = Dj, \text{ where } \quad \begin{array}{l} j = R^\dagger N^{-1} d \\ D = \left(S^{-1} + R^\dagger N^{-1} R\right)^{-1} \end{array}$$



Reconstructing statistically isotropic Gaussian signals from noisy data with uncertain signal and noise covariances

$$S(\hat{n}, \hat{n}') = \int \mathcal{D}s \; s(\hat{n}) s(\hat{n}') \mathcal{P}(s)$$

$$\Rightarrow S_{(\ell m),(\ell' m')} = \int \mathcal{D}s \; s_{\ell m} s^*_{\ell' m'} \mathcal{P}(s)$$

Reconstructing statistically isotropic Gaussian signals from noisy data with uncertain signal and noise covariances

$$
\begin{aligned}
S(\hat{n}, \hat{n}') &= \int \mathcal{D}s \; s(\hat{n}) s(\hat{n}') \mathcal{P}(s) \\
&= S(\hat{n} \cdot \hat{n}') \\
\Rightarrow S_{(\ell m),(\ell' m')} &= \int \mathcal{D}s \; s_{\ell m} s^{*}_{\ell' m'} \mathcal{P}(s) \\
&= \delta_{\ell \ell'} \delta_{m m'} C_{\ell}
\end{aligned}
$$

Reconstructing statistically isotropic Gaussian signals from noisy data with uncertain signal and noise covariances

$$
\begin{aligned}
S(\hat{n}, \hat{n}') &= \int \mathcal{D}s \; s(\hat{n}) s(\hat{n}') \mathcal{P}(s) \\
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\Rightarrow S_{(\ell m),(\ell' m')} &= \int \mathcal{D}s \; s_{\ell m} s^*_{\ell' m'} \mathcal{P}(s) \\
&= \delta_{\ell\ell'} \delta_{mm'} C_\ell
\end{aligned}
$$

$$
N_{ij} = \delta_{ij} \sigma_i^2
$$

(uncorrelated noise)

Reconstructing statistically isotropic Gaussian signals from noisy data with uncertain signal and noise covariances

$$S_{(\ell m),(\ell' m')} = \delta_{\ell\ell'}\delta_{mm'} C_\ell \qquad N_{ij} = \delta_{ij}\eta_i\sigma_i^2$$

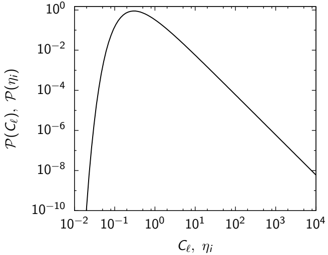
Reconstructing statistically isotropic Gaussian signals from noisy data with uncertain signal and noise covariances

$$S_{(\ell m),(\ell' m')} = \delta_{\ell\ell'}\delta_{mm'} C_\ell \qquad N_{ij} = \delta_{ij}\eta_i\sigma_i^2$$

assume priors for parameters

$$\mathcal{P}\left((C_\ell)_\ell\right) = \prod_\ell \frac{1}{q_\ell \Gamma(\alpha_\ell - 1)} \left(\frac{C_\ell}{q_\ell}\right)^{-\alpha_\ell} \exp\left(-\frac{q_\ell}{C_\ell}\right)$$

$$\mathcal{P}\left((\eta_i)_i\right) = \prod_i \frac{1}{q_i \Gamma(\alpha_i - 1)} \left(\frac{\eta_i}{q_i}\right)^{-\alpha_i} \exp\left(-\frac{q_i}{\eta_i}\right)$$



$\Rightarrow$ marginalize over all possible parameters

Problem: $\mathcal{P}(s|d)$ is non-Gaussian.
Solution: Find Gaussian $\mathcal{G}(s - m, D)$, that best approximates $\mathcal{P}(s|d)$.

Extended Critical Filter

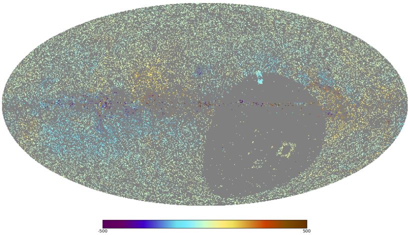
$$m = Dj, \quad D = \left[\sum_\ell C_\ell^{-1} S_\ell^{-1} + \sum_i \eta_i^{-1} R^\dagger N_i^{-1} R \right]^{-1},$$

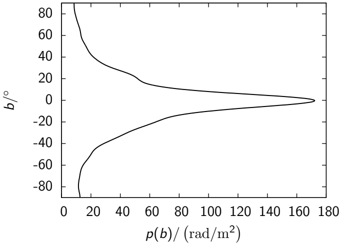
$$j = \sum_i \eta_i^{-1} R^\dagger N_i^{-1} d$$

$$C_\ell = \frac{1}{\alpha_\ell + \ell - 1/2} \left[q_\ell + \frac{1}{2} \mathrm{tr} \left(\left(m m^\dagger + D \right) S_\ell^{-1} \right) \right]$$

$$\eta_i = \frac{1}{\alpha_i} \left[q_i + \frac{1}{2} \mathrm{tr} \left(\left((d - Rm)(d - Rm)^\dagger + D \right) N_i^{-1} \right) \right]$$

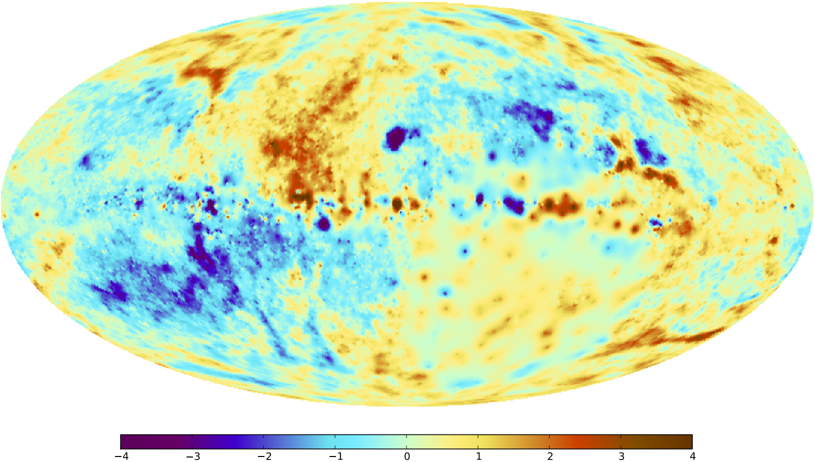
The Images





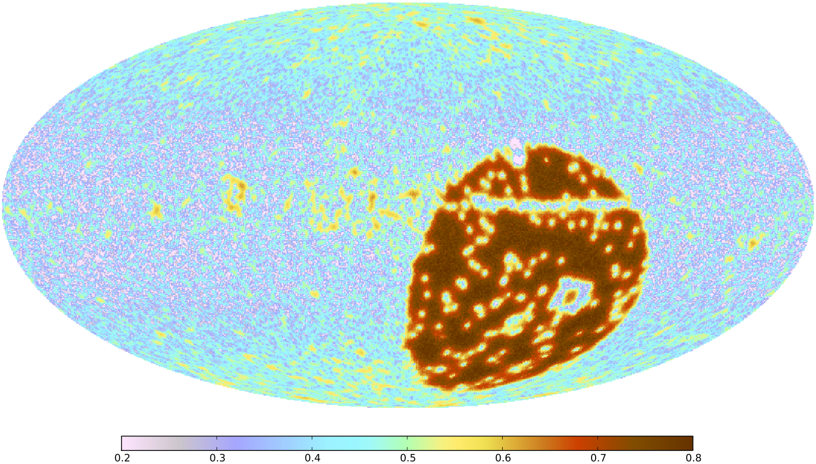
- Approximate $s(b,l) := \frac{\phi(b,l)}{p(b)}$ as a statistically isotropic Gaussian field
- R: multiplication with $p(b)$ and projection on directions of sources
- $N_{ij} = \delta_{ij}\eta_i\sigma_i^2$

posterior mean of the signal



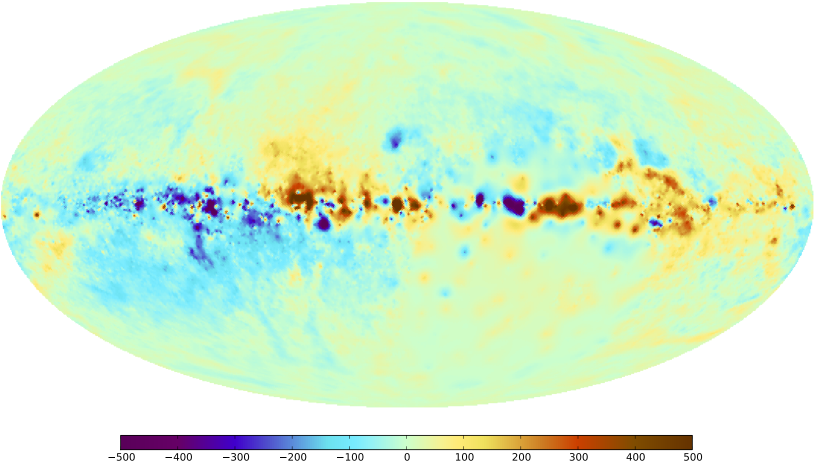
m

uncertainty of the signal map



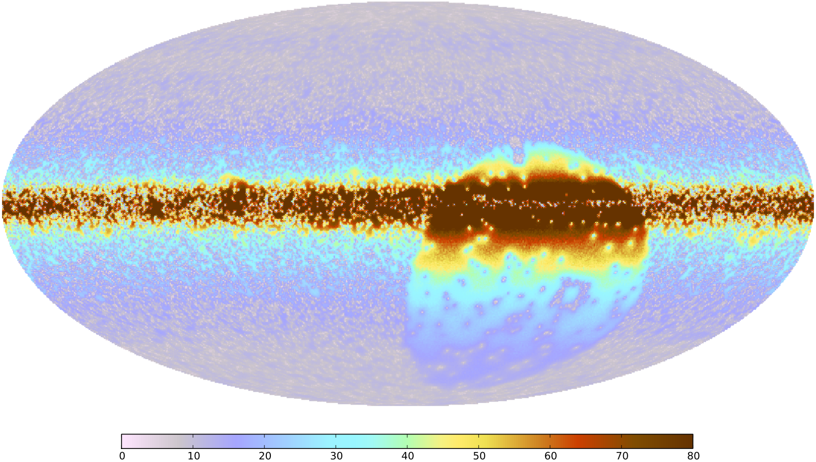
$$\sqrt{\operatorname{diag}(D)}$$

posterior mean of the Faraday depth



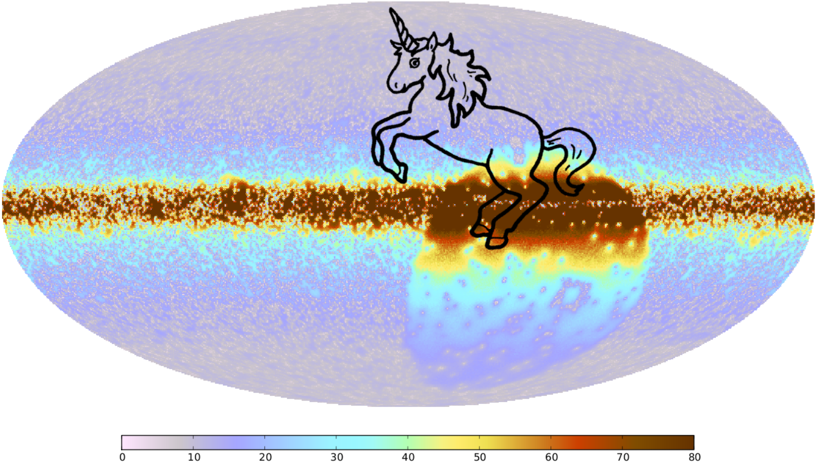
pm

uncertainty of the Faraday depth

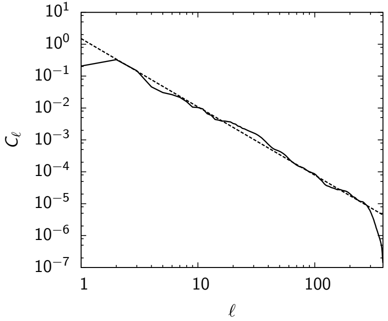


$$p\sqrt{\mathrm{diag}\left(D\right)}$$

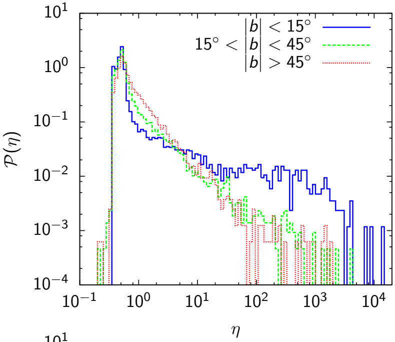
uncertainty of the Faraday depth

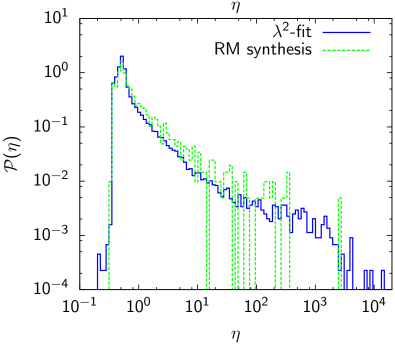


$$p\sqrt{\mathrm{diag}\left(D\right)}$$



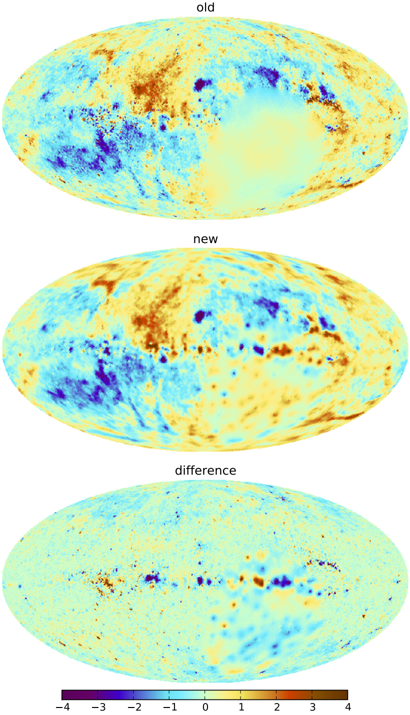
$$C_\ell \propto \ell^{-2.14}$$





- N. Oppermann, G. Robbers, T.A. Enßlin:
  "Reconstructing signals from noisy data with unknown signal and noise covariance"
  Physical Review E, vol. 84, Issue 4, id. 041118
  arXiv:1107.2384
- N. Oppermann, H. Junklewitz, G. Robbers, M.R. Bell, T.A. Enßlin, A. Bonafede, R. Braun, J.C. Brown, T.E. Clarke, I.J. Feain, B.M. Gaensler, A. Hammond, L. Harvey-Smith, G. Heald, M. Johnston-Hollitt, U. Klein, P.P. Kronberg, S.A. Mao, N.M. McClure-Griffiths, S.P. O'Sullivan, L. Pratley, T. Robishaw, S. Roy, D.H.F.M. Schnitzeler, C. Sotomayor-Beltran, J. Stevens, J.M. Stil, C. Sunstrum, A. Tanna, A.R. Taylor, C.L. Van Eck:
  "An improved map of the Galactic Faraday sky"
  arXiv:1111.6186
- All results available at
  http://www.mpa-garching.mpg.de/ift/faraday/

Backup


old
new
difference
−4
−3
−2
−1
0
1
2
3
4