Nonlinear Intermittent Field Dynamics in the Early Universe

by

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Abstract

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We explore nonequilibrium dynamics of inhomogeneous nonlinearly interacting scalar fields in several cosmological settings. In particular, we explore the effects of (initially small) asymmetric fluctuations around field configurations with a high degree of spatial symmetry. These symmetric configurations include planar domain wall and SO(2,1)vacuum bubble collisions, and oscillations of a homogeneous scalar field condensate after inflation. In the last of these, it is well known that linear fluctuations may experience a variety of dynamical instabilities — a process known as preheating.

By extending methods from the Floquet theory of ODEs to PDEs, we demonstrate that linear fluctuations around the planar wall and vacuum bubble collisions also experience exponential growth. This effect has been ignored in the existing literature. We use sophisticated numerical lattice techniques to study the full (3+1)-dimensional dynamics of the collisions. Once the fluctuations begin to interact nonlinearly, the original spacetime symmetries of the collision are badly broken. In each model we study, this symmetry breaking occurs through an inhomogeneous annihilation of the walls. As a result of this annihilation, a collection of oscillons is produced in the collision region.

In our study of preheating, we focus on entropy generation as the field transitions from a homogeneous, coherent state to an inhomogeneous, incoherent state. We introduce a coarse-graining procedure based on maximizing the (differential) Shannon entropy subject to a collection of observational constraints. Using lattice simulations, we find a sharp spike in the entropy production rate around the onset of strong nonlinearities amongst the fluctuations. Based on an analogy with a hydrodynamic shock as randomization front, we dub this sudden increase in entropy as the "shock-in-time".

In each of the systems considered, the new dynamics we discover suggest potential observational signatures. Perhaps the most interesting new signatures are the production of gravitational waves from individual vacuum bubble collisions and the production of nonGaussian density perturbations during preheating. A novel aspect of these signatures is that they are spatially (or temporally) intermittent. Although this thesis focusses on the underlying dynamics responsible for these signatures, it suggests interesting future work developing the necessary methods to observationally constrain them.

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Chapter 1

Introduction

Our current picture of cosmology has emerged from the tremendous wealth of information extracted from measurements of the cosmic microwave background (CMB), big bang nucleosynthesis (BBN), large-scale structure (LSS), weak gravitational lensing (WL), Lyman- α forest (Ly- α) and supernovae (SN). These measurements have allowed us to probe physics at energy scales beyond those currently accessible in terrestrial experiments and provide a glimpse into the universe in its infancy over 13 billion years ago. Measurements of the CMB, in particular, have allowed us to develop a well-tested and consistent picture of the early universe, including information about its composition, spatial curvature and the properties of the initial density fluctuations.

The current observational era began with the experimental discovery of the CMB by Penzias and Wilson, which provided the first direct evidence for Big Bang cosmology. Detection of temperature anisotropies at the level $\frac{\delta T}{T} \sim 10^{-5}$ by COBE, and the subsequent detection and confirmation of the acoustic peaks in the temperature-temperature (TT) autocorrelation spectrum by balloon-borne missions such as BOOMERanG and MAX-IMA and satellites such as WMAP established the adiabatic nature of the primoridial perturbations and provided experimental support for the inflationary paradigm. Most recently, the Planck satellite and ground based telescopes such as ACT and SPT have provided an exquisite picture of the TT power spectrum up to multipoles of order $\ell \sim 10000$. Current experiments such as Planck, ACTPol, and BICEP as well as future experiments such as Spider and CMBPol are searching for anisotropies in the B-mode polarization of photons originating from primordial sources. If detected, these would provide our most convincing test of inflation, including a direct measurement of the inflationary energy scale.

To date, these experiments have revealed a remarkably simple picture of the universe, whose bulk properties can be described with a handful of phenomenological parameters.

Regarding the very early universe, the most important findings are that the density fluctuations in the primordial plasma are adiabatic and well-described by a homogeneous and isotropic Gaussian random field with amplitude $\frac{\delta\rho}{\rho} \sim 10^{-5}$ and nearly scale invariant power spectrum. Inflationary cosmology predicts this form for the fluctuations, which thus provide the strongest evidence for an early inflationary phase. With the basic picture now established, the next goal in cosmological exploration is to either discover or constrain subdominant features in the primordial fluctuations. Indeed, the robust nature of the inflationary predictions creates huge degeneracies in the leading order inflationary observables that standard parameterizations of the data are unable to break. Finding new and novel signatures could help to break these degeneracies. Much effort currently focusses on either detecting or improving constraints on primordial B-mode polarization of the CMB photons. There are also several low- ℓ anomalies in the currently available CMB temperature data: the cold spot, hemispherical power asymmetry and an apparent deficit of fluctuation power on large scales. While the anomalies may simply be statistical flukes or some currently unappreciated systematic, they may also hint at something about the inner workings of inflation. In particular, since the anomalies are intermittent, either in spatial location or in scale, a breaking of the homogeneous and near scale invariant nature of inflation is needed. An explanation for a single one of the anomalies may not be convincing. However, a mechanism that could simultaneously produce several of the anomalies, thus unifying them into a single theoretical framework, would be a far more compelling argument. Much of the work undertaken in this thesis has an eye to uncovering novel new observational signatures that have not been considered in detail before.

1.1 A Brief History of the Universe

Despite the tremendous growth in our knowledge of the early universe there are still many unknowns. Here we provide a brief non-technical account of the most widely held current view of the history of universe, being careful to point out elements for which there is theoretical or observational uncertainty.

During the earliest moments for which we have observational evidence the universe was expanding at an accelerated rate. This epoch is known as inflation and is generally believed to have been driven by the condensate of some scalar field, although the precise microphysics is unknown. While the universe is inflating, subhorizon quantum fluctuations in the scalar field and the metric are stretched to scales larger than the Hubble radius where they freeze out. These fluctuations will eventually leave their imprint in the CMB and matter power spectrum, giving us an observational window into this early evolution.

Eventually, inflation ends and the universe stops accelerating and begins to decelerate. The energy in the inflaton condensate is transferred into a hot dense plasma of Standard Model particles and possibly additional degrees of freedom such as supersymmetric partners. The details of this transition are subject to great uncertainty, and even basic details, such as the temperature at which the plasma first reaches local thermodynamic equilibrium (LTE), are unknown. Baryogenesis may have occurred and dark matter may have been generated prior to the establishment of LTE. However, these events may also have occurred during the subsequent cooling of the plasma.

The universe continues to expand, with the hot plasma adiabatically cooling as it dilutes. During this expansion, the universe may have undergone a series of phase transitions, and baryogenesis may have produced the slight matter-antimatter asymmetry we observe today. Dark matter may also have been produced by the thermal freezeout of a weakly interacting massive particle (WIMP). Since the initial temperature of the plasma and the microphysics above energy scales of order a TeV are unknown, no definitive statement can be made about the exact sequence of events that occurred.

Once the plasma reaches a temperature $T \sim 10 MeV$ we once again reach an epoch which has been confirmed by observations. Unlike the inflationary phase, we have a thorough understanding of the underlying microphysics. The plasma now consists of photons, electrons, positrons, three species of neutrinos, protons and neutrons. The protons and neutrons begin to fall out of nuclear statistical equilibrium and the proton to neutron ratio freezes in at $T \sim MeV$. While this process is occurring, the electroweak interactions become too weak to keep the neutrinos in equilbrium and they decouple at $T \sim MeV$. Subsequently, the electrons and positrons annihilate, but since the neutrinos have already decoupled the energy from the annihilations is transferred entirely to the photons, resulting in the neutrino temperature being smaller than the photon temperature. Finally, BBN completes at $T \sim 0.1 MeV$, with protons and neutrons combining to form light elements. Most of the neutrons combine with protons to form helium, although smaller amounts of other light elements such as deuterium, tritium and lithium are also synthesized. The universe is now an ionized plasma of light nuclei, electrons and photons, with additional energy in the decoupled neutrinos, dark matter and dark energy. At $T \sim eV$ the energy density in the radiation (photons and neutrinos) drops below the energy density in dark matter, the so called epoch of matter-radiation equality. From this point, perturbations in the dark matter begin to grow with gravitational potential wells remaining constant instead of damping on subhorizon scales. The baryons remain tightly coupled to the photons and thus continue to experience acoustic oscillations rather than collapse. When the temperature reaches $T \sim 3000K$, the photons are no longer energetic enough to ionize hydgrogen and the electrons and protons form hydrogen atoms. With no charged particles to interact with, the photons then free stream and redshift for the next 13 billion years, forming the CMB as detected in terrestrial experiments.

The baryons are now free to form structure and they fall into the potential wells created by the dark matter. Baryonic matter begins to form collapsed structures, and from these clouds of collapsed gas the earliest stars form and synthesize the first heavy elements. The death of these stars often lead to violent supernovae explosions, creating more heavy elements and spreading them into the interstellar medium. The process of structure formation continues, with successively larger structures collapsing and additional production of heavy elements from star formation. Planets form from gravitational collapse of the heavy elements formed from previous generations of stars.

Quite recently, the final element in our modern cosmological picture comes into play. The dark energy comes to dominate the energy density and the universe begins a new phase of accelerated expansion. The origin and properties of the dark energy are still mysterious. At present, observations are consistent with a constant contribution to the energy density $\rho_{\Lambda} \sim 10^{-120} M_P^4$ currently comprising $\sim 70\%$ of the cosmic energy budget. Due to the accelerated expansion, large scale subhorizon gravitational potentials decrease in size (rather than remaining constant as during the matter dominated phase) and the epoch of large-scale structure formation ends. Previously collapsed structures that decoupled from the cosmic expansion will persist, but no new clusters or superclusters form.

Aside from the formation of structure from gravitational collapse, the above is a description of the homogeneous universe. However, when the primordial plasma first formed its density varied from location to location in the universe.¹ While the wavelength of these perturbations remained larger than sound horizon of the fluid, they remained frozen. However, as the Hubble rate decreased, the modes began to move inside the horizon one by one, leading to damped acoustic oscillations in the primordial plasma. These oscillations continue until recombination, when the baryon-photon fluid decouples. The oscillation phase of each mode is determined by the amount of time the mode spent inside the horizon before recombination, leading to a characteristic pattern of acoustic peaks in the CMB.

 $^{^{1}}$ We are making a particular choice of coordinate system with this statement.

1.2 Mathematical Description of Λ CDM Cosmology

Now that we have given a brief nontechnical overview of the thermal history of our observable universe, we present a brief review of the mathematical description of the homogeneous expanding universe. One of the fundamental postulates of modern cosmology, confirmed to high accuracy by data, is that on large-scales the universe is homogeneous and isotropic. This assumption might break on scales much larger than our current Hubble volume, but is sufficient for describing current observations. The most general homogeneous and isotropic metric can be written in the form

$$ds^{2} = -dt^{2} + a^{2}(t) \left(\frac{1}{1 - kr^{2}}dr^{2} + r^{2}(d\theta^{2} + \sin^{2}\theta d\phi^{2})\right).$$
(1.1)

The constant k determines the curvature of spatial three-slices, a is known as the scale factor and the spatial coordinates are comoving. Positive, zero and negative k correspond to a closed, flat or open universe respectively, with spatial three-slices having the geometry of spheres, planes or hyperboloids. It is also convenient to define the conformal time $d\tau = adt$. The dynamics is governed by the Hubble constraint equation, also known as the Friedman equation,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_P^2} - \frac{k}{a^2} \tag{1.2}$$

where $M_p^{-2} = 8\pi G_N$ is the reduced Planck mass, G_N is Newton's gravitational constant, ρ is the total energy density of the universe and the Hubble constant is $H = \dot{a}/a$. The energy density evolves according to

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{1.3}$$

which has solution

$$\rho = \rho_0 e^{3\int (1+P/\rho)d\ln a} \,. \tag{1.4}$$

For separate components interacting only gravitationally $(\rho = \sum_i \rho_i)$ we have further

$$\dot{\rho}_i + 3H(\rho_i + P_i) = 0.$$
(1.5)

From this we see that for a component with a constant equation of state $w_i \equiv P_i/\rho_i$

$$\rho_i = \rho_{i,0} a^{3(1+w_i)} \,. \tag{1.6}$$

For a given value of the Hubble constant, it is convenient to define the critical density

$$\rho_{crit} = 3M_P^2 H^2 \tag{1.7}$$

and the corresponding energy density fraction

$$\Omega_i \equiv \frac{\rho_i}{\rho_{crit}} \tag{1.8}$$

as well as

$$\Omega_k \equiv -\frac{k}{a^2 H^2} \tag{1.9}$$

in terms of which the Friedman equation becomes

$$\sum_{i} \Omega_i + \Omega_k = 1.$$
 (1.10)

In Λ CDM cosmology, the energy density in the universe consists of a cosmological constant Ω_{Λ} , cold dark matter Ω_{cdm} , baryonic matter Ω_b , radiation (photons) Ω_{γ} , and neutrinos Ω_{ν} .

1.3 Inflation's Role in Cosmology

Inflation was originally introduced in an attempt to solve several problems associated with the homogeneous universe: the horizon, curvature and monopole problems. However, it was soon realized that inflation does something far more important. By stretching subhorizon quantum mechanical fluctuations to superhorizon scales it is able to produce density perturbations that seed subsequent structure formation in the universe. Here we briefly review the generation of inflationary perturbations, more detailed accounts can be found in [1, 2, 3, 4].

We write the perturbed metric in longitudinal gauge, assuming a spatially flat universe

$$ds^{2} = a^{2}(\tau) \left(-(1+2\Phi)d\tau^{2} + ([1-2\Psi]\delta_{ij} + h_{ij})dx^{i}dx^{j} \right)$$
(1.11)

where we have dropped the vector perturbations which are not produced in most inflationary models. The tensor h_{ij} is traceless $\delta^{ij}h_{ij} = 0$ and transverse $\partial_i h_j^i$ with respect to the flat spatial metric. Similarly the perturbed inflaton is

$$\phi = \phi(\tau) + \delta\phi. \tag{1.12}$$

Thus, on the surface we have 2 tensor and 3 scalar degrees of freedom. However, the constraints for general relativity remove two of the scalar degrees of freedom leaving us with a single physical scalar mode. It is convenient to define a gauge invariant combination of the scalars \mathcal{R} , which does not transform under linear variable changes

$$\mathcal{R} = \Phi + \frac{\mathcal{H}}{\bar{\phi}'} \delta \phi \,. \tag{1.13}$$

A convenient parameter describing the rate of change of the inflationary expansion is

$$\epsilon_H = -\frac{d\ln H}{d\ln a} = -\frac{\dot{H}}{H^2} = \frac{\rho + P}{2M_P^2 H^2} = \frac{\dot{\phi}^2}{2H^2} \,. \tag{1.14}$$

After a straightforward, but tedious, expansion of the action to second-order we obtain the following equations for the scalar and tensor degrees of freedom

$$(z\mathcal{R})'' + \left(k^2 - \frac{z''}{z}\right)(z\mathcal{R}) = 0$$
(1.15)

and

$$(ah_{\alpha})'' + \left(k^2 - \frac{a''}{a}\right)(ah_{\alpha}) = 0$$
(1.16)

respectively, where $' = \partial_{\tau}$ is a derivative with respect to conformal time. h_{α} represents the amplitude of one of the tensor mode (graviton) polarization states appearing as $h_{ij} = h_1 e_{ij}^1 + h_2 e_{ij}^2$ with e_{ij}^{α} fixed transverse-traceless (TT) polarization tensors satisfying $\sum_{ij} e_{ij}^{\alpha} e_{ij}^{\alpha'} = \delta^{\alpha \alpha'}/2$. We have defined $z = \frac{a\phi'}{\mathcal{H}} = \sqrt{2\epsilon_H a}$. In spatially flat gauge the inflaton perturbations have the standard form for a scalar field

$$\widetilde{\delta\phi}_k|_{k\gg aH} = \frac{a^{-3/2}}{\sqrt{2k/a}M_P} \left(\alpha_k^{\delta\phi} e^{-i\int \frac{k}{a}dt} + \beta_k^{\delta\phi} e^{i\int \frac{k}{a}dt} \right) \,. \tag{1.17}$$

However, in this gauge $\mathcal{R} = \frac{\mathcal{H}}{\phi'} \delta \phi$, so

$$k^{3/2} \mathcal{R}_k|_{k\gg aH} = \frac{k/a}{2\sqrt{\epsilon_H}M_P} \left(\alpha_k^{\mathcal{R}} e^{-i\omega_S \tau} + \beta_k^{\mathcal{R}} e^{+i\omega_S \tau} \right) \,. \tag{1.18}$$

Meanwhile, the tensor modes are initially given by

$$k^{3/2}\tilde{h}_k = \frac{2k/a}{M_P} \left(\alpha_k^h e^{-i\omega_T \tau} + \beta_k^h e^{+i\omega_T \tau} \right)$$
(1.19)

where $|\alpha_k|^2 - |\beta_k|^2 = 1$ for pure states. Most often, the Bunch-Davies vacuum is se-

During inflation, we have $a(\tau) \approx \tau^2$ and to leading order in slow-roll $z''/z = 2/\tau^2$. Therefore, at leading order both the scalars and tensors obey

$$f'' + \left(k^2 - \frac{2}{\tau^2}\right)f = 0 \tag{1.20}$$

which has the general solution

$$f = C_1 e^{-ik\tau} \left(1 - \frac{i}{k\tau} \right) + C_2 e^{ik\tau} \left(1 + \frac{i}{k\tau} \right) .$$
(1.21)

At leading order, we can approximate the amplitude of the mode functions once they cross the horizon using the value of the Hubble constant at horizon crossing. Matching and making use of $\mathcal{H} = aH = -\tau^{-1}$ we have in the $k\tau \ll 1$ limit

$$\langle |\tilde{\mathcal{R}}_k|^2 \rangle = \frac{H^2}{4\epsilon M_P^2 k^3} \bigg|_{aH=k}$$
(1.22)

and

$$\langle |\tilde{h_{\alpha}}|^2 \rangle = \frac{4H^2}{k^3 M_P^2} \Big|_{aH=k}.$$
 (1.23)

Observational constraints are usually expressed in terms of the scalar and tensor power spectra

$$\mathcal{P}_S = \frac{k^3}{2\pi^2} \langle |\tilde{\mathcal{R}}_k|^2 \rangle = \frac{H^2}{8\pi^2 M_P^2 \epsilon} \bigg|_{aH=k}$$
(1.24)

$$\mathcal{P}_{T} = \frac{k^{3}}{2\pi^{2}} \sum_{ij} \langle |\tilde{h}_{ij,k}|^{2} \rangle = \frac{k^{3}}{2\pi^{2}} \langle |\tilde{h}_{k}^{2}| \rangle = \frac{2H^{2}}{\pi^{2}M_{P}^{2}} \bigg|_{aH=k}$$
(1.25)

Since the Hubble rate evolves slowly in standard slow-roll inflation, it is convenient to expand the spectra in terms of the Hubble slow-roll parameters $\epsilon_H^{(1)} = -d \ln H/d \ln a$ and $\epsilon_H^{(i+1)} = d \ln \epsilon_H^{(i)}/(d \ln a)$. For convenience, we will define $\eta_H = \epsilon_H^{(2)}$. At leading order, we retain the first two of these parameters

$$\epsilon_H = \frac{-\dot{H}}{H^2} \qquad \eta_H = \frac{d\ln\epsilon}{d\ln a} \approx 4\epsilon_V - 2\eta_V \tag{1.26}$$

where we have also introduced the potential slow-roll parameters

$$\epsilon_V = \frac{M_P^2}{2} \left(\frac{V'}{V}\right)^2 \approx \epsilon_H \qquad \eta_V = M_P^2 \frac{V''}{V} \approx 2\epsilon_H - \frac{\eta_H}{2}. \tag{1.27}$$

For standard single-field slow-roll inflation, $\epsilon_H \ll 1$ and $\eta_H \ll 1$. If more precise analytic predictions are desired, higher order terms in the slow-roll expansion may be included.

Due to the smallness of ϵ_H and η_H , it is convenient to expand $\ln \mathcal{P}$ as a power series in $\ln k$. Motivated by this, the primoridal power spectra are usually parameterized as

$$\mathcal{P}_{S} = A_{S} \left(\frac{k}{k_{pivot}}\right)^{n_{S}-1+\frac{1}{2}n_{run}\ln k/k_{pivot}}$$
$$\mathcal{P}_{T} = rA_{S} \left(\frac{k}{k_{pivot}}\right)^{n_{T}}$$
(1.28)

when comparing with observations. Here k_{pivot} is some pivot scale around which to expand $\ln \mathcal{P}$ and is often taken to be $0.05 Mpc^{-1}$ of $0.002 Mpc^{-1}$.

Four of these parameters arise at leading order in single-field slow-roll inflation: the scalar amplitude

$$A_S = \mathcal{P}_S(k_{pivot}) = \frac{V}{24\pi^2 M_P^2 \epsilon_H}, \qquad (1.29)$$

the tensor to scalar ratio

$$r = \frac{\mathcal{P}_T}{\mathcal{P}_S}\Big|_{k_{pivot}} = 16\epsilon_H, \tag{1.30}$$

the scalar tilt

$$n_S - 1 = \frac{d \ln \mathcal{P}_S}{d \ln k} \bigg|_{k_{pivot}} = -2\epsilon_H - \eta_H, \qquad (1.31)$$

the tensor tilt

$$n_T = \frac{d\ln \mathcal{P}_T}{d\ln k} \bigg|_{k_{pivot}} = -2\epsilon_H = -\frac{r}{8},$$
(1.32)

and the running of the scalar spectral (n_{run}) first appears at the next order in the expansion. All of the slow-roll parameters are evaluated at the time that k_{pivot} exited the horizon during inflation. Constraints on inflationary models are usually expressed in the (n_S, r) plane, with the well-measured scalar amplitude used to determine the position on the potential at which the slow-roll parameters are evaluated. The total number of efolds since the end of inflation enters in order to relate the pivot scale k_{pivot} to a given comoving scale during inflation. Due to poor constraints from the data, it is common to set $n_{run} = 0$ and $n_t = 0$ or $n_t = -r/8$. As well, since the initial Bunch-Davies fluctuations are Gaussian, and the evolution equations are linear, the primordial fluctuations themselves must be Gaussian to very high accuracy. One further property of the fluctuations is that they are adiabatic, leading to the characteristic acoustic peaks seen in the CMB TT-autocorrelation function. Our most convincing evidence for inflation is that the parameterization (1.28) provides an excellent fit to the data on scales much larger

than the causal horizon at the time of recombination (assuming a radiation dominated universe back to the big bang), which are Gaussian and adiabatic.

1.4 Inflation in the Multiverse

Despite the confirmation of inflationary predictions by observations, we are still far from a microphysical theory for the inflaton. Since an early era of accelerated expansion of the universe was first introduced [5, 6, 7, 8, 9] there have been numerous proposals for the microphysical description of the inflaton. These models have developed in conjunction with ideas from theoretical particle physics. Recently, much of the top-down approach to inflationary model building has focussed on embedding an inflationary phase into various string theory constructions. Rather than producing a single possible inflationary phase, this has lead to the discovery of many stringy inflationary models.

The four-dimensional effective theories derived from string theory generally have a plethora of scalar fields. Common examples include moduli fields associated with the sizes of holes in compactified extra dimensions and the overall size of the compactification. Separations between extended objects, such as Dbranes, in the extra dimensions also lead to additional scalar degrees of freedom. Further, in order to stabilize the compactification, various fluxes must be introduced, leading to more model building freedom. By adjusting these fluxes it is possible to obtain quasi-stable potential minima with positive vacuum energy (so called deSitter minima), each with their own collection of low energy fields, vacuum energy and fluxes [10]. Current wisdom suggests that string theory permits a staggeringly large number of deSitter minima, with oft-quoted estimates of 10⁵⁰⁰ such vacua.

The presence of many effective scalar degrees of freedom, metastable minima, and possible inflationary solutions has lead to a picture of cosmology that we refer to as the landscape paradigm [11, 12]. Although current theoretical motivation for this comes from string theory, the only important ingredients for what follows are that at some highenergy scale the universe is adequately described by General Relativity coupling to many effective scalar degrees of freedom described by Quantum Field Theory with a mechanism to seed random initial conditions. In this picture, the structure of the ultra-large scale universe is determined by the evolution of an ensemble of initial values for the many scalars, fluxes and other degrees of freedom that arise in the high energy theory describing our universe. Different spatial locations have different randomly chosen initial conditions and thus undergo a different cosmic evolution. Each of the self-consistent inflationary mechanisms in the high energy theory (say string theory) is then realized somewhere in this landscape. In a simplified view where we only consider the scalar degrees of freedom and ignore gradients, we can view this evolution as a collection of balls moving on some high-dimensional potential surface. Each ball will move under the combined influence of classical forces, quantum diffusion [13, 14], and quantum tunnelling events. Under the assumption that probabilities should be weighted by physical (rather than comoving) volume, balls along inflating trajectories would reproduce during the evolution. In particular, regions undergoing eternal inflation (either stochastic or false vacuum) would continue to reproduce indefinitely. A consequence of this is that our current vacuum is randomly selected from one of the many possible deSitter minima, forming the basis for anthropic approaches to the cosmological constant problem. However, this picture is far richer than simply allowing for a dynamical (and stochastic) selection of vacua. The trajectories of the balls as they move along the potential surface give possible cosmological histories.

We have not specified how the initial conditions arise. Presumably this would require a deep understanding of the full quantum mechanical properties of the high energy theory. As one example, we could imagine that some portion of the wavefunction for the high energy degrees of freedom leads to a description of general relativistic spacetime coupled to fields described by quantum field theory. This portion of the wavefunction may then decohere producing a set of stochastic initial conditions for the subsequent evolution of the universe.

Given that our observations are restricted to our local Hubble volume, the possibility of testing this scenario might seem hopeless. However, certain subensembles of trajectories may produce unique signals that we can look for. A detection of such a signal would provide evidence for that subensemble of possible trajectories. One such class of trajectories that has received considerable attention recently is what we will call false vacuum eternal inflation on the landscape. In this scenario, a patch of the universe becomes stuck in one of the many possible metastable deSitter minima. This may occur due to either initial conditions or dynamical evolution. While it remains trapped in the minimum, this patch of the universe will experience inflationary expansion. Since the minimum is only metastable, decays are possible through, for example, Coleman-deLuccia (CdL) tunnelling [15] or Hawking-Moss decay [16]. When CdL is the dominant mechanism, this leads to the creation of bubbles nucleating within the ambient false vacuum. The interior of each of these bubbles can be described with an open FRW cosmology, and our observable universe is then contained within one of these bubbles.

In this scenario, our local cosmological history thus proceeds as follows. A bubble nucleates from some ambient deSitter minima that we denote dS_{parent} , leading to the

creation of our observable universe. This nucleation leads to a short period of curvature dominated evolution inside the bubble, followed by $N \gtrsim 60$ efolds of slow-roll inflation to seed the initial perturbations that eventually collapse to form structure and to dilute the effects of spatial curvature Ω_k . Inflation then ends as we approach a new deSitter minima (dS_{local}) with vacuum energy $\rho_{\lambda} \sim 10^{-120} M_P^4$ in accord with current observations. Reheating occurs followed by the standard hot big bang cosmology.

Several key features of this scenario lead to claims that it can be observationally tested: the spatial sections are hyperbolic and thus our observable universe has negative curvature and $\Omega_k > 0$ [17, 18]. Since inflation is supposed to begin shortly after the nucleation of the bubble claims have also been made that we may be able to see remnants of the earliest stages of slow-roll inflation in the CMB [17, 19, 20, 21]. Perhaps the most unique signature is the imprint of a collision between our bubble and another one nucleating within dS_{parent} . The standard estimate for the number of potentially observable collisions given the assumption that we do indeed inhabit a bubble nucleated within some parent deSitter false vacuum is [22, 23, 24]

$$\mathcal{N}_{col} \sim \gamma \frac{\rho_{dS_{parent}}}{\rho_{inf}} \sqrt{\Omega_k^0} \tag{1.33}$$

where $\gamma = \Gamma H_{dS_{parent}}^{-4}$, Γ is the decay rate of dS_{parent} per unit four-volume and $\rho_{dS_{parent}}$ and ρ_{inf} are the energy densities in the parent deSitter minimum and during slow-roll inflation respectively. The value of Ω_k is measured today. For CdL tunnelling in the semiclassical limit $\Gamma \sim B^2 \left| \frac{\det' S''(\phi_{bounce})}{\det S''(\phi_{fv})} \right|^{-1/2} e^{-B}$ with $B = S_{bounce} - S_{fv}$ the difference between the Euclidean actions of the corresponding CdL instanton and the field sitting at the false vacuum minimum. The prefactor arises from the second variation of the action around the bounce solution. This estimate does not account for the probability that False Vacuum Eternal Inflation of the Landscape is itself the correct description of our universe. Needless to say, detailed studies of this scenario are still mostly phenomenological.

The scenario outlined above is one of the primary motivations for the work in chapter 4, where we study collisions between vacuum bubbles, and indirectly for the companion work in chapters 2 and 3, in which we study collisions between planar walls.

1.5 Preheating

If we accept that inflation happenned within our Hubble patch then we know with certainty that the homogeneous inflaton energy must be transferred to the dense primordial plasma of Standard Model particles (quarks, gluons, leptons, photons, weak bosons) and possibly additional degrees of freedom described by some beyond the Standard Model physics. At present, the only observational knowledge we have of this transition is that the plasma must reach thermal equilibrium with a temperature $T \gtrsim 10 MeV$ in order for standard big bang nucleosynthesis to occur. Once the plasma reaches local thermal equilibrium, it subsequently evolves adiabatically as the universe expands and cools. The subsequent standard thermal history is then characterized by a constant entropy (possibly with the exception of entropy release at a high-temperature phase transition or by the decay of a thermally decoupled relic), at least until the universe becomes matter dominated and the growth of structure via gravitational collapse begins.

The earliest attempts to describe this transition are now known as the perturbative theory of reheating [25]. This approach is still widely used, and in its simplest form the decay of the inflaton ϕ is described using a phenomenological decay parameter Γ

$$\ddot{\phi} + (3H + \Gamma)\dot{\phi} + V'(\phi) = 0$$
 (1.34)

$$\dot{\rho}_R + 4H\rho_R = \Gamma \dot{\phi}^2 \tag{1.35}$$

where ρ_R is the energy density of a radiation bath. In this approximation, neither the inflaton condensate ϕ nor the radiation bath it decays into develop subhorizon inhomogeneities as a result of the reheating process.

It was later realized by Kofman, Linde and Starobinski [26] (and worked out in great detail for a specific model in [27]) that this transition may instead occur in a very different manner — the rapid nonperturbative production of inhomogeneous fluctuations (see also [28, 29]. This process is known as preheating to distinguish it from (perturbative) reheating. During the early stages of the instability fluctuations in fields coupled to the inflaton obey

$$\delta \ddot{\chi}_{i,k} + 3H\delta \dot{\chi}_{i,k} + \left(\frac{k^2}{a^2}\delta_{ij} + V_{ij}(\bar{\phi})\right)\delta\chi_{j,k} = 0$$
(1.36)

$$\ddot{\phi} + 3H\dot{\phi} + \partial_{\phi}V(\bar{\phi}) = 0 \tag{1.37}$$

where we have assumed canonical kinetic terms for the field χ_i . After inflation the mean field ϕ generally oscillates around the minimum of some potential, and as a result the fluctuations $\delta\chi_{i,k}$ obey the equation for a harmonic oscillator with time-dependent mass. For simplicity, the following discussion will assume that the eigenvectors of the Hessian matrix of the potential $V_{ij} \equiv \partial_i \partial_j V$ are time-independent. Thus, we can diagonalize the equations of motion via a time-independent redefinition of the fluctuations. For a wide range of models, the fluctuations $\delta\chi$ may experience instabilities. Roughly, these

instabilities can be classified into three general categories depending on the evolution of the effective frequency $\omega_{i,k}^2 \equiv \frac{k^2}{a^2} + \partial_{ii}V(\bar{\phi}) \equiv \frac{k^2}{a^2} + m_{eff}^2$.² In the first case, we have m_{eff}^2 < 0 shortly after inflation but with $\bar{\phi}$ far from its minimum [30, 31, 32]. This occurs when inflation ends via a second-order phase transition (as in hybrid models) or after small-field inflation. The modes with $k^2 \leq a^2 m_{eff}^2$ experience a spinodal instability and grow as $e^{|m_{eff}|t}$. In the second, known as broad parametric resonance, we have $\omega_{eff}^2 \geq 0$ but short periods of $\dot{\omega}_{eff}/\omega_{eff}^2 \gtrsim 1$ [33, 34, 27]. Typical examples include models where the inflaton couples to the preheat field via $g^2 \phi^2 \chi^2$. In this case, certain bands of wavenumbers will undergo damped oscillations with nonadiabatic kicks during which the amplitude experiences a rapid step-like increase. Finally, in tachyonic resonance the effective frequency again oscillates but now there are intervals with $\omega_{eff}^2 < 0$ [35, 36]. This occurs when the inflaton-preheat field coupling is of the form $\sigma\phi\chi^2$. $\delta\chi$ then alternates periods of damped oscillations with periods of exponential growth. Other instabilities, such as weak parametric resonance, may also occur, but the expansion of the universe often renders these instabilities inefficient in a cosmological setting [37]. Various combinations of these effects may be relevant in a given model. Cases also arise in which none of them are effective and we obtain perturbative decay.

When fluctuations grow sufficiently large, rescattering effects become important and the inflaton ϕ and preheat fields χ_i fracture into a highly inhomogeneous state. This fracturing process is both strongly nonlinear and highly inhomogeneous and therefore requires a numerical approach to investigate fully. The use of lattice numerics to investigate this process is now a well-developed field. LATTICEEASY [38] was the first publically available code designed to study dynamical evolution of scalar fields coupled to gravity. This was followed by a clever high-order symplectic time-integrator and improved spatial discretization in DEFROST [39]. HLATTICE [40] also used a symplectic integrator and allowed for evolution of the second-order gravitational equations. Improvements to the spatial discretization were made using pseudospectral methods in PSpectRe [41], although with a significantly less accurate staggered-leapfrog or RK4 time-stepping method compared to the symplectic schemes. Performance gains based on parallelization were implemented in a port of LATTICEEASY called CLUSTEREASY [42], and through GPGPU computing with PYCOOL [43]. Finally, GABE [44] uses a second-order explicit Runge-Kutta time integration to evolve arbitrary non-canonical scalar fields.

Using these codes, many interesting possible dynamics for the preheating transition have been discovered, including production of topological defects [45, 46], production of superheavy particles, creation of a second inflationary stage, production of oscillons and

²For time-dependent Hessian eigenvectors, additional types of instabilities may also arise.

Q-balls, baryogenesis [47, 48], and many more. Perhaps the most interesting of these possible dynamics are the ones that lead to potential observational signatures, including the production of gravitational waves [49, 50, 51, 52, 53, 54] and nongaussian density perturbations [55, 56, 57, 58, 59, 60, 61]. However, despite much progress, there are still many mysteries surrounding the process of reheating. In chapter 5 we will provide a new characterization of the preheating process in terms of entropy production, and also demonstrate a new mechanism for creating density perturbations from preheating.

The results in this thesis make extensive use of a new lattice code that attempts to combine the best elements of previous approaches. Novel new features include a highorder symplectic time integrator for arbitrary scalar field theories and options for either finite-difference or pseudospectral discretization. Also included are a suite of analysis routines previously unavailable to the preheating community such routines for calculation of cross-power spectra, discrete wavelet transforms of the fields, smoothing routines, and calculation of various fields not usually considered in the preheating literature. The code also comes fully MPI and OpenMP parallelized out of the box. The numerical methods are sufficiently general to allow for the study of DBI-type scalar field theories as well as non-linear sigma models. Despite their importance in high-energy model building, the preheating dynamics of these types of theories have largely been neglected in the literature. As well, the numerical techniques should also allow for a full general relativistic treatment of the problem. Work in this direction is currently underway.

1.6 Parametric Resonance and Floquet Theory

A common thread running through this thesis is the presence of resonant instabilities in the fluctuations of scalar fields around dynamical but highly symmetric backgrounds. In early universe cosmology, these instabilities are familiar from the theory of preheating as outlined in section 1.5, where the homogeneous oscillating inflaton resonantly excites inhomogeneous fluctuations in another field or itself. When preheating proceeds via the resonant excitation of fluctuations via a homogeneous oscillating inflaton, the linear dynamics are adequately described by Floquet theory. Chapter 2 will apply similar techniques to the case of fluctuations around an *inhomogeneous* oscillating background. The nonlinear dynamics explored in chapters 3,4 and 5 are also preceded by a stage of linear instability that is well approximated by Floquet theory. Here we review the basics of Floquet theory to set the stage for future chapters. Floquet theory applies to linear equations of the form

$$\frac{d\mathbf{y}}{dt} = L(t)\mathbf{y} \tag{1.38}$$

where \mathbf{y} is a vector and the matrix L is periodic L(t + T) = L(t). Here we will collect some useful properties of solutions to these types of equations. The key result for this thesis is that solutions to (1.38) of the form $\mathbf{y}(t) = e^{\mu t} \mathbf{p}(t)$ exist, with $\mathbf{p}(t + T) = \mathbf{p}(t)$ a periodic function and $\mu \in \mathbb{C}$.

Consider a fundamental matrix solution F(t) of (1.38), which is nonsingular and satisfies $\dot{F} = LF$, thus providing a complete set of solutions to (1.38). Defining the Wronskian $W(t) \equiv \det(F(t))$, we have $W(t) = W(t_0)e^{\int_{t_0}^{t} L(s)ds}$ as can easily be seen by differentiating Tr ln W. The complete dynamics of the system can be extracted from the eigenvalues and eigenvectors of the monodromy matrix

$$M \equiv F^{-1}(0)F(T)$$
 (1.39)

which can easily be shown to satisfy F(t+T) = F(t)M. Denote the eigenvalues of Mby $\Lambda_i \equiv e^{\mu_i T}$ where we have defined the Floquet exponents $\mu_i \in \mathbb{C}$. The real parts of the Floquet exponents $\mathbb{R}(\mu_i)$ are often referred to as Lyapunov exponents. We have

$$\sum_{i} \mu_{i} T = \ln \det(M) = \int_{0}^{T} dt \operatorname{Tr} L(t) \qquad \operatorname{Tr}(M) = \sum_{i} \Lambda_{i}$$
(1.40)

Let **b** be an eigenvector of M with eigenvalue $e^{\mu t}$. Consider $\mathbf{x}(t) = F(t)\mathbf{b}$, which is a solution to (1.38) and satisfies $\mathbf{x}(t+T) = F(t+T)\mathbf{b} = F(t)M\mathbf{b} = e^{\mu T}\mathbf{x}(t)$. Defining $\mathbf{p}(t) \equiv e^{-\mu t}\mathbf{x}(t)$, it is easy to show that $\mathbf{p}(t)$ is periodic. Therefore, for each eigenvector of the monodromy matrix with eigenvalue μ , we have a solution $x(t) = e^{\mu t}\mathbf{p}(t)$ with $\mathbf{p}(t+T) = \mathbf{p}(t)$.

1.7 Structure of Thesis

The theoretical ideas and analytic framework for the four papers comprising the body of this work were developed through an intense collaborative process. In addition to the people listed below, many of the ideas permeating the results presented here were influenced by other projects and discussions with Lev Kofman, Andrei Frolov, Zhiqi Huang and Neil Barnaby. In addition to being a major contributor in these collaborative efforts, I performed all of the numerical calculations presented in this thesis including the development of the required software.

Chapter 2 contains the first installment of a three-part study of colliding domain walls and will appear as *Cosmic Bubbles and Domain Walls I: Parametric Amplification of Linear Fluctuations* in collaboration with J. Richard Bond and Laura Mersini-Houghton. We consider the evolution of linear fluctuations around planar and SO(2,1)symmetric backgrounds appropriate for the case of planar domain wall and bubble collisions respectively. A series of reasonable approximations for the evolution of the symmetric background allow the fluctuations to be studied through Floquet theory for a system with many degrees of freedom. We find that fluctuations generically experience strong exponential amplification during the collisions, indicating that the problem must be treated using full lattice simulations if the walls bounce a few times.

Chapter 3 examines the full nonlinear dynamics of colliding parallel scalar field domain walls with near planar symmetry and will be submitted as *Cosmic Bubbles and Domain Walls II: Fracturing of Colliding Walls.* Using lattice simulations, we find that the exponential growth of fluctuations studied in chapter 2 completely changes the collision dynamics. The onset of nonlinearities leads to a complete breakdown of the planar symmetry, and the walls undergo a highly inhomogeneous annihilation process. At the end of this a collection of oscillons are formed from the energy stored in the domain walls.

Chapter 4 is the final installment in our study of the role instabilities play in the dynamics of colliding domain walls and will be submitted as *Cosmic Bubbles and Domain Walls III: The Role of Oscillons on Three-Dimensional Bubble Collisions*. We consider collisions between bubbles of true vacuum using three-dimensional simulations. As in the planar case, the assumption of symmetry (in this case SO(2,1)) breaks down shortly after the collision due to the onset of nonlinearity amongst the fluctuations for collisions in double-well potentials.

Chapter 5 studies the production of entropy from preheating instabilities at the end of inflation and will appear as *The Shock-in-Times of Post-Inflation Preheating* in collaboration with J. Richard Bond. We present a formalism to study entropy in systems far from equilibrium and find that for preheating models based on broad-band parametric resonance entropy production occurs during a short time interval around the onset of nonlinearities amongst the fluctuations-the so called shock-in-time. An exploration of the entropy generation under various measurement assumptions demonstrates that the shock is a robust feature. We then use this observation to study the production of nongaussian density perturbations from the modulation of coupling constants driving the preheating instability.

Chapter 6 includes conclusions and possible directions for future work.

Chapter 2

Cosmic Bubble and Domain Wall Instabilities: Parametric Amplification of Linear Fluctuations

In this chapter we consider the behaviour of linearized fluctuations around colliding scalar field domain walls in situations where the colliding walls possess a high degree of spatial symmetry. Domain walls arise when discrete symmetries are spontaneously broken and are familiar from the magnetic domains formed in ferromagnets. In the context of the early universe they can form in high-temperature or vacuum phase transitions: either through self-ordering dynamics following a rapid quench or as the walls of nucleated bubbles during a first-order transition. However, observations place restrictions on the stability of walls produced in this fashion. Domain walls and similar objects such as Dbranes are also a common ingredient in early universe model building. Examples include braneworld cosmologies in which our observable dimensional space [62, 63, 64, 65, 66, 67], inflationary cosmologies including stacks of Dbranes [68, 69], and some cyclic cosmologies [65].

In a complete theory the domain walls interact and posess their own dynamics, either inherited from an underlying scalar field theory or intrinsically in the case of Dbranes. When several such walls are present, the dynamical evolution may result in collisions. In some cases, such as the self-ordering dynamics after a quench or a rapid percolating firstorder phase transition, the domain walls form a complicated network with interactions and collisions occuring in a wide variety of orientations. However, in other scenarios the collisions posess a large amount of symmetry, such as planar symmetry or SO(2,1)symmetry. Such a highly symmetric configuration may arise from tuning of the initial conditions as in braneworld cosmologies. In other cases, the dynamics naturally leads to symmetric collisions, although the underlying theory might still require tuning to realize the appropriate dynamics. An example of this latter scenario is a first order phase transition where the bubbles expand to several times their initial radius before colliding.

In this chapter we focus on the particular case of colliding parallel planar walls formed by the condensate of some scalar field ϕ . The qualitative behaviour of the fluctuations around the planar walls also carries over to the case of collisions with an SO(2,1) symmetry. These two symmetry assumptions—planar and SO(2,1)—are widely invoked to study collisions in braneworld scenarios [70, 67, 71, 72] and false vacuum decay [73], respectively. In both cases, assuming so much symmetry reduces the underlying field equations to a one-dimensional nonlinear wave equation, which greatly simplifies the problem and has been central to many past studies of domain wall collisions. We use the solutions to these one-dimensional nonlinear wave equations as backgrounds, around which we expand our fluctuations.

An important difference between the classical and quantum problems, which has been neglected in previous work, is the extent to which the dynamical evolution preserves the initially assumed symmetries. While the *classical* dynamics may possess exact planar or SO(2,1) symmetry, the quantum fluctuations only respect these symmetries in a statistical sense. In particular, individual realizations of the quantum fluctuations will not respect the spatial symmetry exactly, although this breaking may initially be very small. These fluctuations possess their own dynamics, and it is important to test their stability. Provided the fluctuations are initially small, we can describe the first stages of their dynamics using linearized equations. If the fluctuations grow, they may have nonnegligible backreaction and rescattering effects on the symmetric part of the field, which can significantly modify the overall dynamics. A proper treatment of these effects requires studying the nonlinear problem and is the subject of chapters 3 and 4 respectively.

We restrict our considerations to two different scalar field theories possessing domain wall solutions. For simplicity, we only consider single-field models which we denote by ϕ and will refer to as the symmetry breaking field. The background spacetime is assumed to be Minkowski throughout. In addition to the choice of underlying theory, the evolution of the fluctuations depends on the particular background around which we expand. Therefore, we consider a variety of collisions in each potential. We will show that nonplanar fluctuations in ϕ can experience exponential instabilities for a broad class of collisions.

The analysis is performed using Floquet theory applied to a non-separable PDE. This approach generalizes the techniques used in preheating, where Floquet theory is applied to ODEs in order to study fluctuations around a spatially homogeneous background. For different choices of the background evolution, we find generalizations of broad parametric resonance and narrow parametric resonance to the case of fluctuations around a spatially inhomogeneous background.

Although we focus on two specific scalar field models, the dynamical mechanism that leads to the rapid growth of fluctuations is much more general. As we will explicitly demonstrate, the broad parametric resonance instability is essentially particle production in the Bogoliubov sense for fluctuations bound to the walls. These fluctuations are the transverse generalization of the Goldstone mode arising from the spontaneous breaking of translation invariance by the domain wall. Therefore, these modes exist for *any* membrane-like structure appearing in a translation invariant theory. Such membrane-like structures can include domain walls in other field theories, or Dbranes in string theory. When the two "branes" are well separated, the fluctuations are trapped by an effective potential well. As long as the shape of these wells are modified by the collision, then we expect similar instabilities to arise regardless of the underlying theory.

The remainder of this chapter explores the rich dynamics of linear fluctuations around colliding domain walls. We first introduce our two models in section 2.1 and present the domain wall solutions that each potential supports. In section 2.2 we introduce our decomposition of the field into a background and fluctuations, followed by a review of the background dynamics. The central analysis in contained in section 2.3, where we use Floquet theory to understand the dynamics of the fluctuations. We provide instability charts for the fluctuations and study the mode functions in detail. We also comment on the applicability of our results to a broader class of theories. Finally, in section 2.4 we briefly comment on the implications for SO(2,1) bubble collisions and conclude in section 2.5. Some of the more technical details explaining the construction of approximate background solutions are contained in appendix A. Details of our numerical methods and convergence tests demonstrating their superb accuracy and convergence properties are in appendix B.

2.1 Model Lagrangians and Domain Wall Solutions

We start by introducing the two potentials we will consider and reviewing the domain wall solutions they support, as well as the types of perturbations that exist around these solutions. Since we will ultimately be working in three spatial dimensions, we also discuss the embedding of lower-dimensional domain wall solutions in three dimensions. Our first choice of potential is the sine-Gordon model

$$V(\phi) = 1 - \cos(\phi)$$
. (2.1)

This potential supports a family of static inhomogeneous solutions — kinks — with profiles given by

$$\phi_{kink}^{SG}(x) = 4 \tan^{-1}(e^{x-x_0}) + 2\pi n, \qquad n \in \mathbb{Z}.$$
(2.2)

These solutions interpolate between neighbouring minima of the potential (2.1) with $\phi(\infty) = \phi(-\infty) + 2\pi$, and are the one dimensional version of domain walls. Here x_0 determines the spatial position of the kink and n is an integer determining which minima the kink interpolates between. There is also a corresponding antikink solution which is obtained by the substitution $(x - x_0) \rightarrow -(x - x_0)$. Kinks moving at a constant velocity can be obtained by Lorentz boosting the static solution.

At linear order in one spatial dimension the only normalizable localized perturbation of the kink is the zero mode corresponding to an infinitesimal translation of the center of mass

$$\delta \phi_{\text{trans}} \propto \partial_x \phi_{\text{kink}} \propto \operatorname{sech}(x - x_0).$$
 (2.3)

We will later consider planar kink solutions in 2 or more spatial dimensions, in which case these localized perturbations give rise to a spectrum of bound state fluctuations with dispersion relationship $\omega = k_{\perp}$, where k_{\perp} is the wavenumber along the directions parallel to the wall.

As a second example of a potential supporting domain wall solutions we consider the double-well

$$V(\phi) = \frac{\lambda}{4} \left(\phi^2 - \phi_0^2\right)^2 - \delta \lambda \phi_0^3 \left(\phi - \phi_0\right) + V_0$$
(2.4)

depicted in figure 2.1, with δ an adjustable parameter controlling the difference between the false and true vacuum energies and V_0 a constant.¹ As long as δ is not too large, this potential supports spatially dependent field configurations in which the field is localized near each of the minima in different regions of space, with the requisite domain-wall structures interpolating between the different regions.

A well known example occurs for $\delta = 0$ in one spatial dimension. In this case, the

¹Unless explicitly indicated, for the remainder of the paper we measure all dimensionful quantities in units of $m \equiv \sqrt{\lambda}\phi_0$, with the exception of the fields measured in units of ϕ_0 and the potential in units of $m^2\phi_0^2$.



Figure 2.1: Plots of the double-well potential for several choices of the parameter δ controlling the difference in potential energies between the two wells. Domain wall solutions interpolate between spatial regions where the field is near the false vacuum at $\phi_{false} \approx -1$ and regions where it is near the true vacuum $\phi_{true} \approx 1$.

kink solution located at x_0 is given by

$$\phi_{kink}^{DW}(x) = \phi_0 \tanh\left(\frac{\sqrt{\lambda}\phi_0(x-x_o)}{\sqrt{2}}\right) \tag{2.5}$$

with a corresponding antikink solution again obtained by replacing $(x - x_0) \rightarrow -(x - x_0)$. Once again, moving kinks are obtained by Lorentz boosting the above solution. Unlike the sine-Gordon model, which possesses a single bound state excitation, the one-dimensional double-well kink has two localized normalizable linear perturbations $\delta\phi(x,t)$. There is also a continuum of delocalized radiative modes with frequencies $\omega^2 \geq 2$. The localized perturbations are often referred to as the translation mode

$$\delta\phi_0 \propto \partial_x \phi_{kink} \propto \operatorname{sech}^2\left(\frac{x-x_o}{\sqrt{2}}\right)$$
 (2.6)

and the shape mode

$$\delta\phi_1 \propto \cos(\omega t) \sinh\left(\frac{x-x_0}{\sqrt{2}}\right) \operatorname{sech}^2\left(\frac{x-x_0}{\sqrt{2}}\right) \quad \text{with} \quad \omega^2 = \frac{3}{2}.$$
 (2.7)

As before, the translation mode corresponds to a spatial translation of the center of the kink and is thus analogous to the sine-Gordon zero mode. The shape mode is an internal excitation which can be thought of as an oscillating wall width.

Kink-like solutions continue to exist as we deform the potential by increasing δ . How-

ever, the potential energy difference between the two minima causes the "wall" to accelerate, so the kink solutions are no longer time-independent in inertial reference frames.

In three spatial dimensions domain walls become embedded two-dimensional hypersurfaces with some small but finite width. In this chapter we will consider two cases that possess a high degree of spatial symmetry. The first case, which is our main focus, is planar walls generated by extending the sine-Gordon and double well kink solutions discussed above in the additional two spatial directions. Second, we will study bubbles of "true vacuum" nucleating within the false vacuum in the double-well potential, restricting to choices of δ in the double-well potential for which these false vacuum bubbles are well described by the Coleman-deLuccia (CdL) instanton [74, 75, 15]. In Minkowski space at zero temperature, the most likely initial bubble profile possesses an SO(4) symmetry in Euclidean signature [76] with profile determined by

$$\frac{\partial^2 \phi}{\partial r_E^2} + \frac{d}{r_E} \frac{\partial \phi}{\partial r_E} - V'(\phi) = 0$$
(2.8)

where $r_E^2 = \mathbf{r}^2 + \tau^2$, τ is the Euclidean time and d is the number of spatial dimensions. The initial bubble profile is obtained by analytically continuing back to real time. In the thin wall limit (valid if the initial radius of the bubble is much greater than the thickness of the wall), the friction term in (2.8) is dropped and we are left with the same equation as for a domain wall in the corresponding 1 + 1-dimensional theory. In this limit, the initial bubble radius is given by

$$R_{init} = \frac{3\sigma}{\Delta\rho} = \frac{\sqrt{2}}{\delta},\tag{2.9a}$$

$$\sigma = \int dr [\partial_r \phi(t=0)]^2 = 2\sqrt{2}\phi_0^2 m/3, \qquad (2.9b)$$

where $\Delta \rho$ is the difference in energy density between the true and false vacuum and σ is the surface tension of the wall. In the final equalities on each line we made use of the specific form of the potential (2.4).

Our choice of planar walls and vacuum bubbles is motivated in part by their prevalence in many cosmological scenarios. In particular, planar walls form the basis for braneworld cosmological models [63, 62] and proposals for bouncing cosmologies [65]. As well, stacks of parallel D-branes hidden in some extra dimensions are an integral part of brane inflation models [69, 68]. Meanwhile, vacuum bubbles arise in open inflationary models based on false vacuum decay [77, 78, 79, 80, 81]. Similarly, first-order high temperature phase transitions proceed via the nucleation of bubbles (albeit without the boost symmetries of the vacuum case) [82, 83]. Thus, backgrounds with a high degree of symmetry are relevant in a variety of contexts while also leading to a tractable analysis for the fluctuations.

2.2 Dynamics of Planar Symmetric Collisions

We now study (nonplanar) fluctuations around colliding parallel planar domain walls. We treat this case first because the scale associated with the overall radius of the bubbles does not enter the problem, so parallel domain walls constitute a slightly simpler arena in which to illustrate the underlying fluctuation dynamics. In the limit that the bubbles have radii much larger than any other relevant scale in the problem we also expect bubble collisions to be reasonably approximated by two colliding planar walls. However, we should mention that our analysis will be restricted to mildly relativistic wall collisions, while (vacuum) bubbles in the limit of large radii have walls that move at very nearly the speed of light. Thus, a direct application of our results is not possible, although we believe that all of the qualitative features we discuss will continue to hold.

2.2.1 General Formalism

Our setup consists of a kink starting at $x = -x_{init}$ and moving to the right and an antikink starting at $x = x_{init}$ and moving to the left. For ease of nomenclature, we will refer to this as a $K\bar{K}$ pair. We take the collision axis to be the x direction and split the field as

$$\phi(x, y, z, t) = \phi_{bg}(x, t) + \delta\phi(x, y, z, t)$$
(2.10)

where the fluctuations satisfy $\langle \delta \phi(x, y, z, t) \rangle = 0$. The planar symmetry allows us to approximate ensemble averages with averages over the (y, z) plane. Before solving for the fluctuation field we must find the background solution around which we will perturb. If backreaction and rescattering effects are ignored the background field ϕ_{bg} undergoes the same dynamic evolution as in 1+1-dimensions, namely

$$\frac{\partial^2 \phi_{bg}}{\partial t^2} - \frac{\partial^2 \phi_{bg}}{\partial x^2} + V'(\phi_{bg}) = 0$$
(2.11)

with initial conditions given by the $K\bar{K}$ pair. Meanwhile, the linearized equation for the fluctuations is

$$\frac{\partial^2 \widetilde{\delta \phi}_{\mathbf{k}_\perp}}{\partial t^2} - \frac{\partial^2 \widetilde{\delta \phi}_{\mathbf{k}_\perp}}{\partial x^2} + \left(k_\perp^2 + V''(\phi_{bg}(x,t))\right) \widetilde{\delta \phi}_{\mathbf{k}_\perp} = 0$$
(2.12)

where $\widetilde{\delta\phi}_{\mathbf{k}_{\perp}} = \frac{1}{2\pi} \int dy dz e^{i(k_y y + k_z z)} \delta\phi$ is the 2d Fourier transform of $\delta\phi$ in the directions transverse to the collision axis and $k_{\perp}^2 \equiv k_y^2 + k_z^2$. For the remainder of the paper, we refer to these planar symmetry breaking flucutations as transverse. At this level of approximation, fluctuations $\delta\phi$ behave as a free field with a time and x-dependent effective mass $(V''(\phi_{bg}(x,t)))$ determined independently by the background evolution. For intuition, it is perhaps easiest to discretize the x direction and view the system (for each choice of k_{\perp}) as a collection of coupled oscillators. In real space this coupling occurs via our choice of discretization of the laplacian term, while in momentum space (along x) the oscillators couple via the Fourier transform of V''(x,t). This line of thinking suggests that for a given V''(x,t) the (time-dependent) 'normal' mode oscillations yield a simple description of the system. After considering the typical behaviour of solutions to the background equations, we will show that this approach can be carried out approximately and provides very useful insight into the behaviour of the fluctuations. Of course, the rate at which our discretized system converges to the continuum result depends upon our choice of spatial discretization. In this paper we use a Fourier pseudospectral approximation to discretize ∂_{xx} , thus obtaining exponential convergence as we increase the number of grid sites for a fixed box size. Details about our precise numerical procedures as a well as a demonstration of the superb convergence properties of our techniques can be found in appendix B.

2.2.2 Dynamics of the Background

We now briefly review the background dynamics for colliding planar domain walls in our two chosen potentials, and determine the typical form of $V''(\phi_{bg}(x,t))$ to use as input in the fluctuation equation (2.12). As noted above the assumed planar symmetry allows us to reduce the background dynamics to the study of a $K\bar{K}$ pair interacting in 1+1-dimensions. The sine-Gordon model is particularly useful because in one spatial dimension it is integrable and the kink and antikink solutions are true solitons, which interact with each other while preserving their shapes at early and late times. More importantly, there exist analytically known periodic breather solutions

$$\phi_{breather} = 4 \tan^{-1} \left(\frac{\cos(\gamma_v vt)}{v \cosh(\gamma_v x)} \right) , \qquad (2.13)$$

where v > 0 is a free parameter determining the properties of the breather solution and we have defined $\gamma_v \equiv (1+v^2)^{-1/2}$ [84]. In the case $v \ll 1$ the kink and antikink pair are well separated at t = 0 and have initial positions $x_{init} \approx \pm \sqrt{1+v^2} \ln(0.5v)$. For larger values
of v, the kink and antikink are much more tightly bound and the breather is instead a localized oscillating blob of field with size $r_{breather} = \sqrt{1 + v^2} \cosh^{-1}(1/v \tan(\phi_{edge}/4)) \sim$ $-\sqrt{1 + v^2} \ln(\phi_{edge}v/8)$, where we define the edge of the breather as the point where $\phi_{breather}(t = 0, r_{breather}) = \phi_{edge}$ and assumed $\phi_{edge}v \ll 1$ in the last approximate equality. Some representative examples are shown in Fig. 2.2.



Figure 2.2: Sample breather dynamics for three representative cases corresponding to $v = (\sqrt{2} - 1)^{-1}, 1, 0.01$: initial ϕ profiles (top left), $\phi_{breather}$ for $v = (\sqrt{2} - 1)^{-1}$ (top right), v = 1 (bottom left) and v = 0.01 (bottom right).

Now we consider the more complicated case of the double well (2.4). Because we do not have exact solutions to (2.11) we must resort to numerical simulations. We use a Gauss-Legendre time-integration combined with a Fourier pseudospectral discretization, allowing us to obtain machine precision results for both the spatial discretization and time-evolution. Details can be found in appendix B. This problem has been studied by many authors and leads to a rich phenomenology [85, 86, 87, 88, 89]. The kinks in this model are solitary waves rather than true solitons, and thus they emit radiation when they interact. Combined with excitation of the internal mode during collisions, this means that the motion is no longer exactly periodic. We restrict our review to a kink-antikink pair described by some initial separation and relative velocities, and cover only a few salient features of the interaction. For more general setups a wide range of interesting phenomenology arises; the interested reader should consult the previously cited works for more details.

To simplify the analysis, we work in the center of mass frame and take the initial kink and antikink speeds v (not to be confused with the breather parameter above) and separations as free parameters. Some illustrative examples of the dynamics are shown in Fig. 2.3. In the symmetric well the attractive force between the kink and



Figure 2.3: Some sample dynamics of kink-antikink collisions in the double-well potential. We plot the value of the field ϕ as a function of time and position along the collision axis. Red corresponds to field values on the true vacuum side of the potential, blue to values on the false vacuum side, and white to values near the top of the potential. The spatial coordinates are chosen so that the collision occurs at the origin. Three choices of initial velocity in the symmetric well illustrate three different types of behaviour: for v = 0.05 (top left) the $K\bar{K}$ pair capture each other and form a long-lived bound state rather than immediately annihilating. For v = 0.2 the kinks are in an escape resonance (top right), and a collision above the critical escape velocity (v = 0.3) appears in the bottom left. In the bottom-right is the behaviour for an asymmetric double well with $\delta = 1/30$ and the $K\bar{K}$ pair starting from rest at a separation $md_{sep} = 16$. In all cases, the oscillation of the internal shape mode is visible as an oscillating wall width.

antikink decreases exponentially at large separations, and as a result unbound motions with the kink escaping back to infinity are possible. For low initial velocities ($v \leq$ 0.15), the $K\bar{K}$ pair always capture each other after colliding and do not escape back to infinity – see Fig. 2.3, top left, for an example with v = 0.05. Rather than immediately

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annhilating into radiation, the kinks bounce off each other several times and then settle into a long-lived oscillatory blob known as an oscillon (here living in one dimension). During each oscillation, some energy is radiated away, so this localized state eventually decays. However, the rate of energy loss is slow and the oscillons can persist for thousands of oscillations (or more) before finally disappearing.

As the incident velocity increases, the kinks enter a "resonant escape" regime characterized by bands of incident velocities in which the two kinks eventually escape back to infinity separated by bands in which they trap each other, as in the low velocity limit [88]. Within each escape band, the number of bounces the walls undergo before escaping back to infinity is a very complicated function of the incident velocity. This seemingly strange behaviour is usually attributed to the shape mode. At each collision, nonlinear interactions transfer some of the kinks' translational kinetic energy into a homogeneous excitation of the shape mode or vice versa. The direction of energy transfer depends on the oscillation phases of the two shape modes (one on the kink and one on the antikink). As a result, the kinetic energy of the kinks can decrease in one bounce as the shape mode is excited causing the $K\bar{K}$ pair to become temporarily trapped. In the subsequent collision some of this stored energy can then be transferred back into overall translational motion, giving the kink and antikink enough translational kinetic energy to escape back to infinity. An example of this behaviour for v = 0.2 is illustrated in the upper right panel of Fig. 2.3. The $K\bar{K}$ pair collides, losing some energy to radiation and exciting the internal shape mode. At the second bounce, energy is transferred from the shape mode back into translational kinetic energy and the kinks escape each other.

Finally, there is a critical velocity above which the $K\bar{K}$ pair will always interact exactly once before escaping back to infinity. During the collision, some of the energy escapes as radiation. The remaining energy flows between the shape and translational modes of the kink. Provided that the shape modes are not initially excited, the outgoing velocities of the two walls are thus always less than their incident velocities. This sort of interaction appears in the bottom left panel of Fig. 2.3.

When we take $\delta > 0$, i.e. make the double well asymmetric, the difference in vacuum energies across the kink causes an acceleration toward the false vacuum side, and the kink and antikink experience an approximately constant attractive force at large separations. As a result, the kinks are no longer able to escape back to infinity and will always undergo multiple collisions while slowly radiating energy. Eventually, an oscillon forms at the location of the original collision. See the bottom right panel of Fig. 2.3 for an example of kinks interacting in an asymmetric double well.

In all cases discussed above the shape mode is excited by the collisions. This is visible

as an oscillating wall thickness, as seen in Fig. 2.3.

2.3 Dynamics of Linear Fluctuations

In the previous section we briefly reviewed the dynamics of the planar symmetric background field. The key feature for our analysis is the presence of oscillatory behaviour: the walls typically bounce off each other many times or settle into a localized bound state rather than immediately annihilating, and internal vibration modes of the walls are excited in the collisions. We now study the transverse fluctuations described by (2.12) in these types of oscillating backgrounds, focusing on whether small initial (transverse) perturbations to the background dynamics illustrated in Fig. 2.2 and Fig. 2.3 can be amplified to the extent that they become important to the full field dynamics. The results presented in this section are the main results of our analysis. We will present (in)stability charts for linear fluctuations around various choices of planar collision backgrounds in both the sine-Gordon and double-well potentials. In addition we study several representative choices of the unstable mode functions, which provides insight into the qualitative structure of the instability charts. We find inhomogeneous generalizations of both broad and narrow parametric resonance, demonstrating that the transverse fluctuations in ϕ can indeed grow rapidly as a result of collisions.

Amplification can happen several ways. Radiative modes can be excited in the collision region and then subsequently propagate into the bulk, which is the transverse generalization of the outgoing radiation seen in the (1+1)-d simulations above. However, these excitations carry energy away from the collision and cannot experience sustained growth. More interesting is the pumping of fluctuations bound to the kinks (in cases where the walls separate widely), or bound in the oscillating effective mass well created by the late-time one-dimensional oscillons. A similar effect for fermionic degrees of freedom has been studied by several authors [90, 91, 92], and for an additional field coupled to ϕ in [93]. We only consider fluctuations in the field ϕ itself, which is required in a consistent quantum treatment, and we do not appeal to additional fields coupled to ϕ in order to obtain growing fluctuations. As well, unlike the authors of [93] we include the deformation of the spatial structure in V'' and couplings between the bound modes and radiative modes.

Although strictly true only for the case of the breathers, we approximate the oscillatory evolution as periodic, allowing us to use Floquet theory to quantitatively study instabilities in $\delta\phi$. The resulting Floquet modes then provide a time-dependent normal mode decomposition in which the evolution of the system is simple. The periodic approximation is very accurate for the late-time bound states and excitations of the shape mode in the planar background, so that the Floquet analysis is well justified for those two cases. When the walls repeatedly bounce off of each other, we will show that the amplification happens in a very short time interval around the time of collision. The short bursts of growth in the fluctuations remain when the bouncing is not periodic, so again the Floquet analysis provides an accurate account of the full evolution.

In early universe cosmology and nonequilibrium field theory, Floquet theory is familiar from the theory of preheating [26, 29, 27, 94]. For these problems the background is spatially homogeneous, which leads to two simplifications. First, the background evolution is described by a nonlinear ODE instead of a nonlinear PDE, and analytic solutions are known in many interesting cases [33, 34, 35, 36]. Second, the equation for the fluctuations is diagonalized by the 3D Fourier transform, so instead of involving a large number of coupled oscillators the problem reduces to a collection of decoupled oscillators with periodically changing masses $\ddot{\phi}_k + (k^2 + m_{eff}^2(t))\delta\phi_k = 0$ where $m_{eff}^2(t) = V''(\phi_{bg}(t))$ depends only on time and $k^2 = k_x^2 + k_y^2 + k_z^2$ is now the full three-dimensional wavenumber. Solutions to this equation have the well known form $\delta \phi_k(t) \sim P(t) e^{\mu t}$ with the (possibly complex) exponent μ , known as a Floquet exponent, and P(t) a function satisfying P(t+T) = P(t), where T is the period of the oscillation in V''. The μ 's depend parametrically on k as well as the functional form of m_{eff}^2 . Typically bands of stable $(\max(Re(\mu_k) = 0))$ and unstable $(\max(Re(\mu_k)) > 0)$ modes appear as we vary k while holding the form of m_{eff}^2 fixed.² There are a variety of underlying mechanisms that can be responsible for the instability, such as tachyonic resonance, narrow resonance and broad parametric resonance [29, 27, 35].

For a spatially dependent effective mass, which is the case we consider, the form of the decoupled solutions generalizes to $\delta \phi = F(x,t)e^{\mu t}$, where the profile function F(x,t) satisfies F(x,t+T) = F(x,t). This is easily seen by discretizing the x direction and placing the system in a finite box. The equations of motion then take the form $\dot{\mathbf{v}} = M(t)\mathbf{v}$, with M(t) a periodic matrix and $\mathbf{v} = (\delta \vec{\phi}^T, \delta \vec{\phi}^T)^T$. Such equations fall within the domain of Floquet theory and solutions of the form given above thus exist (modulo issues of convergence on taking the continuum limit).

To treat the case of spatially inhomogeneous masses, we first discretize the fields

²This condition, rather than $\max(Re(\mu_k) < 0)$, is true for the wave equations we study here. This follows from the fact that the linear operator $M(t) = \begin{pmatrix} 0 & \mathbb{I} \\ \partial_{xx} - V'' & 0 \end{pmatrix}$ describing the evolution of $(\delta\phi, \dot{\delta\phi})^T$ has Tr(M) = 0. We must then have $\sum_i \mu_i = 0$ so that the presence of a single negative Lyapunov exponent necessarily creates a positive Lyapunov exponent. For the homogeneous case this requirement becomes $\max(Re(\mu_k) = 0)$.

 $\delta \phi_{k_{\perp}}$ on a lattice of N points labelled by index *i*. We denote the field value at the *i*th lattice site by ϕ_i and the corresponding field by the vector $\delta \phi$. Next, we consider 2N linearly independent solutions $(\delta \phi^{(j)}(t), \delta \dot{\phi}^{(j)}(t))$ to (2.12). Using this set of solutions, we form a 2Nx2N fundamental matrix solution F(t). For simplicity, we choose our initial conditions such that the *j*th row of F(t) is given by the solution with initial condition

$$\delta\phi_i^{(j)}(0) = \begin{cases} \delta_{i,j} & \text{for } j \le N \\ 0 & \text{for } j > N \end{cases} \qquad \delta\dot{\phi}_i^{(j)}(0) = \begin{cases} 0 & \text{for } j \le N \\ \delta_{i,j} & \text{for } j > N \end{cases}$$
(2.14)

Of course, we can construct our fundamental matrix from any complete set of initial states, and we verified that choosing an orthonormal basis in Fourier space reproduced the results we present in the remainder of this chapter. Finally, the Floquet exponents are related to the eigenvalues Λ_n of $F(0)^{-1}F(T)$ via $\Lambda_n = e^{\mu_n T}$, with the initial conditions for the mode functions given by $F(0)\mathbf{b}^{(n)}$, where $\mathbf{b}^{(n)}$ is an eigenvector of $F(0)^{-1}F(T)$ with corresponding eigenvalue Λ_n . For each choice of k_{\perp}^2 and effective mass there are 2N such exponents, but we focus on the exponent with the largest real part (*i.e.* the largest Lyapunov exponent). For notational simplicity we refer to this maximal real part of a Floquet exponent as $\mu_{max} \equiv \max Re(\mu)$. This method allows us to find any exponentially growing instabilities given some fixed background evolution, but it is completely blind to other more slowly growing instabilities. In particular, power law growth results when there are degenerate Floquet exponents, and we might expect this to be common in the continuum limit given that we then have infinitely many oscillators. However, we are ultimately interested in the quantum problem where we must integrate over all possible modes, and therefore we expect the exponentially growing ones to be the most important dynamically and thus most interesting for our purposes.

We first consider fluctuations in the sine-Gordon model, using the breather solution as a background. We then consider the behaviour of fluctuations around various backgrounds supported by the double well potential.

2.3.1 Sine-Gordon Potential

Using the planar symmetric sine-Gordon breather as the background circumvents the problem of finding appropriate approximations to ϕ_{bg} . Furthermore, as bound states of a kink-antikink pair, breathers resemble the states arising from collisions in the double well and can be used to gain some qualitative insight into the dynamics of the fluctuations for

the double well as well. The exact equation for linear fluctuations around the breather is

$$\frac{\partial^2 \widetilde{\delta \phi}_{\mathbf{k}_{\perp}}}{\partial t^2} - \frac{\partial^2 \widetilde{\delta \phi}_{\mathbf{k}_{\perp}}}{\partial x^2} + \left[k_{\perp}^2 + \cos\left(4 \tan^{-1}\left(\frac{\cos(\gamma_v vt)}{v \cosh(\gamma_v x)}\right)\right) \right] \widetilde{\delta \phi}_{\mathbf{k}_{\perp}} = 0.$$
(2.15)

The initial profile of V''(x,t) for several representative choices of v can be found in the right panel of Fig. 2.4, with the subsequent time evolution for these same three choices in Fig. 2.5, Fig. 2.8 and Fig. 2.10.

The left panel of Fig. 2.4 is an instability chart in which $\mu_{max}T_{breather}$ appears as a function of $k_{\perp}^2(1+v^2)/v^2$ as we vary v^{-1} . There are several generic features of note. First, $\mu_{max} = 0$ when $k_{\perp}^2 = 0$ for all values of the parameter v considered. Since the $k_{\perp} = 0$ modes are part of the planar symmetric system, this is a validation of our numerical procedure since no exponentially growing modes exist in the sine-Gordon model in 1+1-dimension.³ For $v^{-1} < 1$, V''(x,t) has the form of a single well whose depth oscillates with time. There is a single instability band, the width and strength of which increase monotonically as v decreases, which corresponds to moving to the right on the instability chart. Once $v^{-1} > 1$ the kink and antikink become less tightly bound and V'' develops a pair of local minima separated by a small bump. An additional instability band then appears on the chart. As we continue to increase v^{-1} (decrease v), the kink and antikink begin to separate from each other during the motion, and the two local minima in V'' develop into two well separated wells as seen for v = 0.01 in the right panel of Fig. 2.4. Additional instability bands being to appear. Each of these bands grows quickly with increasing v^{-1} and approach a nearly constant width, although the strength of the instability (per breather period) within each band continues to increase. However, T_{breather} is simultaneously increasing $\sim v^{-1}\sqrt{1+v^2}$ and this results in a decrease of the maximal floquet exponent if measured in units of t.

We now study the properties of the mode functions in various regimes of the instability chart. This gives us insight into the qualitative features of the instability chart, and also sheds light on the dynamical mechanism responsible for the amplification. First we consider the $v \ll 1$ limit. The breather is effectively a weakly bound kink-antikink pair which repeatedly approach and pass through each other. The resulting evolution of V'' is illustrated in the left panel of Fig. 2.5 for the specific case of v = 0.01. Due to the reflection symmetry of the potential about any of its minima, the period $T_{V''} = \pi \sqrt{1 + v^2}/v$ of the effective mass is half the period $T_{breather} = 2\pi \sqrt{1 + v^2}/v$ of the breather. For most of

³Of course, weaker non-exponential instabilities or transient instabilities are still allowed. For example, for a given v we could add a perturbation such that the new field configuration corresponds to a breather with $\tilde{v} \neq v$. Since the period of the breather depends on v, the perturbation $\delta \phi = \phi_{breather,\tilde{v}} - \phi_{breather,v}$ will grow initially but will not experience unchecked exponential growth.



Figure 2.4: Left: The largest real part of a Floquet exponent per period of the breather $(\mu_{max}T_{breather})$ for $\ddot{f} - f_{xx} + \left[k_{\perp}^2 + \cos\left(4\tan^{-1}\left(\frac{\cos(vt/\sqrt{1+v^2})}{v\cosh(x/\sqrt{1+v^2})}\right)\right)\right]f = 0$ as a function of $k_{\perp}^2(1+v^2)/v^2$ and v^{-1} . The left portion of the chart $(v^{-1} \leq 1)$ corresponds to the small amplitude breathers while the right portion $(v^{-1} \gtrsim 1)$ corresponds to the large amplitude breathers. Right: $V''(\phi_{breather}(x,0))$ for three representative choices of v. For $v \geq 1$, there is a localized blob with a single minimum. We have plotted $v = (\sqrt{2} - 1)^{-1}$ and v = 1 corresponding to the cases when the middle of the breather just reaches $V''(\phi) = 0$ and $V''(\phi) = -1$ respectively. For smaller v, the single blob instead develops a pair of minima, with the formation of two distinct wells when $v \ll 1$ corresponding to the well-separated kink and antikink.

the evolution, V'' has two distinct wells corresponding to the well separated kink and antikink. In the 1d field theory, each of these wells has a zero mode (the translation mode) associated with it. When we allow for the transverse fluctuations, the zero mode leads to a continuum of bound excited states with dispersion relationship $\omega_{bound} = k_{\perp} > 0$. As the breather evolves, the two wells in V'' periodically come together and "annihilate" each other as seen in the left panel of Fig. 2.5. During these brief moments when the two wells have disappeared, the bound states cease to be approximate eigenstates of the system. This temporary annihilation of the wells allows for the rapid amplification of bound fluctuations. Whether or not a particular k_{\perp} receives coherent contributions to its amplitude at successive annihilations depends on the phase $\omega_{bound}T_{V''} \sim k_{\perp}\sqrt{1+v^2}/v$ accumulated between collisions. In the stability chart this dependence on the accumulated phase manifests as the repeating structure in $k_{\perp}^2(1+v^2)/v^2$. Of course, this process is very similar to the familiar case of broad parametric resonance [27], in which short but large violations of adiabaticity $(|\dot{\Omega}/\Omega^2| \gtrsim 1)$ of a harmonic oscillator with a timedependent frequency $\Omega(t)$ lead to bursts of bound state "particle production" in the Bogolyubov sense. For the familiar homogeneous case the nonadiabaticity is captured by a time-dependent frequency alone, while for the inhomogeneous background we have the additional effect of a strong deformation of the spatial properties of the effective mass.

To illustrate this process with a specific example, we plot various aspects of the fastest growing transverse Floquet mode for a breather with v = 0.01 and $k_{\perp}^2 = 0.05$ in Fig. 2.5 and Fig. 2.6. As evident in the bottom panel of Fig. 2.5 the mode function is strongly peaked around the locations of the kink and antikink. The mode function also experiences large jumps around each collision, exactly as expected from our discussion in the previous paragraph. To further illustrate the rapid amplification of bound fluctuations at the collisions, the top left panel of Fig. 2.6 shows the value of $\delta\phi$ at the leftmost instaneous minimum of V'' as a function of time, along with the value V''_{min} at this minimum. During the collision, V''_{min} rapidly changes from -1 to 1 and $\delta\phi$ experiences a nearly instantaneous increase in its amplitude. To quantify the growth of fluctuations, it is useful to introduce the effective particle number

$$n_{eff}^{\omega_{bound}} + \frac{1}{2} = \int dx \frac{1}{2k_{\perp}} (k_{\perp}^2 \delta \phi_{k_{\perp}}^2 + \delta \dot{\phi}_{k_{\perp}}^2)$$
(2.16)

which (modulo contributions from the bulk) gives the occupation number of massless transverse fluctuations bound to either the kink or antikink. For a bound fluctuation on an isolated kink with transverse wavenumber k_{\perp} this quantity is constant, so changes in its value allow us to track the production of fluctuations. From the top right panel of Fig. 2.6, we see that between collisions $n_{eff}^{\omega_{bound}}$ is constant and undergoes nearly discontinuous jumps during the short collisions between the kink and antikink.

The above analysis confirms our intuitive explanation that the exponential growth is associated with production of bound state fluctuations. However, several additional questions remain that we will now address. First, the reflection symmetry of the mode functions around the origin, as seen in the bottom panel of figure 2.5, somewhat obscures the process by which the amplification is occurring. For example, we don't know if amplification occurs from a pumping of bound modes on both the kink and antikink at the same time, from a pumping of only modes on the kink (or antikink) at all times, or from a reflection and growth of the bound modes in only the left (or right) well, because a symmetric mode function cannot distinguish between these processes.⁴ Second, it is also unclear whether the stability bands arise primarily from dissipation of fluctuations into the bulk, or primarily from phase interference between bound fluctuations. Finally, a somewhat surprising feature of the the mode function is the long radiative tail that

⁴Actually, we can distinguish the first case from the other two by looking at the second largest eigenvalue. If pumping only occurred on one kink at a time, then the appropriate linear combinations of only pumping on the kink or only pumping on the antikink would result in two nearly degenerate Floquet exponents. Although we don't include the results here, we have done this for the sine-Gordon model. Within the instability bands, the second largest Floquet exponent was always much smaller than the largest one.



Figure 2.5: Effective mass and mode functions for a large amplitude v = 0.01 breather and $k_{\perp}^2 = 0.05$. Top left: $V''(\phi_{breather})$ for the given breather background and the instantaneous location of the kink, $x_{kink} = -\frac{\operatorname{sign}(\cos(\gamma vt))}{\gamma} \cosh^{-1}\left(\frac{|\cos(\gamma vt)|}{v}\right)$ (red line). Bottom: The corresponding mode function, illustrating the constant amplitude oscillations around the kink and antkink while they are well-separated, and the rapid growth in the short interval during which they collide. Radiation moving away from the collision is also visible. Top right: Initial conditions for the mode function, illustrating both the localization near the kink and antikink, and the extended radiating tail.



Figure 2.6: Two illustrations of the growth of bound state fluctuations around a v = 0.01 breather with $k_{\perp}^2 = 0.05$. Left: Mode function evaluated at the instantaneous location of the leftmost kink, defined as the leftmost minimum of V''. For reference we also include the value of V'' at the minimum. Right: The occupation number of bound state fluctuations as measured by (2.16). Since we use unnormalized mode functions, we don't have $n_{eff} = 0$ initially.

extends far from the spatially localized breather into the bulk.

To gain further insight into these issues, we consider the evolution of the fluctuations given an initial condition $\delta\phi_{init} = \operatorname{sech}(x - x_K)$ with $x_K = -\sqrt{1 + v^2} \log(v/2)$ corresponding to a bound state fluctuation on the leftmost kink (or antikink) of the breather solution. Fig. 2.7 shows the evolution of this initial state for several choices of k_{\perp}^2 , illustrating various aspects of the behaviour. For each run, we show the evolution of $n_{eff}^{\omega_{bound}}$ defined in (2.16). Since our setup is no longer symmetric about the origin, we also plot n_{eff}^{left} and n_{eff}^{right} , in which we perform the integral only over the left half and right half of the domain respectively. For $k_{\perp}^2 = 0.83$, a large amount of radiation is produced in the collisions, so we have also calculated n_{eff} over the entire integration domain (which had length $L \approx 1174$) and also restricted to the interval |x| < 25 in order to separate out the bound fluctuations from the radiation. To illustrate the growth of the bound fluctuations, we also plot $\delta\phi_{wall}$ evaluated at the instantaneous left and right minima of V''.⁵ Finally, we show the profile of $\delta\phi$ either for the entire simulation or else several times around the first collision of the kink and antikink.

As can be seen, in each collision some radiation is released from the collision region. As k_{\perp} is increased the fluctuations appear to be less tightly bound to the kink, and the amplitude of radiation produced in the collisions tends to be larger. This is likely the origin of the decreasing amplitude of μ_{max} at the center of the instability bands as

⁵This provides a useful definition of the instantaneous location of the kink and antikink.

we increase k_{\perp} while holding v constant. Incidentally, this emitted radiation clarifies the origin of the long radiative tail we found for the mode function above. Since the fluctuations obey a linear equation, the amount of radiation produced depends on both the oscillation phase and the amplitude of the bound mode. As the mode function grows, this leads to the production of radiation with an identical spatial profile to that produced in the previous collision but a larger amplitude. These figures also suggest that whether or not a particular value of k_{\perp} will experience an exponential instability is primarily determined by the phase interference between the bound state fluctuations. This can occur either because the fluctuations are not excited in any of the collisions (as in the case $k_{\perp}^2 = 4.006 \times 10^{-4}$), or because the fluctuations are excited in one collision, but then de-excited due to phase interference in the subsequent collision (as in the case $k_{\perp}^2 = 0.83$). Finally, as expected from studying the second largest Floquet exponent (see footnote 4), we also see that for the unstable modes the excitation tends to occur on both the kink and antikink simultaneously. On the other hand, when the modes are drawn from a stability band the fluctuations on the kink and antikink no longer experience the same degree of excitation or deexcitation at each collision.

We now turn to the case of larger v's, where we no longer have a well-defined kink and antikink at any point in the breather's motion. First take v = 1 and $k_{\perp}^2 = 0.35$, which is located near the center of the instability band. For this choice of v, the middle of the breather just makes it to a maximum of the potential every half oscillation. Therefore, V''(x,t) has the form of a single oscillating well whose middle oscillates between -1 and 1 and asymptotes to 1 at ∞ , as illustrated in Fig. 2.8. The kink and antikink are now so tightly bound that they never have separate identities and the notion of particles bound to the kink and particles bound to the antikink is ill-defined. As a result, our previous intuition based on the creation of fluctuations bound to the individual kink and antikink no longer applies. Instead, we expect the pumping to occur more smoothly and be localized in the region of the breather. As seen in the bottom panel of Fig. 2.8, this is indeed the case. The mode function looks like a smoothly oscillating function whose amplitude grows exponentially and satisfies $\delta \phi(x, t+T_{V''}) = -\delta \phi(x, t)e^{\mu T_{V''}}$, so the period of the oscillation is the period of the breather not the period of the effective mass. One way to see this is to consider the quantity

$$n_{eff}^{(\omega_{breather})} + \frac{1}{2} \equiv \frac{1}{2\omega_{breather}} \int dx \left(\delta\dot{\phi}^2 + \omega_{breather}^2\delta\phi^2\right)$$
(2.17)

which is an effective particle number, like (2.16) for the $v \ll 1$ breathers, but modified to account for the fact that in this case the oscillation frequency of the breather dominates



Figure 2.7: Evolution of $\delta\phi$ for initial condition $\delta\phi_{\text{init}} = \operatorname{sech}(x - x_k)$ with $x_k = \sqrt{1 + v^2} \log(v/2)$. We have taken v = 0.01 and four choices of k_{\perp}^2 , illustrating various types of behaviour. For each case, we plot the effective particle number defined in (2.16) (*left*), the value of the field at the instantaneous locations of the two minima in V'' (*middle*), and a measure of the local "particle density" ($\delta\phi^2 + k_{\perp}^2\delta\phi^2$) for several times around the first collision (*right*). In the top line we take $k_{\perp}^2 = 4.006 \times 10^{-4}$ which is located in the first stability band. The second line has $k_{\perp}^2 = 6 \times 10^{-4}$, which is near the maximum of the second instability band. In the third line we have $k_{\perp}^2 = 0.05$ which is in one of the higher order instability band. Finally, in the fourth line we take $k_{\perp}^2 = 0.83$, which is located in one of the higher order stability bands. For the purposes of plotting, we have defined $T_b = T_{breather}$ in the right panels.

the oscillation frequency of the mode function. The top right panel of Fig. 2.8 demonstrates that $n_{eff}^{(\omega_{breather})}$ increases as a smooth exponential, with some small subleading oscillations. It is also useful to recall that the Floquet mode can be written in the form



Figure 2.8: In the top left panel we plot V'' for a breather with v = 1. In the bottom panel we show the resulting fastest growing mode function for $k_{\perp}^2 = 0.35$. The growth is localized around the location of the breather and is much smoother than the $v \ll 1$ case. In the top right panel, we illustrate this smooth growth by plotting $n_{eff}^{(\omega_{breather})}$.

 $\delta \phi = e^{\mu t} P(x,t)$ and to decompose the periodic function P(x,t) as

$$\delta\phi(x,t)e^{-\mu t} = \sum_{\omega_m} |\delta\tilde{\phi}_{\omega_m}(x)|\cos(\omega_m t + \theta_m(x)) \qquad \omega_m = \frac{2\pi m}{T_{breather}}.$$
 (2.18)

The results of performing this decomposition are shown in Fig. 2.9. This clearly demonstrates that the frequency content of the mode function is indeed dominated by the frequency of the breather, with smaller subleading contributions from higher harmonics, thus justifying our introduction of the quantity $n_{eff}^{(\omega_{breather})}$ in (2.17). We now see that the small oscillations around pure exponential growth arise from this subleading frequency content. As expected, the mode function is concentrated in the region around the well created by the breather. Once we move further away from the breather, we see that the spatial phase in each frequency varies linearly. This is consistent with the production of radiation near the location of the breather which then travels off to infinity.



Figure 2.9: Several aspects of the frequency content of the periodic factor $\delta \phi e^{-\mu t}$ of the fastest growing Floquet mode for the v = 1 breather with $k_{\perp}^2 = 0.35$. Left: The amplitude of oscillation $|\delta \tilde{\phi}_{\omega_i}(x)|$. The overall normalization is arbitrary. Middle: The oscillation phase (defined as $\theta_i(x) \equiv \tan^{-1}(\operatorname{Im}(\delta \tilde{\phi}_{\omega_i})/\operatorname{Re}(\delta \tilde{\phi}_{\omega_i})))$ for the same frequencies. Right: The relative amplitudes of the three largest frequencies normalized to $\sigma_{\omega}^2(x) \equiv \sum_i |\delta \tilde{\phi}_{\omega_i}(x)|^2$.

Finally, we consider the case $v = 1/(\sqrt{2}-1)$ and $k_{\perp}^2 = 0.2$. As in the case v = 1, there is now a single oscillating well. However, the maximum excursion of the field is $\pm \pi/2$, so that the effective mass for the fluctuations is positive semidefinite, as illustrated in the top left panel of Fig. 2.10. We see that the resulting fastest growing Floquet mode is qualitatively very similar to the v = 1 case. There is an isolated blob that oscillates with a frequency determined by $\omega_{breather}$ whose amplitude grows smoothly as an exponential. Looking at the frequency content in Fig. 2.11, we see that the mode function again consists of a large amplitude part concentrated in the potential well of the breather and what appears to be a much smaller radiative tail. The oscillation frequencies in the two regions are even more monochromatic than the v = 1 cases, with the frequency of the radiative part being different than the frequency of the bound part. In both of these cases, the growth of mode functions is analogous to the case of narrow resonance for a single oscillator.

2.3.2 Symmetric Double Well

We now consider the symmetric double well. As demonstrated earlier, for certain choices of the initial kink speed the background solutions can undergo oscillatory motion. For the case of the asymmetric well, this oscillatory motion is generic due to the attractive force experienced by a well separated $K\bar{K}$ pair resulting from the potential energy difference



Figure 2.10: The effective mass (top left), mode function (bottom) and effective particle number (2.17) (top right) for a small amplitude $v = (\sqrt{2} - 1)^{-1}$ breather with $k_{\perp}^2 = 0.2$.

between the two wells. We will not consider the asymmetric well explicitly, but the behaviour of the fluctuations is qualitatively the same as the cases we study in the symmetric well.

The oscillatory motion of the background solution comes in three forms: repeated collisions of the walls, oscillations of the internal shape mode, and evolution of a latetime oscillon. The first of these is analogous to the sine-Gordon breathers with $v \ll 1$, while the last is qualitatively similar to the breathers with $v \ge 1$. Compared to the breathers, the only new feature here is that we can have $V''(x = 0) > V''(x = \infty)$ for a portion of the evolution. We will study each of these independently, although there are several caveats to this approach that the reader should be aware of. First of all, the repeated collisions and shape mode oscillations typically occur together in actual background solutions. In the homogeneous case, interference effects mean that the stability chart for a pair of oscillating masses with different frequencies is not the same as a superposition of the charts for each individual oscillator [37]. Despite this,



Figure 2.11: Frequency content of the fastest growing Floquet mode for $v = (\sqrt{2} - 1)^{-1}$ and $k_{\perp}^2 = 0.2$. In the left panel we show the oscillation amplitude for the three largest harmonics, while the center panel shows the initial phase of oscillation. The full solution takes the form $\delta \phi e^{-\mu t} = \sum |\delta \tilde{\phi}_{\omega_i}(x)| \cos(\omega_i t + \theta_i(x))$. In the right panel we instead show the relative amplitudes, normalized to the total power at position x.

from the sine-Gordon analysis above we expect that the growth of fluctuations due to the collision will occur in a very short time interval during the actual collision. The collision also excites the (planar) shape mode in the background, which is then free to pump excitations during the much longer intervals while the walls separate from each other. As a result, we can gain very good qualitative understanding by considering these processes in isolation. The resulting interference from the two mechanisms could then be done using projections on the appropriate Floquet basis (either for the wall collisions or the shape oscillations) just before and after each collision. Of course, since the exact background is not exactly periodic, this destructive interference will tend to get smeared out leading to a smoothing of the instability diagram, analogous to what happens in the homogeneous case.

Keeping the above caveats in mind, we begin with the case where the $K\bar{K}$ separate widely from each other between collisions. Unlike the sine-Gordon breather, we have no analytic solutions for this case. Also, due to the emission of radiation and the excitation of the shape mode, the background motion is no longer periodic. In order to create a periodic approximation that is amenable to our Floquet analysis, we now introduce our most questionable approximation and take the background to have the following form

$$\phi_{bg} = -\tanh\left(\frac{\gamma}{\sqrt{2}}(x-r(t))\right) + \tanh\left(\frac{\gamma}{\sqrt{2}}(x+r(t))\right) - 1, \qquad (2.19)$$

where r(t) is taken to be our dynamical variable and $\gamma = (1 - \dot{r}^2)^{-1/2}$. This ansatz ignores the production of radiation, excitation of the planar shape mode and any additional deformation of the kink profile due to interactions. After several further simplifying



Figure 2.12: Real parts of Floquet exponents for several choices of initial separation r_{max} of the kink and antikink in the double-well potential. In the left panel we show the largest Floquet exponent $\mu_{max}T_{walls}$, while in the right we show the second largest $\mu_{max,2}T_{walls}$. Unlike the sine-Gordon model, here we have regions where there are two unstable modes rather than just one. We see the expected banding structure with bands evenly spaced in $k_{\perp}T_{walls}$.

assumptions, we arrive at the following approximate solution for r(t)

$$r(t) = r_{max} + \frac{1}{2\sqrt{2}} \log \left(\cos^2 \left(\frac{\pi t}{T_{walls}} \right) + e^{-2\sqrt{2}(r_{max} - r_{min})} \right) \qquad T_{walls} = \frac{\pi}{2\sqrt{6}} e^{2\sqrt{2}r_{max}}$$
(2.20)

where r_{min} is the (minimum) solution to $V_{eff}(r_{min}) = V_{eff}(r_{max})$ with $V_{eff}(r)$ defined in (A.7) of appendix A. Further details of this construction are provided in appendix A. An alternative approach would have been to simply insert a periodic function for r(t) and study the resulting fluctuation behaviour. However, (2.20) is perhaps better justified than a completely adhoc choice for r(t) because it partially incorporates the full field dynamics.⁶

Fig. 2.12 shows the resulting Floquet exponents for several choices of r_{max} . As expected, the bands are distributed evenly in the phase accumulated by bound fluctuations in subsequent collisions $k_{\perp}T_{walls}$. There is a very strong instability as $k_{\perp} \rightarrow 0$, which is not unexpected given that our approximation ignores the radiation and planar shape mode that are excited during the collisions. One interesting new feature is the presence of a second growing mode, which we illustrate by plotting the second largest real part of a Floquet exponent in addition to the largest one.

Next consider the pumping of fluctuations by oscillations of the shape mode. This is a new effect that is not present in the sine-Gordon model. Since the shape mode is

⁶As a check, we did insert several other parameterized choices for r(t) "by hand" and found banding structure as we varied the parameters.

generically excited (or de-excited) during collisions, this amplification will usually occur in conjunction with the nonadiabatic production of fluctuations due to the colliding walls described above. We parameterize this motion as

$$\phi_{bg} = \tanh\left(\zeta\right) + A_{shape} \cos(\omega_1 t) \frac{\sinh(\zeta)}{\cosh^2(\zeta)}$$
(2.21)

where we have defined $\zeta = m(x - x_0)/\sqrt{2}$ and $\omega_1^2 = 3m^2/2$. Plugging this into the equation for fluctuations, we find

$$\delta\ddot{\phi} - \partial_{xx}\delta\phi + \left[k_{\perp}^2 - 1 + 3\left(\tanh\left(\frac{mx}{\sqrt{2}}\right) + A_{shape}\cos\left(\frac{\sqrt{3}mt}{\sqrt{2}}\right)\frac{\sinh(mx/\sqrt{2})}{\cosh^2(mx/\sqrt{2})}\right)^2\right]\delta\phi = 0$$
(2.22)

The resulting Floquet chart is given in Fig. 2.13. As in the case of the repeated collisions described above, we emphasize that this is not an exact solution of the 1-d field equations. Nonlinear couplings in the potential will modify the oscillation frequency of the shape mode and also cause it to radiate. Hence the amplitude will gradually decrease in time, leading to a slowly changing oscillation frequency. It is even possible to tune the amplitude so that the subsequent evolution leads to the creation of a kink-antikink pair in addition to the kink that was initially present [89]. However, provided the amplitude is not too large, the time-varying amplitude and frequency can be approximated as an adiabatic tracing of modes on the instability chart. Hence the instability chart and corresponding mode functions provide a heuristic understanding of the evolution of the fluctuations. In the top left panel of Fig. 2.14 we plot V''(x,t) for $A_{shape} = 0.5$. As expected from the interpretation of the shape mode as a perturbation to the width of the kink, V'' looks like a well whose width oscillates in time. There is also some additional oscillating side-lobe structure. The remaining panels in Fig. 2.14 show the fastest growing Floquet mode, the effective particle number and the various frequency components of the solution as in the case of the small-amplitude breather. Because the mode function completes only half an oscillation per period of the shape mode, the frequency used in the definition of n_{eff}^{ω} is $\omega = \pi/T_{shape}$. The mode function displays more structure than for the sine-Gordon breathers, in particular in the higher harmonics. This additional structure is probably due to the additional sidelobe structure of V''. However, the solution is still well described by a single oscillation frequency near the core of the kink, with a different harmonic becoming important as we move towards the sidelobes of the oscillations, and finally a third harmonic arising as we move into the radiating region.

Finally consider the late-time (1-dimensional) oscillon state that develops. In order to



Figure 2.13: Instability chart for planar oscillations of the shape mode. Shown is $\mu_{max}T_{shape}$ for $\ddot{f} - f_{xx} + \left[\frac{2}{3}(k_{\perp}^2 - 1) + 2\left(\tanh(x/\sqrt{3}) + A_{shape}\cos(t)\frac{\sinh(x/\sqrt{3})}{\cosh^2(x/\sqrt{3})}\right)^2\right]f = 0$ and various choices of the parameters k_{\perp} and A_{shape} .

approximate this motion, we first expand the background solution about the true vacuum minimum as $\phi_{bg} = \phi_{min} + \bar{\phi}(x,t)$. Up to a constant, the potential for $\bar{\phi}$ then takes the form $V(\bar{\phi}) = \frac{\bar{m}^2 \bar{\phi}^2}{2} + \frac{\sigma}{3!} \bar{\phi}^3 + \frac{\bar{\lambda}}{4!} \bar{\phi}^4$ with $\bar{m}^2 = 2\lambda \phi_0^2$, $\sigma = 6\lambda \phi_0$ and $\bar{\lambda} = 6\lambda$. In order to extract an approximate background solution, we follow [95] and perform an asymptotic expansion in some small parameter ϵ and define new time and space coordinates $u = \sqrt{1 - \epsilon^2} \bar{m}t$ and $w = \epsilon \bar{m}x$. Expanding the solution as $\bar{\phi}(u,w) = \sum_{n=1}^{\infty} \epsilon^n \phi_n(u,w)$ and solving the resulting equations order by order in ϵ , we find

$$\bar{\phi}_{\text{osc}} = \left(P(x) + g(\sqrt{2}mx)\right)\cos(\sqrt{2}\sqrt{1-\epsilon^2}mt) + \frac{3}{2\phi_0}P(x)^2\left(\cos(2\sqrt{2}\sqrt{1-\epsilon^2}mt) - 3\right)\right) + \mathcal{O}(\epsilon^3)$$
(2.23)

$$P(x) = \frac{4c}{\sqrt{\alpha}} \operatorname{sech}(\sqrt{2\epsilon mx})$$
$$\partial_{ww}g - g + 3\alpha g^3 = 0$$

where we have defined $\alpha \equiv \frac{5\sigma^2}{3\bar{m}^4} - \frac{\bar{\lambda}}{\bar{m}^2}$. For the particular form of the double well potential, we have $\alpha = 12/\phi_0^2$. In the literature, two choices for how to treat the function g have been considered. Fodor et. al [95] assumed that no bounded solutions exist and set g = 0, while Amin [96] instead demanded $\phi_2(t=0) = 0$ to set $g(x) = 3P(x)^2/\phi_0$. We make the second of these two choices when we plot the second order oscillon instability chart. The



Figure 2.14: The same series of plots are shown as for the Floquet modes of the v = 1 (Fig. 2.8, Fig. 2.9) and $v = (\sqrt{2}-1)^{-1}$ (Fig. 2.10, Fig. 2.11) sine-Gordon breathers. In the definition of n_{eff}^{ω} , we used the frequency $\omega = \pi/T_{shape}$. As for the sine-Gordon breathers, the mode function consists of a well-defined core region near the location of the kink as well as a much small radiative component. Due to the additional spatial complexity of V'' in this case, the mode function displays more spatial and frequency structure than for the breathers.



Figure 2.15: Floquet chart for the small amplitude "breather"-like solution in the doublewell potential, eqn. (2.23). Left: Floquet chart is for the leading order in ϵ solution Middle: Floquet chart for the second order in ϵ solution. Right: Instability of the $k_{\perp}^2 = 0$ mode as a function of ϵ for both the leading and second order background solutions.

resulting equation for the transverse fluctuations is

$$\frac{\partial^2 \delta \phi}{\partial t^2} - \frac{\partial^2 \delta \phi}{\partial x^2} + \left[k_2^2 - 1 + 3(1 + \bar{\phi}_{\rm osc})^2\right] \delta \phi = 0.$$
(2.24)

In Fig. 2.15 we show the corresponding stability chart for the fastest growing mode taking both the leading order and second-order in ϵ approximations to the background.⁷ As can be clearly seen, the detailed structure of the instability does display some sensitivity to our choice of approximation for the background. For $k_{\perp} = 0$ and $\epsilon \leq 0.2$, the higher order approximation removes a weak instability that was present in the leading order approximation, indicating that it is indeed a better approximation to the background at small ϵ . However, for larger ϵ the higher-order background is actually more unstable than the leading order approximation. This is merely a reminder that our approximation is asymptotic rather than convergent. When we consider $k_{\perp} \neq 0$, we see that the additional oscillating frequencies in the second-order background lead to several weak instability bands at small epsilon. Meanwhile, for larger ϵ the main instability band extends for a wider range of k_{\perp} and has larger Floquet exponents. Again, this is not surprising since the oscillation amplitude is larger in this approximation and we would thus expect it to drive stronger instabilities.

2.3.3 Comments on Collisions of other Membrane-like Objects

Although we have focussed on two specific scalar field models, the dynamical mechanism that leads to rapid growth of the fluctuations is much more general. In particular, for

⁷The particular choice of oscillon profile is not essential. We also ran simulations using Gaussian profiles $\phi_{bg} = A \cos(\omega t) e^{-x^2/w^2}$ for various choices of ω and w taking A as a parameter and again found a similar structure for the Floquet chart.

the well-separated walls the explosive growth of fluctuations relied only on the presence of bound fluctuations around each of the kinks and the periodic violation of adiabaticity for these bound states. Recall that these fluctuations arise as the transverse generalization of the Goldstone mode (i.e. the translation mode) resulting from the spontaneous breaking of translation invariance by the kink. However, an equivalent Goldstone mode will occur for kinks in any translation invariant theory, and thus these bound fluctuations are ubiquitous. For kink-antikink type collisions such as those studied here, we typically expect large deformations in the shape of V'' as well as adiabaticity during each collision and thus the parametric amplification of wall-bound fluctuations we have found here should be very common. It would be interesting to consider instead kink-kink collisions, where the effective potential wells resulting from each of the kinks will not completely annihilate during collision, but we leave this to future work.

As well as this rather direct extension to other scalar field theories, we believe our results also have some relevance to collisions of other membrane-like objects such as Dbranes. When the kink and antikink are wellseparated, the transverse translation modes simply describe a spatially dependent location for the center of the kink, and are thus well-described by an effective action for a membrane. If two such membranes are in close proximity to each other, it is natural to expect them to interact. For the case of Dbranes, this interaction is usually described in terms of the excitation and production of string modes. Since string production is a local process from the viewpoint of a field theory on the brane, we expect that the resulting fluctuations will be inhomogeneous and analogous to the excitation of our transverse modes. We will show in chapters 3 and 4 that the inhomogeneity of the growing fluctuations dramatically changes the full three-dimensional dynamics as compared to the case when the backreaction is assumed to have planar symmetry. Although this is highly dependent on the details of the highenergy theory (in our case a scalar field theory), we expect brane collisions to be strongly affected by inhomogeneity and amplification of fluctuations for the same reasons that gave rise to the rich phenomena from domain wall collisions presented in this work.

2.4 Comments on Fluctuations Around Colliding Bubbles

Having explored instabilities around colliding domain walls, we now briefly comment on the case of two colliding vacuum bubbles. Beginning with the work of Hawking, Moss and Stewart [73], this problem has been explored by many authors [97, 98, 99, 100, 22, 101, 102, 103, 104]. A common feature of these analyses is the assumption of SO(2,1) symmetry for the field profiles. This is motivated by the SO(4) symmetry of the minimum action bounce solution for a single bubble [76], which translates into an SO(3,1) symmetry for the nucleation and subsequent expansion of the bubble in real time. The nucleation of a second such bubble destroys the boost symmetry along the axis connecting their centers, leaving a residual SO(2,1) symmetry for the 2-bubble solution. Making this assumption allows for the treatment of the problem as effectively 1 + 1-dimensional, which greatly eases the numerical challenges and has even lead to a general relativistic treatment [105].

However, as in the case of the domain walls studied above, this is not the full story. Quantum fluctuations are inevitably present; in fact it is these field fluctuations that are responsible for the bubble nucleation in the first place. For a discussion of these perturbations in the presence of a single bubble see [106, 107, 108]. Motivated by the presence of these fluctuations, we now make some brief comments on the linear fluctuation dynamics in the background of the pair of colliding bubbles. Results for the full dynamics of bubble collisions are presented in chapter 4.

2.4.1 Background Dynamics of Highly Symmetric Collisions

As in the case of the planar walls, we first decompose our problem into a highly symmetric background field and a nonsymmetric fluctuation. This allows us to connect with previous treatments as well as the foregoing sections of this chapter. A convenient set of coordinates is given by

$$t = s \cosh \chi$$

$$x = x$$

$$y = s \sinh \chi \cos \theta$$

$$z = s \sinh \chi \sin \theta$$
(2.25)

where we align our coordinates such that the centers of the two bubbles both lie on the x-axis. The SO(2,1) symmetry is now manifest as the background depends only on s and x. As for the case of planar symmetry, we separate the field into a symmetric background and symmetry breaking fluctuations $\phi = \phi_{bg}(s, x) + \delta\phi(s, x, \chi, \theta)$. Ignoring backreaction of the fluctuations, the background solution obeys

$$\frac{\partial^2 \phi_{bg}}{\partial s^2} + \frac{2}{s} \frac{\partial \phi_{bg}}{\partial s} - \frac{\partial^2 \phi_{bg}}{\partial x^2} + V'(\phi_{bg}) = 0$$
(2.26)

and the linearized fluctuations evolve according to

$$\frac{\partial^2(sA_\ell)}{\partial s^2} - \frac{\partial^2(sA_\ell)}{\partial x^2} + \left(\frac{\ell^2}{s^2} + V''(\phi_{bg})\right)(sA_\ell) = 0.$$
(2.27)

Here we have factored the perturbation as $\delta \phi = \sum_{\ell,n} A_{\ell}(s,x) C_{\ell,n}(\chi) e^{in\theta}$, with $C_{\ell,n}$ obeying

$$\frac{1}{\sinh(\chi)}\frac{d}{d\chi}\left(\sinh(\chi)\frac{dC_{\ell,n}}{d\chi}\right) = \left(-\ell^2 + \frac{n^2}{\sinh^2(\chi)}\right)C_{\ell,n}$$
(2.28)

and *n* an integer. $\sum_{l,n}$ represents an integral over the continuous part of ℓ and a sum over *n* and any discrete normalizable modes $C_{\ell,n}$. The curvature of the bubble walls thus influences the fluctuation dynamics in three ways: damping the overall amplitude as s^{-1} , redshifting the effective transverse wavenumber squared as s^{-2} and modifying the dynamics of ϕ_{bg} and by extension $V''(\phi_{bg})$.

Treatments that assume SO(2,1) symmetry restrict themselves to studying (2.26) with no consideration of the fluctuations which evolve (initially) according to (2.27). A sample collision between two such bubbles is shown in Fig. 2.16. As in the case of the



Figure 2.16: Collision of two thin-wall vacuum bubbles in the asymmetric well (2.4) with $\delta = 0.1$. The color coding indicates the value of the scalar field. Red indicates it is near the true vacuum minimum, blue shows regions where it is near the false vacuum, and the location of the bubble wall is white. At early times, the acceleration of the walls and corresponding Lorentz contraction is visible. As in the planar symmetric case, the two walls bounce off of each other multiple times rather than immediately annihilating. During this process, scalar radiation is emitted from the collision region.

kink-antikink collisions, the bubble walls undergo multiple collisions, each time opening up a pocket where the field is localized near the false vacuum minimum. The bouncing behaviour we observe is characteristic of bubble collisions in double-well potentials, and was first noted by Hawking, Moss and Stewart [73].

Considering the implications of this behaviour for the full 3+1-dimensional evolution suggests that two instabilities may occur. The first is the generalization of our previous results to the SO(2,1) symmetric rather than the planar symmetric case. Given the background evolution depicted in Fig. 2.16, we see that (2.27) again describes a field with an oscillating time and space-dependent mass. Further consideration of Fig. 2.16 reveals the presence of another possible instability. Due to the SO(2,1) symmetry, each pocket with the field near the false vacuum corresponds to a torus with growing radius centered on the initial collision in the full 3-dimensional evolution. Roughly, we can think of this torus as containing false vacuum with a thin membrane separating it from the true vacuum on the outside. The energy difference between the false and true vacuum leads to a pressure acting normal to the local surface of the membrane. Since this pressure wants to push the membrane into the false vacuum, this will tend to cause small initial ripples on the surface of the torus to grow. Of course, the surface tension of the membrane and the stretching of the torus as it expands will tend to counteract this effect, so that a three-dimensional calculation is required to determine the ultimate fate of these ripples. As we will see in chapter 4, both of these instabilities do indeed manifest themselves in the fully 3+1-dimensional problem.

2.5 Conclusions

In this chapter we studied the dynamics of linear asymmetric fluctuations around highly symmetric collisions between planar domain walls and vacuum bubbles. Parallel planar walls are a common ingredient in cosmological model building based on braneworlds, and SO(2,1) bubble collisions are generally believed to be an accurate description of individual bubble collisions during false vacuum decay. Our results thus have potential implications for these cosmological scenarios.

Fluctuations are generically present in the field that forms the domain wall and therefore must be included in a quantum treatment of the problem. However, nearly all past studies of planar wall and vacuum bubble collisions dynamics have used symmetry to reduce the problem to a 1+1-dimensional PDE, thus excluding the fluctuations a priori. Assessing the validity of this drastic reduction in the effective number of dimensions requires a more sophisticated treatment of the problem, and this chapter provides the first step in such a treatment.

Once we have fixed the symmetric background dynamics for the collision, the fluctuations behave as a free scalar field with a time and space dependent effective frequency. Using Floquet theory and extending well-developed methods for ODEs to PDEs, we were able to show that the time-dependence of the effective frequency can lead to exponential growth of the symmetry breaking fluctuations. We also studied the spatial structure of the amplified modes to obtain an understanding of the mechanism responsible for the amplification. We found generalizations of both broad parametric resonance and narrow parametric resonance. Due to the spatial dependence of the effective frequency, the amplified modes are localized along the collision direction and have a spread of characteristic wavelengths in the transverse directions. For collisions between well defined wall-antiwall pairs, the resulting amplification can be conveniently interpreted as Bogoliubov particle production for particles bound to the walls.

Although we focussed on two specific scalar field models, a detailed study of the unstable modes revealed that the dynamical mechanism responsible for the rapid growth of fluctuations is much more general. In particular, for collisions between a pair of well defined walls, the most strongly amplified modes are the transverse generalization of the Goldstone mode resulting from the spontaneous breaking of translation invariance by the domain wall. These modes are present on *any* membrane-like object in a translation invariant theory. The amplification only relied on a strong deformation of the effective potential binding these fluctuations to the wall. Such deformations will be extremely common in domain walls formed in scalar field theories, and should also arise in collisions between other membrane-like objects such as Dbranes. Therefore, we expect qualitatively similar results to be obtained in a wide variety of collisions involving membrane-like objects.

The linearized approach taken here cannot tell us what the ultimate fate of the exponentially growing modes will be. Since the modes have $k_{\perp} \neq 0$, they do not respect the planar symmetry of the background, which suggests that this symmetry will be badly broken once the fluctuations become large. The treatment of the full nonlinear field evolution in the regime where the fluctuations become nonlinear will be the subject of chapters 3 and 4. We will explore nearly planar symmetric domain wall and nearly SO(2,1) symmetric bubble collisions using high resolution lattice simulations, thus allowing us to explore the full three-dimensional field dynamics. These investigations will demonstrate that the nonlinear evolution leads to a complete breakdown of the original symmetry of the background, including a dissolution of the walls and production of a population of oscillons in the collision region. This entire process is unique to more than one spatial dimension and is a completely new effect that has not previously been considered in either domain wall or bubble collisions.

Chapter 3

Cosmic Bubble and Domain Wall Instabilities II: Fracturing of Colliding Walls

3.1 Introduction

This chapter extends the study begun in chapter 2 of linear fluctuations around colliding planar domain walls. While chapter 2 considered linearized fluctuations, in the present chapter we include the full nonlinear dynamics using lattice simulations. The results for SO(2,1) bubble collisions are presented in chapter 4.

When considering the fully nonlinear dynamics between extended objects, the highenergy completion is needed rather than just the effective theory for small fluctuations in the planar shape. We will focus on domain walls formed by the condensate of some scalar field, and thus we take the high-energy completion of our theory to be a scalar field theory with canonical kinetic terms and symmetry breaking potential. We will refer to this field as the symmetry breaking scalar and denote it by ϕ .

The case of interacting parallel planar walls in this class of theories has been considered by many authors, usually under the assumption that the nonlinear dynamics can be treated as planar thus reducing the system to one living in a single spatial dimension. This assumption is tenable in a *classical* field theory. However, as noted in the previous chapter, once the theory is quantized, individual realizations of quantum fluctuations will break the planar symmetry, even though the statistics as a whole do not. These quantum fluctuations can experience resonant instabilities when the domain walls are allowed to collide with each other. These instabilities have important consequences, which we explore in this chapter.

Several authors have explored the amplification of *additional* fields coupled to the symmetry breaking scalar, including the cases of additional fermionic fields [92, 90, 91] and additional scalar fields [93]. However, these studies assumed the background maintained planar symmetry and did not investigate the onset of nonlinearities amongst the fluctuations. We instead take a more minimal approach and study the amplification of fluctuations in the symmetry breaking field ϕ . Since we require the field ϕ to exist in order to have domain walls in the first place, these fluctuations must be present for a consistent quantum treatment of the problem. As foreshadowed in the previous chapter we find that accounting for the dynamics of these fluctuations can drastically change the collision dynamics between the walls, in the process completely invalidating the original assumption of planar symmetry. Throughout this chapter, we will refer to this situation as a breaking of the symmetry by fluctuations.

The remainder of the chapter is organized as follows. In section 3.2 we present our scalar field models and review the domain wall solutions they support. We also briefly review how these solutions interact in the limit of exact planar symmetry. Section 3.3 constitutes our main results. We use lattice simulations to study the full collision dynamics between domain wall-antiwall pairs for two different potentials and several choices of initial conditions. A generic outcome of these collisions is the rapid amplification of nonplanar fluctuations, eventually leading to an inhomogeneous dissolution of the wall and antiwall. While some of the energy is released into the bulk as radiation during the dissolution, some of it remains trapped in the collision region in the form of localized oscillating blobs of field called oscillons. Motivated by the creation of oscillons from domain wall collisions, section 3.4 looks at some of their properties. Finally, we provide a brief qualitative summary of the full collision dynamics in section 3.5 and then conclude.

3.2 Review of (Early Time) Linear Fluctuation Dynamics

As in our study of linear fluctuations in chapter 2, we consider two potentials that support kink solutions in one-dimension: the sine-Gordon model

$$V(\phi) = \Lambda \left[1 - \cos\left(\frac{\phi}{\phi_0}\right) \right]$$
(3.1)

and the double-well potential

$$V(\phi) = \frac{\lambda}{4} \left(\phi^2 - \phi_0^2\right)^2 - \delta \lambda \phi_0^3 \left(\phi - \phi_0\right) + V_0.$$
 (3.2)

For the double-well, δ controls the difference between the false and true vacuum energies $\Delta \rho \approx 2\delta \lambda \phi_0^4$ and V_0 is a constant. We restrict to Minkowski space, so V_0 will not play any role in the dynamics. Unless explicitly indicated, throughout the chapter we will express the fields in units of ϕ_0 , spacetime coordinates in units of m_{eff}^{-1} , and energy densities in units of $m_{eff}^2 \phi_0^2$ where m_{eff} refers to the effective mass defined for the sine-Gordon or double well potential as appropriate (see (3.3) and (3.4)). Both of these potentials have solutions (known as kinks and antikinks in one-dimension) interpolating between neighbouring minima of the potential. For the sine-Gordon model and degenerate double-well ($\delta = 0$), they are given by

$$\phi_{kink}^{SG} = 4\phi_0 \tan^{-1} \left(e^{m_{SG}(x-x_0)} \right) \qquad m_{SG} = \sqrt{\Lambda} \phi_0^{-1} \tag{3.3}$$

and

$$\phi_{kink}^{DW} = \phi_0 \tanh\left(\frac{m(x-x_0)}{\sqrt{2}}\right) \qquad m = \sqrt{\lambda}\phi_0 \tag{3.4}$$

respectively. The antikinks are obtained via the replacement $(x - x_0) \rightarrow -(x - x_0)$. For the slightly asymmetric well ($\delta \ll 1$) stationary kink solutions no longer exist as the pressure differential between the false and true vacuum causes the kinks to accelerate. In this case, we will take the following as an approximate initial profile interpolating between the two minima

$$\phi_{kink}^{DW} = \frac{\phi_{true} - \phi_{false}}{2} \tanh\left(\frac{m(x - x_0)}{\sqrt{2}}\right) + \frac{\phi_{true} + \phi_{false}}{2} \tag{3.5}$$

where ϕ_{false} and ϕ_{true} are the locations of the false and true vacua repectively. In onedimension the energy of the sine-Gordon and double well kinks are $E_k^{SG} = 8m_{SG}\phi_0^2$ and $E_k^{DW} = \frac{2\sqrt{2}}{3}\sqrt{\lambda}\phi_0^3$ respectively.

In this chapter our focus is collisions between a pair consisting of a single kink and a single antikink, which together carry no net topological charge. Since we are interested in the three-dimensional problem, we extend the kink and antikink in the additional transverse spatial dimensions. We refer to this setup as a wall-antiwall pair to distinguish it from the one-dimensional case. Under the approximation of exact planar symmetry, the field obeys

$$\frac{\partial^2 \phi_{bg}}{\partial t^2} - \frac{\partial^2 \phi_{bg}}{\partial x^2} + V'(\phi_{bg}) = 0$$
(3.6)

with the initial condition

$$\phi_{bg}(t=0) = \phi_{kink}(x-x_0) + \phi_{antikink}(x+x_0) + \phi_{\infty}$$
(3.7)

where we align our axes so that the kink and antikink move along the x direction and collide at x = 0. The constant ϕ_{∞} is chosen so that the field is sitting at the desired minimum of the potential at infinity. For our purposes, the important aspect of these collisions is that they tend to produce oscillatory behaviour in the motion of the fields. This oscillatory motion comes in three forms: repeated collisions between the kink and antikink, formation of localized pseudostable nearly periodic blobs of field, and for the double well the vibration of internal modes of excitation of the individual kink and antikink. When we consider the extension of the one-dimensional kinks to planar walls in higher spatial dimensions, small nonplanar fluctuations obey

$$\partial_{tt}\delta\tilde{\phi}_{k_{\perp}} - \partial_{xx}\delta\tilde{\phi}_{k_{\perp}} + \left(k_{\perp}^2 + V''(\phi_{bg}(x,t))\right)\delta\tilde{\phi}_{k_{\perp}} = 0$$
(3.8)

where $\delta \tilde{\phi}_{k_{\perp}}$ is the Fourier transform of the fluctuations in the additional orthogonal directions and k_{\perp} is the transverse wavenumber. The time-dependence of the kinkantikink solution in one-dimension leads to a space and time dependent effective mass $k_{\perp}^2 + V''(\phi_{bg}(x,t))$ for the fluctuations. In chapter 2 we performed a detailed analysis of the fluctuations accounting for this time-dependent effective mass induced by the background evolution. We found that the oscillations in the background drive resonant instabilities in the fluctuations causing certain transverse wavenumbers k_{\perp} to grow exponentially. Eventually these fluctuations become sufficiently large that the assumption of linearity fails. At this stage we have to solve the full nonlinear three-dimensional problem and waive any additional assumptions of symmetry. This full problem will be the focus of the remainder of this chapter.

3.3 Nonlinear Dynamics of Planar Domain Walls with Non-Planar Fluctuations

In this section we present the results for the full three-dimensional nonlinear field dynamics. We only consider choices of the couplings for which the fluctuations become highly excited while still in the linear regime. Thus, the system transitions to the semiclassical wave limit before interactions between the fluctuations become important. Invoking the standard assumption that the system remains in the semiclassical limit after the onset of strong nonlinearities, we can then use classical statistical simulations as an approximation to the full quantum evolution. We use a high resolution numerical lattice code, with second-order accurate and fourth-order isotropic finite-differencing stencils and sixth-order accurate Yoshida integrators for the time-evolution. Since we are interested in studying spatially localized objects, we also implement absorbing boundary conditions [109] along the collision direction

$$\left[\partial_t \phi - \partial_{x_{\parallel}} \phi \Big|_{x_{\parallel}=0} = 0 \qquad \left[\partial_t \phi + \partial_{x_{\parallel}} \phi \Big|_{x_{\parallel}=L_{\parallel}} = 0 \tag{3.9}\right]$$

in order to remove energy released from the collision region. We have chosen coordinates along the collision axis to range from 0 to L_{\parallel} and denoted the coordinate along this axis x_{\parallel} . Quantum effects are incorporated through the initial conditions

$$\phi_{init}(\mathbf{x}, t = 0) = \phi_{bg}(x, t = 0) + \delta\phi(\mathbf{x})$$

$$\dot{\phi}_{init}(\mathbf{x}, t = 0) = \dot{\phi}_{bg}(x, t = 0) + \delta\dot{\phi}(\mathbf{x})$$
(3.10)

where ϕ_{bg} is the initial profile of the desired classical background field (here a pair of walls as in (3.7)) and $\delta\phi$ and $\delta\dot{\phi}$ are realizations of a random field with the same statistics as the quantum fluctuations around the walls. Although this approach cannot capture the final thermalization of the modes to the Bose-Einstein distribution, it is capable of describing all forms of nonlinear mode-mode coupling, including both mean-field like backreaction (as is included in the Hartree approximation) as well as rescattering effects and the development of nongaussian field statistics. Since we focus on the dynamical regime well before the system reaches thermal equilibrium, the lack of proper thermalization on the lattice is not a serious limitation.

An important ingredient in this framework is the spectrum of the initial fluctuations. Although we don't engage in a full study of the initial fluctuations here, as long as the exact initial state has a nonzero projection onto the linearly unstable Floquet modes we will obtain the same qualitative behaviour. For simplicity, we primarily consider two choices for the fluctuations. The first is to take $\delta\phi$ to be a homogeneous Gaussian random field with spectrum

$$\langle |\delta \tilde{\phi}_k|^2 \rangle \sim \frac{1}{2\sqrt{k^2 + V''(\phi_{true})}}$$

$$\langle |\delta \dot{\tilde{\phi}}_k|^2 \rangle \sim \frac{\sqrt{k^2 + V''(\phi_{true})}}{2}$$
(3.11)

which corresponds to the correct fluctuations if the background were homogeneous and sitting at its true vacuum minimum. For the second choice we initialize each individual kink as

$$\phi_{init}(\mathbf{x}, t = 0) = \phi_{kink}(\gamma(x + \delta x))$$

$$\dot{\phi}_{init}(\mathbf{x}, t = 0) = -\gamma(v + \delta v)\phi'_{kink}(\gamma(x + \delta x))$$
(3.12)

with $\delta x(y, z)$, $\delta v(y, z)$ two dimensional Gaussian random fields with spectra $\langle |\tilde{\delta x}_{k_{\perp}}|^2 \rangle \sim \frac{1}{2k_{\perp}\sigma_{kink}}$ and $\langle |\tilde{\delta v}_{k_{\perp}}|^2 \rangle \sim \frac{k_{\perp}}{2\sigma_{kink}}$ where $k_{\perp}^2 = k_y^2 + k_z^2$ and $\sigma_{kink} = \int dx_{\parallel} (\partial_{x_{\parallel}} \phi)^2$ is the surface tension of the stationary kink. The Lorentz contraction factor is given by $(1 - v^2)^{-1/2}$. We only consider initial velocities with $v \ll 1$, so the inclusion of fluctuations does not lead to any superluminal wall propagation speeds. In order to fix notation, we introduce an amplitude parameter \mathcal{A}_b and initialize the fluctuations as

$$\delta x(x_{\perp}) = \frac{\mathcal{A}_b}{L_{\perp}} \sum_{\mathbf{k}_{\perp}} \frac{\alpha_{\mathbf{k}_{\perp}}}{\sqrt{2k_{\perp}}} e^{i\mathbf{k}_{\perp}\cdot\mathbf{x}_{\perp}} \qquad \delta v(x_{\perp}) = \frac{\mathcal{A}_b}{L_{\perp}} \sum_{\mathbf{k}_{\perp}} \beta_{\mathbf{k}_{\perp}} \sqrt{\frac{k_{\perp}}{2}} e^{i\mathbf{k}_{\perp}\cdot\mathbf{x}_{\perp}} \tag{3.13}$$

with $\alpha_{\mathbf{k}_{\perp}}$ and $\beta_{\mathbf{k}_{\perp}}$ complex Gaussian random deviates with $\langle |\alpha_{\mathbf{k}_{\perp}}|^2 \rangle = 1 = \langle |\beta_{\mathbf{k}_{\perp}}|^2 \rangle$ and L_{\perp} the side length of the box in the directions orthogonal to the collision. This amounts to including the fluctuations associated with local translations of the kinks, which are a subset of the full fluctuation content around the kink background. In this case we do not include the remaining bulk fluctuations (and transverse excitations of the shape mode in the case of the double-well) as these require solving an additional eigenvalue problem. From our linear analysis we know that for the case of well separated walls, this second set of fluctuations are precisely the modes which are most strongly amplified by the collision.¹ As well, when absorbing boundary conditions are used, the use of these localized initial conditions avoids an initial spurious loss of energy due to aborption of

¹Although all of the results we present here used one of these two choices, we also tested a variety of other initial conditions and obtained similar results.

bulk fluctuations by the boundaries.

For each sample collision we provide several plots illustrating different aspects of the dynamical evolution. First, we slice the field along two orthogonal planes. The first slice is parallel to the collision axis, providing a view of the effective one-dimensional dynamics at early times, the production of outgoing radiation and the rippling of the walls as the transverse fluctuations are amplified. The second slice is orthogonal to the collision axis and centered at either the collision point or the instantaneous location of one of the planar walls. This slice provides a full two-dimensional view of the development of the transverse instability. As illustrated in the examples that follow, this is especially useful to study the nonlinear evolution of the fluctuations. The next set of figures are contour plots of the energy density $\rho \equiv -T^{00} = \frac{\dot{\phi}^2}{2} + \frac{(\nabla \phi)^2}{2} + V(\phi)$. At early times these clearly show the locations of the two walls as well as the bumps that develop due to the linear fluctuations. During the nonlinear stages these plots provide a very clear picture of how the system is evolving, including the localization of structures in three-dimensions. Finally, to study the spectral content of the transverse fluctuations, we include the twodimensional angle averaged power spectrum of ρ for the transverse wavenumbers k_{\perp} as a function of position along the collision axis. Explicitly, we compute

$$\mathcal{P}_{\rho}^{2d}(k_{\perp}, x_{\parallel}) \equiv \frac{L_{\perp}^{2} k_{\perp}^{2}}{N_{\perp}^{2}} \left\langle |\tilde{\rho}_{k_{\perp}}^{2d}(x_{\parallel})|^{2} \right\rangle_{\perp}$$
(3.14)

where $\langle \cdot \rangle_{\perp}$ represents an averaging performed in the plane orthogonal to the collision axis at position x_{\parallel} along the collision axis and $N_{\perp} = N_y N_z$ is the number of lattice sizes in each of these planes. Our discrete Fourier transform convention is $\tilde{\rho}_{k_{\perp}}^{2d}(x_{\parallel}) =$ $\sum_i e^{i\mathbf{k}_{\perp,n}\cdot\mathbf{x}_{\perp,i}}\rho(x_{\perp,i},x_{\parallel})$ with $x_{\perp} = (y,z)$ the coordinates in the directions orthogonal to the collision axis. This gives us a very clear view of the spatial localization (along x_{\parallel}) of the amplified fluctuations as well as their typical transverse wavenumber.

Before presenting the results of our simulations, we briefly summarize the outcome of the collisions in order to orient the reader and unify the discussion to follow. While the precise details depend on the choice of potential and planar symmetric background ϕ_{bg} we perturb around, the qualitative details are essentially the same for every case we consider. Initially, the two walls are well-described by the planar ansatz, and due to our choice of initial setup undergo multiple collisions or else capture each other to form an oscillating bound state. In either case, the initially small planar symmetry breaking fluctuations experience exponential growth as described by the linear theory in chapter 2. Once the fluctuations have grown large enough they begin to interact nonlinearly. In every case involving the dynamics of a wall-antiwall pair, we found that the stage of nonlinear interactions leads to a complete breakdown of the original planar symmetry. This occurs by an inhomogeneous annihilation between the wall and antiwall which results in the production of a population of oscillating blobs of field known as oscillons. These oscillons are distributed homogeneously in the transverse directions to the collision, but are highly localized at the collision site in along the collision axis. Details and illustrations that will clarify this picture are presented below.

3.3.1 Sine-Gordon Potential

We begin with the sine-Gordon model and consider two distinct classes of background solutions. The first is the planar symmetric breathers indexed by the parameter v

$$\phi_{breather} = 4 \tan^{-1} \left(\frac{\cos(\gamma_v vt)}{v \cosh(\gamma_v x)} \right) \qquad \gamma_v \equiv (1+v^2)^{-1} \,. \tag{3.15}$$

We restrict ourselves to $v \gtrsim 1$ so that the background field configuration is a localized oscillating blob. For the second case, we instead set the background solution to be a kink-antikink pair approaching each other with nonzero initial velocity. Since the 1d sine-Gordon kinks are true solitons they interact while preserving their shapes and velocities, acquiring only an overall phase shift due to the interaction. Energy conservation then dictates that after the collision this wall-antiwall pair will again move off to infinity, at least in the absense of fluctuations. When the system lives on the infinite interval we can use a Backlund-transformation to find a simple analytic form for this interaction

$$\phi_{k\bar{k}} = 4 \tan^{-1} \left(\frac{\sinh(\gamma v t)}{v \cosh(\gamma x)} \right) \qquad \gamma \equiv (1 - v^2)^{-1/2} \tag{3.16}$$

which describes the passage of the kink and antikink through each other during the collision. In this case the parameter v represents the speed of the kink and antikink at infinity. In order to allow for multiple collisions, we will take the collision direction to be periodic with a linear size less than the transverse directions. As a result the wall and antiwall will alternately collide in the middle of the simulation volume and at the boundaries. Since the collision results in the passage of the wall and antiwall through each other, it is easy to see that each of these collisions will have the same ordering of the wall and antiwall relative to the collision site. Either we always have a kink approaching the collision from the right and antikink approaching the collision from the left, or vice-versa.

The evolution of planar symmetric breathers with small fluctuations are shown in Fig. 3.1



Figure 3.1: Evolution of a breather with $v = (\sqrt{2} - 1)^{-1}$ showing the development of the instability in planar symmetry breaking fluctuations. Top row: The field sliced along a plane parallel to and orthogonal to the collision direction. The orthogonal slice is taken through the center of the breather. White shading corresponds to the field sitting at the origin. Middle row: Contours of the energy density $\rho = \frac{\dot{\phi}^2}{2} + \frac{(\nabla \phi)^2}{2} + V(\phi)$. Bottom row: The dimensionless 2d power spectrum \mathcal{P}_{ρ}^{2d} defined in (3.14). The top panel shows the spectrum as a function of x_{\parallel} and k_{\perp} , while the bottom panel plots the spectrum along the slice through the center of the breather as indicated by the green line in the top panel. The normalization of the power spectrum color scale differs between the left plot and the remaining three. Animations of the field and energy density evolution can be found at www.cita.utoronto.ca/~jbraden/ Movies/sg_vsqrt2_field.aviand www.cita.utoronto.ca/~jbraden/Movies/sg_vsqrt2_rho.avi.

and Fig. 3.2 for the case $v = (\sqrt{2} - 1)^{-1}$ and v = 1 respectively. At early times, the field is an oscillating blob localized along the collision axis with near planar symmetry in the transverse directions. However, transverse fluctuations in a narrow band of k_{\perp} are resonantly amplified by this oscillating background. The fluctuations appear as ripples in the field profile and energy density contours, and as a growing peak in the dimensionless power spectrum. Since the planar breather is an exact solution for the one-dimensional sine-Gordon model, very little radiation is produced during this stage. Of course, the growth of these fluctuations only continues until they begin to interact nonlinearly. At this point the behaviour changes dramatically, and rescattering effects
destroy the clean separation between the planar background and the fluctuations. The ripples in the breather from the transverse fluctuations become very large and pockets of field appear. Outside of the pockets the field is near the origin, while in the interior it is displaced towards one of the two neighbouring vacua. These pockets quickly condense into localized oscillating pseudostable blobs known as oscillons. The oscillons are very long-lived and are held together by a competition between attractive forces from the potential and the dispersion induced by the laplacian. During this condensation, radiation is released into the bulk, with the slow decay of the oscillons also emitting radiation into the bulk after their formation. In these cases, the characteristic transverse scale of the amplifed fluctuations is close the the final size of the oscillons, and the oscillons condense almost instantaneously from the pockets of field formed by the linear instability.



Figure 3.2: Evolution of the sine-Gordon breather with v = 1 including small fluctuations around the planar symmetric solution. The choice of plots are the same as Fig. 3.1, although we have modified the color schemes slightly. Notice the color scale normalization for the power spectrum differs in the leftmost plot compared to the other three. Corresponding animations for the field and energ density evolution may be found at www.cita.utoronto.ca/~jbraden/Movies/sg_v1_field.aviand www.cita.utoronto.ca/~jbraden/Movies/sg_v1_rho.avi.

Now let's consider the case of a colliding kink-antikink pair as illustrated in Fig. 3.3.

The evolution is in many ways similar to the two breathers considered above. Initially the field is well described by a colliding planar symmetric wall-antiwall pair. This time the transverse fluctuations experience an inhomogeneous generalization of broad parametric resonance, with the amplitude of fluctuations bound to the kink (and antikink) making a large jump at each collision. The typical transverse wavelength of these fluctuations is closely related to the period of the background via $k_{\perp} \sim T_{collision}^{-1}$ where $T_{collision}$ is the time between collisions of the kinks. Eventually the fluctuations become large enough that the next collision between the kink and antikink does not occur at the same time everywhere in space. For our sample run, this inhomogeneous collision occurs in the middle of the domain. As a result, punctures form between the two walls that begin to expand outwards. In this case, some segments of the walls pass through each other and manage to collide one more time at the periodic boundary, rather than immediately condensing into oscillons in the center of the domain. This leads to a very inhomogeneous state with large blobs where the field makes large excursions. These blobs eventually collapse into a collection of oscillons, just as in the case of breathers. For this study, we have taken the sine-Gordon model to simply provide the potential for a single-field scalar field model and have not imposed any additional identifications on the field. However, one could imagine that the field ϕ is instead an angular degree of freedom in some two-field model with the radial degree of freedom effectively trapped at the minimum. In this case, we may expect the fracturing process to result in the excitation of the radial degree of freedom and possibly the production of global strings. We do not explore this possibility here, although it could provide potentially interesting phenomenology in models based on small compactified extra dimensions.

3.3.2 Double-Well Potential

Thus far we have demonstrated that in the sine-Gordon model the inclusion of small initial nonplanar fluctuations around colliding planar domain walls can have a drastic effect on the dynamics, leading to a complete breakdown of the initial near planar symmetry of the fields. However, the sine-Gordon model in 1 + 1-dimensions (i.e. the planar limit) is integrable and thus a rather special theory. We now demonstrate that similar conclusions hold for the potential (3.2).

Low Incident Velocity Collision in Symmetric Double Well

As a first example, consider the case of low incident speed v = 0.05 in the symmetric double-well potential. Initially, the fluctuations are small and the system is well described



Figure 3.3: Evolution of repeated wall-antiwall collisions in the sine-Gordon model with periodic boundary conditions $\phi(x_{\parallel} + L_{\parallel}) = \phi(x_{\parallel})$ along the collision direction. In the top row we show slices of the field parallel to and orthogonal to the collision direction. In the middle row, we instead show contours of the energy density. The bottom row shows the evolution of the 2d angle averaged power spectrum \mathcal{P}_{ρ}^{2d} defined in (3.14). The top panel plots \mathcal{P}_{ρ}^{2d} as a function of transverse wavenumber k_{\perp} and position along the collision axis x_{\parallel} . The bottom panel plots the value along the green-line indicated in the top panel. Note the pseudocolor scale is different in the leftmost plot. In all three rows, the data are taken at t = 0,166,192, and 280. Corresponding animations can be found at www.cita.utoronto.ca/~jbraden/Movies/cascade_field_wfluc.avi and www.cita.utoronto.ca/~jbraden/Movies/cascade_contour_wfluc.avi.

by the background configuration of two planar walls as seen in the top panel of Fig. 3.4. The walls then move towards each other and first collide at $mt \approx 110$. After this initial collision the walls never become well separated and they appear as a localized oscillating blob that is very similar to the $v \geq 1$ sine-Gordon breathers above. However, when plotting energy density contours it still appears as though two individual walls are present. A small amount of planar radiation is released during these interactions, an effect which is properly captured by the symmetry reduced (1 + 1)-d dynamics. Far more importantly, there is a range of transverse fluctuations which grow exponentially in the background of this oscillating planar symmetric "blob". As these fluctuations



Figure 3.4: Several snapshots demonstrating various aspects of the time evolution of two colliding domain walls with initial speeds v = 0.05. From the top row to bottom: (a) the field distribution taken on a slice through the center of the collision $(mx_{\parallel} = 16)$, (b) contours of the energy density, (c) the dimensionless 2-d angular averaged power spectrum for the energy density as a function of position along the collision direction. We align our coordinates so that the collision occurs along the x-direction. The simulation parameters were mdx = 0.125, $mL_{\parallel} = 64$, $mL_{\perp} = 256$, mdt = 0.025. Absorbing boundary conditions were used at $mx_{\parallel} = 0, 64$. For this initial speed, the energy density at the center of the wall is $\rho/\lambda\phi_0^4 = 0.5/(1 - v^2) \approx 0.50125$. A description of the dynamics is given in the main text. An animation of the field is available at www.cita.utoronto.ca/~jbraden/Movies/field_v0.05.mp4, the evolving energy density at www.cita.utoronto.ca/~jbraden/Movies/pspec_v0.05.mp4.

grow, they appear as bumps and ripples on the oscillating blob, which are evident in the energy density contours. Eventually, these bumps become large enough that several sections of the planar blob pinch off, forming punctures of true vacuum in the planar symmetric blob. As a result of this, the region sandwiched by the colliding walls, where the field is displaced from the minimum, is threaded by tubes where the field is near the true vacuum. These tubes then begin grow in radius and eventually coalesce, leading to a network of fat filamets with the field near the false vacuum in the interior. This network is contained within a planar region of width $\sim m^{-1}$ along the collision axis and extends indefinitely in the directions orthogonal to collision (due to the original planar symmetry). This process is illustrated in the second and third columns of Fig. 3.4. In Fourier space, the developing network of filaments manifests itself as a rapidly growing tail of fluctuation power that extends to $k \sim 15m$. As the final step in the process, the filaments fracture into localized blobs of field—the oscillons of the double-well potential. Thus, exactly as in sine-Gordon model examples, a population of oscillons is produced in the collision region as the endpoint of the dynamical amplification of the symmetry breaking fluctuations.

An important quantity is the amount of energy that escapes from the collision region as scalar radiation versus the amount that remains stored in oscillons, shown in Fig. 3.5. In Fig. 3.5 we plot the energy density per unit transverse area

$$\sigma_2 = \frac{1}{A_{\perp}} \int d^2 x_{\perp} \int_{-L/2}^{L/2} dx_{\parallel} \rho \qquad A_{\perp} = \int d^2 x_{\perp}$$
(3.17)

remaining in the simulation box as a function of time. The production and longevity of the oscillons prevents a complete release of the energy originally stored in the domain walls into the bulk. However, relative to the case of exact planar symmetry, much more of the energy is released as radiation (at least up until the end of our simulations). Of course, this occurs because in the planar limit the wall and antiwall don't immediately annihilate each other, but instead form a long-lived bound state. The planar bound state is the one-dimensional version of the oscillon for this potential, and since it is very long lived the energy release in the planar case is slow.

Interactions in a resonant escape band

As an example of an even more dramatic effect induced by the breaking of planar symmetry, let's now choose an incident velocity of v = 0.2. In the planar case, the walls now bounce twice before escaping back to infinity, which is illustrated in the top right panel of Fig. 2.3 of chapter 2. During the first collision the planar shape mode is excited, draining energy from the kinetic motion of the (planar) walls. In the second collision, the nonlinear field interactions cause energy to be transferred from the excited shape mode back into translational energy of the (planar) walls. This process clearly requires a tuning between the oscillation period of the shape mode and the time between collisions for the two-walls. If this tuning is disrupted or energy is drained from the planar oscillations of



Figure 3.5: Energy within a slab of width $mL_{slab} = 10$ centered on the initial collision of the two walls for v = 0.05. We have plotted the result for both bulk fluctuations and transverse fluctuations in the wall's location, as well as for a range of box sizes and grid spacings. Also shown for comparison is the result if no fluctuations are included. Unless indicated in the legend, we used mdx = 0.25, $mL_{\perp} = 128$ and $mL_{\parallel} = 64$. If a value of \mathcal{A}_b is listed in the legend we used initial conditions (3.12), while if λ is listed we used (3.11). The oscillations in the planar solution are due to the slab being slightly small that the size of the region occupied by the bouncing walls.

the shape mode, then rather than escape the walls will capture each other instead.

When we include the transverse fluctuations, they experience a nonadiabatic kick at each collision. As well, since the (homogeneous) shape mode is excited during the first collision, further pumping occurs while the walls are separated. The energy required to amplify the fluctuations must be drained from the kinetic energy of the planar background and the oscillations of the planar shape mode. Since the amplification is a linear effect, the size of this backreaction on the planar background increases with the initial amplitude of the fluctuations at the start of the simulation. Therefore, for sufficiently large initial fluctuation amplitudes, the resulting backreaction will prevent the walls from escaping back to infinity. In this case the qualitative behaviour of the system changes; instead of escaping back to infinity, the walls capture each other and fracture into oscillons. To illustrate this, Fig. 3.6 shows the energy per unit area (3.17) within a slab of width $m\Delta x_{\parallel} = 10$ centered on the collision for a range of initial fluctuation amplitudes. For an isolated kink moving at speed v with $L = \infty$ this is $2\sqrt{2}/3\sqrt{1-v^2}$. Two distinct behaviours are visible: either ρ drops abruptly to zero by $mt \sim 200$ corresponding to the case when the two walls escape back to infinity, or else ρ slowly decays for $mt \gtrsim 200$ corresponding to the case when the walls capture and subsequently fracture into oscillons. The transition between these two behaviours occurs as we increase the



Figure 3.6: Energy density within a slab of width $mL_{slab} = 10$ for initial speed v = 0.2 and various initial fluctuation amplitudes $(m\delta x)^2$ for the 2-d gaussian random field corresponding to the local translations in the wall position. For the simulations we used a box of size $L_{\parallel} = 64, L_{\perp} = 128$ and dx = 0.25, and we used the initial fluctuations (3.12) with $\delta v = 0$.

initial amplitude of the fluctuations. We only included fluctuations associated with the translation mode of the walls (3.12) and also set $\delta v = 0$. Hence, this underestimates the true amount of backreaction on the walls and is only an illustration of the dramatic effect that amplified fluctuations can have on the background. We have not carefully explored the space of initial fluctuation amplitudes, so more complicated behaviour such three bounces before escaping or annihilating may also be possible for finely tuned initial fluctuations amplitudes.

3.3.3 Asymmetric Double-Well Potential

Finally, consider collisions in the asymmetric double well. As a concrete example, we take $\delta = 1/30$ and give the walls an initial separation $2mr_{init} = 16$ and initial speeds v = 0. The vacuum energy difference between the two wells causes the walls to accelerate and they collide with an initial speed $v^2 = \frac{\mu(\mu+2)}{(1+\mu)^2} \sim 0.6$ where we've defined $\mu = r_{init}\Delta\rho/E_{kink} \sim 0.57$ with $\Delta\rho \sim 2\delta\lambda\phi_0^4$ and $E_{kink} = 2\sqrt{2}\sqrt{\lambda}\phi_0^3/3$. Because their relative velocity at collision is much larger than the cases considered above, they separate much further from each other between collisions. However, the pressure induced by the nondegeneracy of the two vacua eventually turns the two walls around and prevents them from escaping back to infinity.

The resulting behaviour is illustrated in Fig. 3.7. Initially we have two well-separated walls with small fluctuations. The walls accelerate towards each other due to the non-degeneracy of the vacua, and then bounce off of each other multiple times. At each

bounce, a band of transverse fluctuations are excited. As well, planar symmetric shape modes are excited by the collision. Between the bounces, the two walls separate with the shape mode further pumping transverse excitations of the field. By the time of the final collision, these transverse fluctuations have become quite large. They are visible in the second column of Fig. 3.7 as bumps in the energy density contours, bumps in the field profile, and as peaks in the power spectrum. The nearly planar radiation emitted during the early stages of the evolution is also visible in the field profile.

Aside from the pumping of fluctuations by the planar shape mode becoming a distinct process, the biggest difference between this case and the v = 0.05 walls in the symmetric well is the inhomogeneity of the final disintegration of the walls. In the case we have illustrated, fluctuations with transverse wavelengths much larger than the typical size of the bumps in the walls and the oscillons that eventually form were excited. As a result, the final collision is inhomogeneous on scales much larger than the size of the individual transverse ripples on the walls as seen in the third row. Once again the walls become threaded by tubes where the field is near the true vacuum. These tubes then expand producing a network of filaments with the field trapped near the false vacuum in the interior, just as in the v = 0.05 case. Subsequently these filaments fracture and a collection of oscillons is again formed.

For the case illustrated here, the walls collide five times with the fifth collision resulting in the breakup of the walls and ultimately the production of oscillons. Whether or not a long-wavelength mode was excited in an individual collision depends on the particular realization of the initial fluctuations used in the simulation. An additional interesting feature of this particular run was the ejection of a pair of oscillons into the bulk during the breakup of the filamentary network. These are visible as two isolated peaks at $mx_{\parallel} \sim 15,50$ in the transverse power spectrum in the right column of Fig. 3.7. These are a pair of oscillons that scattered off of each other during the disintegration of the walls. This ejection of oscillons into the bulk was sensitive to the particular realization of the fluctuations.

3.3.4 Growth of fluctuations from shape mode

Thus far we have considered the three-dimensional dynamics of colliding wall-antiwall pairs (the higher dimensional equivalent of a kink-antikink pair in one-dimension) including small initial planar symmetry breaking fluctuations.² Among the cases we considered were the oscillations of tightly bound wall-antiwall pairs (such as the v > 1 sine-Gordon

²Statistically the fluctuations still respect the planar symmetry, but individual realizations do not.



Figure 3.7: Collision dynamics in the asymmetric well with $\delta = \frac{1}{30}$ during several different stages of evolution. The choice of illustrations match those in Fig. 3.1. From left to right, we have snapshots of the initial setup (mt = 0), just prior to the final collision (mt = 108), during the fracturing process (mt = 136) and showing the final population of oscillons (mt = 250). The green line in the pseudocolor plot of the 2d power spectrum indicate both position of the orthogonal slice of the field (top row) as well as the slice along which the 2d spectrum is plotted (bottom panels, bottom row). Animations of the field evolution can be found at www.cita.utoronto.ca/~jbraden/Movies/field_nondegen.avi, the energy density evolution at www.cita.utoronto.ca/~jbraden/Movies/econtour_nondegen.mp4, and the power spectrum evolution at www.cita.utoronto.ca/~jbraden/Movies/pspec_nondegen.mp4.

breathers) and repeated collisions between weakly bound pairs (such as the asymmetric double well). In this latter case, the internal dynamics (specifically planar symmetric excitations of the shape mode) of each individual wall plays an important role both in the dynamics of the background planar solutions as well as the growth of fluctuations. This is especially true for double-well potential at large collision velocities where the walls either escape back to infinity (symmetric well) or become well separated from each other between collisions (asymmetric well). In order to isolate the effects of the internal wall dynamics, we study a single domain wall at rest in a symmetric double well with an initial width different from that of the static kink solution

$$\phi_{init} = \phi_0 \tanh\left(\frac{m(x-x_0)}{mw}\right) + \delta\phi_{fluc} \,. \tag{3.18}$$

where $mw \neq \sqrt{2}$ and $\delta \phi_{fluc}$ is again a realization of random field whose statistics are not important for the qualitative effect we present here. This choice of initial background mimics the excitation of a planar symmetric shape mode.

Just as for collisions, the presence of transverse fluctuations radically modifies the behaviour of the field. A very explicit demonstration of this comes from studying the energy contained within our simulation volume as shown in Fig. 3.8. Once again, initially the energy decreases via emission of planar symmetric radiation that can be captured via 1 + 1-dimensional simulation. However, as the shape mode is oscillating it pumps transverse fluctuations, eventually leading to the onset of strong nonlinearities in the symmetry breaking fluctuations. At this point energy is released much more rapidly and approaches the energy of a single isolated domain wall. As expected, the timing of this transition depends on the initial amplitude and spectrum of the fluctuations, but the subsequent behaviour is rather insensitive to these details.



Figure 3.8: Time dependence of the excess energy density within the simulation volume for a box with length parallel and orthogonal to the collision given by $mL_{\parallel} = 32$ and $mL_{\perp} = 128$. Absorbing boundary conditions were used in the collision direction and the grid spacing was mdx = 0.25. In the left figure we show the result for several choices of the initial wall width. Meanwhile, the right figure shows the result for several choices of initial fluctuations and mw = 2. Also shown for comparison are the results if no initial fluctuations are present. The effect of the fluctuations is clearly visible as a sharp change in the rate of energy loss relative to the homogeneous case.

To shed further light on the field dynamics and the origin of the decay in excess energy



Figure 3.9: Several snapshots of the growth of transverse fluctuations due to excitation the planar shape mode of the wall. *Top row:* Slice of the field ϕ parallel to the collision axis and orthogonal to the collision axis and centered on the middle of the wall. *Bottom row:* Two-dimensional power spectrum \mathcal{P}_{ρ}^{2d} defined in (3.14). An animation of the field evolution can be found at www.cita.utoronto.ca/~jbraden/Movies/field_ shapemode.avi.

bound to the domain wall, Fig. 3.9 shows several snapshots illustrating the evolution of the wall as it passes through the nonlinear phase. Initially the field is nearly homogeneous and looks roughly like a tanh function with a vibrating width. This means that bound fluctuations on the wall are bound in a well whose width (and shape) are oscillating in time. These oscillations pump a narrow band of transverse fluctuations as expected from linear perturbation theory. As these fluctuations grow, a speckled pattern emerges in the field distribution taken along a slice through the middle of the wall. This pattern is superimposed on the overall oscillation of the wall. Eventually, the transverse oscillations enter the nonlinear regime, leading to a slight broadening of the peak in the spectrum and the emission of radiation from the wall. The overall oscillation of the wall disappears and instead bound transverse lumps of field appear along the wall as seen in the third and fourth rows. These transverse lumps persist for times $mt \gtrsim 800$. We can now understand the residual energy relative to an unexcited domain wall seen in Fig. 3.8 as being due to this population of "wall lumps". Comparing the second column of Fig. 3.7 with the second column of Fig. 3.9, we can see that the pumping of fluctuations by the shape mode is indeed an important effect for collisions in the asymmetric well.

3.4 Oscillons as a Long-Lived Intermediate State

By solving for the full nonlinear evolution of wall-antiwall collisions, we found that the stage of rapid growth of linear fluctuations is followed by a very short stage during which the walls dissolve. At the end of this stage, we are left with a population of quasi-stable localized oscillating blobs of field distributed in a narrow plane around the collision site. These field structures are known as oscillons and arise from a dynamical balancing act between the dispersive effect of the laplacian and the attractive force from the potential. Oscillons were first discovered by Bogolyubsky and Makhanov [110] and then rediscovered by Gleiser [111]. There is a large literature devoted to their properties [112, 95, 113, 114, 115, 116, 117, 118, 119] and interactions [120, 121]. Several studies of the classical [122, 123, 124, 125] and quantum [126, 127, 128] decay of these objects have also been performed. A number of production methods have been studied: collapse of subcritical bubbles nucleated during first-order phase transition [129], production from collapsing domain wall networks [120], production from homogeneous field oscillations around a false vacuum minimum as a method of facilitating the formation of true vacuum bubbles [130, 131, 132], amplification of thermal fluctuations during inflation [133], and production as a result of preheating at the end of inflation [134, 135, 136, 137].

In our study oscillons appear in two forms. The first is as the localized blobs of oscillating field in the planar background dynamics, which are the planar equivalent of oscillons in the corresponding one-dimensional field theory. More interesting are the three-dimensional oscillons that form at the end of the fracturing of the walls. The production mechanisms listed above are based on the amplification of fluctuations around a homogeneous field background; thus the resulting oscillons tend to be homogeneously distributed throughout the bulk. In this sense, our mechanism is somewhat different since we have a strong localization along the collision axis. Of course, the oscillons are still distributed uniformly in the directions transverse to the collision axis, and an observer restricted to the plane transverse to collision would indeed view their formation as a homogeneous process.

We now consider some simple oscillon properites of relevance to our domain wall collisions. For simplicity, we will only explicitly consider oscillons in the symmetric double-well. First we demonstrate that an *isolated* localized blob of field displaced from the minimum of the potential can indeed collapse to form an oscillon. This problem has been studied using the assumption of exact spherical symmetry for the blob [129, 138], and also for the case nonspherical blobs in two spatial dimension [139, 124].

As seen from our simulations, oscillons ultimately form from the fracturing of the wall



Figure 3.10: Energy in a sphere of radius 12.5*m* as a function of time for several choices of initial elliptical field profiles given by (3.19). We have parameterized the departure from spherical symmetry by $\epsilon^2 = 1 - R_{min}^2/R_{max}^2$ where $R_{min} = \min(a, b)$ and $R_{max} = \max(a, b)$. On the left we plot the results for cigar shapes with and on the right for pancake shapes. For definitions of the pancake and cigar configurations see the main text.

and subsequent collapse of either a network of tubes or densely packed blobs where the field is displaced from the true vacuum minimum in the interior. As a very rudimentary approximation to this, we consider initial field profiles given by

$$\phi_{init} = \phi_{true} + \left(\phi_{false} - \phi_{true}\right) \exp\left(-\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\right)$$
(3.19)

with $\dot{\phi} = 0$. While this initial profile and the choice $\dot{\phi} = 0$ are simplifications of what is seen in our simulations, whether or not an oscillon forms is not sensitive to the initial field configuration or we would not see them form at all. In order to reduce the phase space of initial configurations, we further impose that $b^2 = c^2$. By choosing $a^2 > b^2$ we get cigar-like initial blobs and $a^2 < b^2$ gives us pancake like configurations.

In Fig. 3.10 we show the energy contained within a sphere of radius $12.5m^{-1}$ centered on the initial blob for several choices of initial asymmetry in the blob. We see that provided the blobs are not too asymmetrical the energy within the sphere reaches a longlived plateau indicating the presence of an oscillon. For all initial conditions that result in an oscillon the plateau energy is the same, suggesting that the final oscillon states are all very similar for this particular model.³

If the final population of oscillons do indeed have the same profile, then the 2d power spectrum at the collision location in the oscillon dominated regime simplifies tremen-

³There exist models for which a range of different oscillons with different radii exist [119].

dously. In particular, since unbound forms of energy such as radiation escape the collision region, the power spectrum will be dominated by the contribution from the remaining population of oscillons. We can approximate the field near the collision site as

$$\phi \approx \sum_{\alpha_i} \phi_{osc,i}(\mathbf{x} - \mathbf{x}_{\alpha_i}, t)$$
(3.20)

where the oscillons are located at positions x_{α_i} and the profile of the *i*th oscillon is given by $\phi_{osc,i}$. Similar expressions hold for other derived fields such as the energy density. The distribution of x_{α} will be determined by the (random) realization of the initial fluctuations, as well as the choice of initial planar background. In the special case that $\phi_{osc,i}$ is independent of *i* (ie. all of the oscillons have the same shape) and the final positions are uncorrelated with the oscillon shape, the resulting power spectrum simplifies tremendously and we obtain

$$\left\langle |\tilde{\phi}_k|^2 \right\rangle = \left\langle |\sum_{\alpha_i} e^{i\mathbf{k}\cdot\mathbf{x}_{\alpha_i}}|^2 \right\rangle \left\langle |\tilde{\phi}_{osc}(k)|^2 \right\rangle \equiv P_{form}(k)P_{oscillon}(k) \tag{3.21}$$

In the general case, this expression becomes significantly more complicated, in particular if there are many possible oscillon profiles and the positions and profiles are correlated. Various properties such as the size and phase of oscillation of each of the oscillons will be randomly drawn from some distribution, and there may be correlations between properties such as the size of an oscillon and its position relative to other oscillons.

However, given the potential to extract information about the distribution of oscillons from measurements of two-point correlations, it is worthwhile to explore the possible spectra of individual oscillons as well as a characterize the range oscillon properties and time-dependence. For our semiclassical simulations, we could of course extract this information directly in real space. However, in numerical methods based on evaluation of a hierarchy of n-point correlation functions (such as the nPI formalism), this direct approach is not available.

A complete characterization of all oscillon properties in an arbitrary field theory is a rather daunting numerical task, so here we simply provide the spectra for a sample oscillon in the double-well potential. The energy plateaus we show above provide some preliminary evidence that the oscillons themselves have a very narrow range of properties and hence studying only a small sample of them may be sufficient. Figure 3.11 shows the energy density and 2d power spectrum for an oscillon formed from an initial Gaussian field blob of radius $mr_{init} = 3$. After a short transient, the field quickly settles down



Figure 3.11: A sequence of snapshots illustrating the time evolution of an oscillon formed from an initially spherical gaussian field profile $\phi_{init} = 1 - 2e^{(r/r_{init})^2}$ with $mr_{init} = 3$ in the symmetric double well. In the top row are 2-dimensional slices of the energy density taken along a slice through the middle of the blob. In the bottow row is the dimensionless 2-dimensional power spectrum as a function of position along the slicing axis (top panel, bottom row) and also along the line indicated by the green line in the top panel (bottom panel, bottom row). An animation of the energy density is available at www.cita.utoronto.ca/~jbraden/Movies/oscillon_rho.aviand the power spectrum at www.cita.utoronto.ca/~jbraden/Movies/pspec_oscillon.avi.

into an oscillon configuration. The energy density oscillates in time and looks like a shell of energy density that begins to expand outward before collapsing to form a sharp peak and subsequently expanding outward again. Comparing the spectra with Fig. 3.7, we see that the two isolated blobs of power away from the collision region in the asymmetric double-well collision are indeed oscillons that were ejected from the collision region.

3.5 Summary of Mechanism

We have explored the full nonlinear three-dimensional evolution of colliding nearly planar domain wall-antiwall pairs for various choices of background solutions in the sine-Gordon and double-well potentials. We now highlight the features of the dynamics that are most important in determining the qualitative outcomes of the collisions. Although the evolution of the fields is complex and difficult to intuit without the aid of numerical simulation, once the results are known the qualitative behaviour of the solutions is simple to understand.

Initially, the system is accurately described by a planar symmetric background $\phi_{bg}(x, t)$ and a collection of small nonplanar fluctuations $\delta\phi$, which couple to the background through an effective mass squared term $V''(\phi_{bg})$. When the walls collide repeatedly or form oscillating bound states, this coupling drives an instability in the fluctuations which eventually pushes the entire system into the nonlinear regime. Since the amplified fluctuations vary in the directions transverse to the collision, when they become large the walls develop noticable bumps and ripples.

The details of the next stage of the evolution depend on the particular choice of planar background and potential. In one scenario, which occurs for the sine-Gordon breathers with $v \ge 1$, the fluctuations "pinch off" to form an egg-carton like structure, leading to a collection of densely packed pockets, with the field near the origin outside the pockets and displaced from the minimum inside the pockets. In a second scenario, seen here in the case where the wall and antiwall repeatedly collide with each other, this stage is instead characterized by a final inhomogeneous collision between the wall/antiwall pair. A heuristic illustration of this final collision is given in Fig. 3.12. During this collision, punctures develop that thread the walls with a region of true vacuum. These punctures then expand and eventually coalesce to leave behind an approximately two dimensional network of tubes. Inside the tubes the field is near the false vacuum, while outside it is near the true vacuum. To visualize this process, from the viewpoint of observations restricted to an orthogonal slice of the field through the collision region it looks like the nucleation, expansion and coalescence of a collection of bubbles in two dimensions.

In either of the two cases outlined above, the collision dynamics ultimately results in the formation of a highly inhomogenous field configuration with many peaks where the field is displaced from the true vacuum. These are distributed in the transverse plane, but strongly localized to the collision center along the collision axis. As demonstrated in 3.4, *isolated* peaks of the field can nonlinearly collapse to form oscillons. Thus, we might expect that when we have a field configuration with many peaks, some of them might also eventually form oscillons. Indeed, we find that this is the case. Of course, nonlinear interactions between the various peaks as the collapse is occurring leads to a more complicated scenario than the case of a single isolated peak, but the effects of these interactions are insufficient to completely disrupt the production of oscillons.



Figure 3.12: A heuristic view of the final collision between a wall-antiwall pair leading to the population of peaks in the field that subsequently condense to form oscillons. White corresponds to the regions in the true vacuum, the green angle hatched pattern to regions near the false vacuum minimum and the red hatched pattern to regions which have experienced a displacement due to the collision (initially described by free passage). In the double well the field in the red region is initially displaced up the potential past the true vacuum, while in the sine-Gordon potential the field will be sitting near another minimum of the potential. The red regions subsequently form the tubes of "true vacuum" that puncture the domain walls. *Left:* The two walls with just before the final collisions. *Center:* The first moments of the final collision. Due to the large fluctuations in the location of the wall, the collision occurs asynchronously at different locations. *Right:* The production of pockets of false vacuum as a result of the inhomogeneous nature of the final collision.

3.6 Conclusions

In this chapter we performed full nonlinear three-dimensional simulations of parallel planar domain wall collisions, including the effects of initially small quantum fluctuations. This allowed us to probe nonlinear regimes not accessible to the linear analysis of the fluctuations in chapter 2. As anticipated from the resonances found in the linear analysis, early in the evolution the fluctuations grow rapidly during collisions between the walls. However, the most interesting phenomenology arises once the fluctuations begin to nonlinearly interact with each other, causing a complete breakdown of the original planar symmetry. For the collisions we considered, this symmetry breakdown results in an extremely inhomogeneous dissolution of the walls and eventually the production of oscillons distributed in the collision plane. This is a radical departure from the behaviour expected from studying symmetry reduced one-dimensional collisions. It is also completely different than the result if the backreaction of the fluctuations on the domain walls is treated as a homogeneous effect in the transverse plane, such as a Hartree-like approximation would assume. Therefore, this is a completely new phenomenology which can only be adequately studied using lattice simulations such as those employed in this chapter.

The dissolution of the walls is a consequence of our restriction to collisions between wall-antiwall pairs, which means there is no topological conservation law preventing the walls from eventually annihilating each other. If we were to consider collisions between walls interpolating between different vacua as $|x| \to \infty$, then the post-collision state must contain domain wall like structures interpolating between the two different vacua at infinity. Although the final dissolution of the walls will not occur in this case, our general finding that the planar symmetry is broken should continue to hold. A specific example of this was our study of the fluctuations about a single isolated domain wall with an oscillating width. In more general collisions, we expect that any walls remaining after the collision will not be produced with perfect planar symmetry, but instead will be bumpy.

In the next chapter we consider the full nonlinear dynamics of SO(2,1) collisions between vacuum bubbles. We will find that similar phenomenology arises when the full nonlinear problem in that case is treated properly.

Chapter 4

Cosmic bubbles and domain walls III: The role of oscillons in three-dimensional bubble collisions

4.1 Introduction

In this chapter we use numerical simulations to study the full nonlinear three-dimensional dynamics of collisions between pairs of true-vacuum bubbles nucleating in an ambient false vacuum. This is the third and final installment in our investigation of collisions between highly symmetric domain walls, which we began in chapter 2 with a linearized analysis and continued in 3 with numerical simulations of colliding planar walls. The collisions studied in this chapter are expected to occur in inflationary models based on the false vacuum eternal inflation scenario [11, 12], which is outlined in section 1.4 of the introductory chapter. If our observable universe underwent first-order phase transitions in its infancy, they would similarly have resulted in bubble collisions. This latter case would involve collisions between many bubbles, although individual collisions would contribute to the dynamics.

In the context of false vacuum eternal inflation, we inhabit an open inflationary universe. Many studies have considered the phenomenology of such open inflationary models [77, 78, 79, 80, 81]. There have also been many studies of collisions between vacuum bubbles, beginning with the work of Hawking, Moss and Stewart [73]. Early studies were motivated by the dynamics of early phase transitions within our horizon, with much of the focus on the production of gravitational waves [140, 97, 98, 141, 142, 143]. More recent studies are often motivated by false vacuum eternal inflation [100, 22, 103, 144]. These investigations have culminated in a general relativistic treatment of the bubble problem [105, 145]. However, all of these studies assume the dynamics obeys an SO(2,1) symmetry, and no consideration is given to perturbations that do not obey this symmetry. Even studies of fluctuations around the SO(2,1) collision restrict to perturbations than can still be evolved using the approximation of 1+1-dimensional field theory [104, 146, 147, 148, 99]. However, for the case of a single bubble, the evolution of a more general set of fluctuations have been considered [106, 107, 108].

In an effort to observationally constrain this scenario, many recent studies have proposed observational signatures from the collision between our vacuum bubble and an external "collision" bubble. Several data searches have also been performed in conjunction with theoretical work, but thus far no signal has been seen [149, 150, 151, 152, 153, 154, 155].

Since observing a collision with another bubble universe would revolutionize our understanding of the cosmos, it is important to properly assess the possible outcomes of collisions. In particular, the validity of the SO(2,1) symmetry assumption has never been explicitly checked. We will demonstrate in this chapter that for a broad class of collisions, the SO(2,1) symmetry is badly broken due to dynamical amplification of quantum fluctuations. Since the SO(2,1) assumption severely constrains the form of the final observable signals, it is possible that the searches performed thus far have been blind to other signatures produced by the collision. The results presented in this chapter are a first step towards a more complete understanding of vacuum bubble collisions. Although we will focus on the dynamics of the collisions rather than the final observable signatures, our results suggest several interesting new observational avenues for testing the false vacuum eternal inflation paradigm.

The remainder of the chapter is organized as follows. In section 4.2 we present our models and outline a pseudospectral approach to solve for the initial bubble profile and evolve the system. Section 4.3 considers the evolution of individual bubbles, both in the thin-wall and thick-wall regimes. A brief review of the collision dynamics under the assumption of SO(2,1) symmetry is given in section 4.4. We demonstrate that our lattice simulations reproduce the expected behaviour and test the sensitivity of the dynamics to the accuracy of the initial bubble profile. Our main results are then given in section 4.5 where we study collisions between bubbles in a variety of single-field models, including small initial fluctuations around the SO(2,1) profile. We extend these findings to a simple two-field model that permits inflation inside the bubble in section 4.6. Some brief comments on possible observable signatures and extensions to other scenarios are given in sections 4.7 and 4.8. Finally we conclude in section 4.9



Figure 4.1: Plots of the potential for several choices of δ . On the left we show the linear symmetry breaking (4.1) and on the the right the cubic symmetry breaking (4.2). The largest value of δ in the linear potential is that for which the second minimum disappears.

4.2 Single Field Models and Description of Initial Conditions

To provide a concrete setting for our investigations, we focus on double well potentials. In order for bounce solutions describing vacuum tunnelling to exist in Minkowski spacetime, the false vacuum and true vacuum must not be degenerate. We make two choices for the symmetry breaking

$$V_{linear}(\phi) = \frac{\lambda}{4} \left(\phi^2 - \phi_0^2\right)^2 - \delta\lambda\phi_0^3 \left(\phi - \phi_0\right) + V_0$$
(4.1)

and

$$V_{cubic}(\phi) = \frac{\lambda}{4} \left(\phi^2 - \phi_0^2\right)^2 + \delta\lambda\phi_0^4 \left(\frac{\phi^3}{3\phi_0^3} - \frac{\phi}{\phi_0} + \frac{2}{3}\right) + V_0$$
(4.2)

which we refer to as the linear and cubic symmetry breaking potentials respectively. In both cases δ controls the difference between the false and true vacuum energies and V_0 can be adjusted to give the desired true vacuum energy. Our investigations here are restricted to Minkowski space, so V_0 will not influence the dynamics.¹ For the linear potential (4.1), the second minimum disappears for $|\delta| \geq 2/\sqrt{3}$, while for the cubic potential (4.2) this occurs when $|\delta| \geq 1$. In the linear potential, the locations of the two minima depend on δ , while they are fixed at $\pm \phi_0$ for the cubic potential. Unless explicitly stated, dimensionful parameters are measured in units of $m \equiv \sqrt{\lambda}\phi_0$ with the exception of the field which is measured in units of ϕ_0 and the energy density ρ measured in units of $\lambda \phi_0^4$.

 $^{^{1}}$ We also performed several runs with a homogeneous background Hubble and found no qualitative change to our results.

4.2.1 Solution of the Instanton Equation

In this subsection we present a new and extremely accurate numerical approach to determine the shape of the nucleated bubbles. Throughout this paper we assume that the *most likely* bubble to nucleate within a surrounding false vacuum possesses an SO(4)symmetry and is described using the bounce formalism of Coleman [74, 75, 15]. For a single scalar field in Minkowski space with potential satisfying some regularity conditions, it has been shown that this is indeed the minimum action solution relevant to false vacuum decay [76]. In Minkowski space, this profile satisfies the Euclidean signature equation

$$\frac{\partial^2 \phi}{\partial r_E^2} + \frac{3}{r_E} \frac{\partial \phi}{\partial r_E} - \frac{\partial V}{\partial \phi} = 0 \tag{4.3}$$

with the boundary conditions

$$\phi(r_E = \infty) = \phi_{\text{false}} \qquad \frac{\partial \phi}{\partial r_E}(r_E = 0) = 0.$$
 (4.4)

Here $r_E^2 = d\tau^2 + d\mathbf{x}^2$ is the Euclidean radius and $\tau = it$ is the Euclidean time.

Before presenting our technique, we briefly review the most common analytic approach (the thin-wall approximation) and numerical approach to solving (4.3). In the thin-wall limit, valid if the initial radius of the bubble is much greater than the width of the wall, we can obtain simple analytic estimates for the bounce profile. For the potentials (4.1)and (4.2), this approximation will be appropriate when $\delta \ll 1$. In the simplest form of the approximation, we drop the friction term and the terms proportional to δ in the equation of motion. This leaves the equation for a domain wall in the degenerate double well $\frac{\lambda}{4} (\phi^2 - \phi_0^2)$. Performing the usual quadrature gives an expression for the field profile, $\phi_{wall} = \phi_0 \tanh((r - R_0)/\sqrt{2})$. We then adjust the multiplier on the tanh and add a constant so that the field interpolates between the false and true vacuum. Finally, conservation of energy gives the initial bubble radius as $R_0 = 3\sigma/\Delta\rho$ where $\sigma = \int dr (\partial_r \phi_{wall}(t=0))^2$ is the surface tension of the bubble wall and $\Delta \rho = V(\phi_{false}) - V(\phi_{false})$ $V(\phi_{true})$ is the difference in potential energy between the false and true vacuum. For the linear potential (4.1) we have $R_0^{linear} = \sqrt{2}\delta^{-1}$ and for the cubic potential (4.2) we have $R_0^{cubic} = \frac{3}{\sqrt{2}} \delta^{-1}$. It should be clear that with some modifications this approximation can be applied to other potentials. In particular, by replacing ϕ_{true} with the location where the field tunnels out to ϕ_{tunnel} , we could apply this in situations where the field does not tunnel out near the true vacuum. Of course, in this case it is necessary to either estimate ϕ_{tunnel} or else determine it by solving the bounce equation (4.3) in order to obtain the initial radius. If the latter approach is used, then the thin-wall approximation itself will provide little utility other than aiding intuition.

Although the simplicity of the final result makes this approach very attractive, there are situations where the thin-wall approximation is either invalid or else insufficiently accurate. When this happens we must turn to a numerical solution of (4.3). In the literature, this equation is usually solved via a shooting method, which recasts the ODE into a root finding problem $\vec{\phi}(r_E = \infty | \vec{\phi}(r_E = 0) = \vec{\phi}_0) - \vec{\phi}_{false} = 0$ for the initial field value $\vec{\phi}_0$. For the single-field case, the root finding is often done via besection and the resulting algorithm is known as the overshoot-undershoot method. To extend this to the multifield case one needs to use a root-finding method that works in multiple dimensions. An obvious choice is a Newton-type method, with the required derivatives obtained by solving for a collection of nearby trajectories and then doing a polynomial interpolation. Of course, the desired solution is a saddle point and care must be taken to evaluate these derivatives before the neighbouring trajectories diverge away from the target solution.

Rather than adopt this method, we instead use a pseudospectral approach and expand the function in even rational Chebyshev functions on the interval $(-\infty, \infty)$

$$\phi_{bounce}(r_E) = \sum_i c_i B_{2i} \left(y \left(\frac{r_E}{\sqrt{r_E^2 + L^2}} \right) \right)$$

$$y(x) = \frac{1}{\pi} \tan^{-1} \left(d^{-1} \tan \left(\pi \left[x - \frac{1}{2} \right] \right) \right) + \frac{1}{2}$$

$$(4.5)$$

where B_n are the Chebyshev polynomials. Since this is a global expansion method, it displays excellent convergence properties as the number of lattice sites is increased and machine-precision accuracy is easily obtained. Without the mapping y(x), this is simply an expansion in the rational Chebyshev functions $TB_n(y) = B_n\left(\frac{y}{\sqrt{y^2+L^2}}\right) =$ $\cos\left(n\cot^{-1}\left(\frac{y}{L}\right)\right)$ on the doubly-infinite interval. L is an adjustable parameter that determines where the oscillations in the rational Chebyshev functions (or equivalently the collocation points) are clustered on the infinite interval. The even TB(x)'s form a complete set for functions which asymptote to a constant at infinity and are symmetric about the origin (such as the bounce), and thus they enforce the $r_E = 0$ boundary condition automatically. We also include an additional mapping y(x) which clusters (repels) points around $r_E = L/\sqrt{3}$ if d < 1 (d > 1) without simultaneously introducing new singularities at the boundaries which would greatly reduce the convergence rate of the expansion. Via a judicious choice of L and d, this allows us to properly resolve instantons to machine precision even in the extremely thin-wall case. We illustrate the efficiency of this expansion in Fig. 4.2, where we show instantons for four extremely thin-walled bubbles as well as the spectral coefficients in the above expansion. Numerically this is the most difficult case if no finesse is used in the solution, so it is encouraging that we are able to tackle this limit. In all cases the round-off plateau for the coefficients is clearly present at $i \approx 100$,



Figure 4.2: Some thin-walled bubbles. In each of the cases shown, the ratio of the wall width $(w = \sqrt{2})$ to the initial bubble radius is $w/R_0 \sim \delta$. In the left panel we show the instanton profiles (zoomed into the region $r_E \leq R_0$), while in the right panel we show the resulting spectral coefficients. For all cases, we took $L = \sqrt{3}R_0$ with $R_0 = \sqrt{2}\delta^{-1}$ for the linear potential and $R_0 = 1.5\sqrt{2}\delta^{-1}$ for the cubic potential. Meanwhile, the "stretching" parameter d was d = 0.022, 0.0045 for $\delta = 0.005, 0.001$ in the linear potential, and d = 0.03, 0.014 for $\delta = 0.01, 0.005$ in the cubic potential.

indicating we have hit the limits of double-precision arithmetic. Through the use of a smart collocation grid, we have managed to achieve this accuracy using fewer modes than the ratio of the bubble radius to its width R_{init}/w . To demonstrate the utility of this approach across a range of potential deformations, Fig. 4.3 shows the instanton profiles in the linear (4.1) and cubic potential (4.2) for a range of δ . As an added bonus, our outer collocation point is located at $r_E = \infty$ so there are no errors introduced by putting the system in a finite box.

Of course, the bounce solution only describes the most likely profile for the field and there are additional fluctuations which we include by taking the initial conditions for a single bubble to be

$$\phi_{init}(\mathbf{x},0) = \phi_{bounce}(r_E = |\mathbf{x}|) + \delta\phi \qquad \dot{\phi}_{init}(\mathbf{x},0) = \delta\dot{\phi}$$
(4.6)

where ϕ_{bounce} is a solution of (4.3) and $\delta \phi$ and $\delta \dot{\phi}$ are realizations of random fields. This is generalized to multibubble initial conditions in the obvious way. The fluctuations $\delta \phi$ encapsulate the effects of quantum fluctuations, including deviations of the nucleated bubble from perfect SO(3,1) symmetry. Since we are taking the viewpoint that these



Figure 4.3: In the top row we show instanton profiles for the linearly asymmetric double well (4.1) for a range of δ values. The left panel shows the instanton profiles (zoomed into the region where the field is varying). Meanwhile the right panel shows $\phi(r_E = 0)$ which becomes the initial field value at the center of the bubble. For comparison we also include the location of the true vacuum minimum. The bottom row shows the same two plots for the cubically broken symmetry (4.2). For $\delta \gtrsim 0.25, 0.4$ the location where the field tunnels out to begins to deviate significantly from the false vacuum.

bubbles have nucleated within some background, it is inconsistent to ignore the fluctuations as it is rare coherent excursions of these fluctuations that allow for nucleation to occur at all.

A proper determination of the initial conditions requires a calculation of how the fluctuations in the original false vacuum are projected into the bubble spacetime by the nucleation event (see e.g. [107, 108, 156]). This calculation is beyond the scope of this paper and would only serve to obscure the essence of our result, so instead we simply initialize the fluctuations as a realization of a homogeneous random field with spectrum

$$\langle |\delta \tilde{\phi}_k|^2 \rangle \sim \frac{1}{2\sqrt{k^2 + V''(\phi_{fv})}} \qquad \langle |\delta \tilde{\phi}_k|^2 \rangle \sim \frac{\sqrt{k^2 + V''(\phi_{fv})}}{2} \tag{4.7}$$

to mimic the fluctuations if the field were sitting at its false vacuum mimimum in Minkowski space.² The overall scale of the potential λ enters into the initial fluctuation amplitude since $\delta\phi/\phi_0 \propto \sqrt{\lambda}$. Since we choose to measure the fields in units of ϕ_0 and time in units of $\sqrt{\lambda}\phi_0$, the vev ϕ_0 only appears in the equations of motion if we consider coupling to gravity where it determines the strength of the gravitational interaction via ϕ_0/M_P . At a technical level, we use the same convolution based method as DEFROST [39, 157] to initialize the fluctuations.

The remainder of the paper presents results obtained from a parallel lattice code written by one of the authors. Hamilton's equations for the discretized system were evolved using a sixth-order symplectic Yoshida integrator [158, 40, 43] and (unless otherwise indicated) a second-order accurate and fourth-order isotropic stencil for the Laplacian [39, 159]. All production runs used lattices with 1024^3 points per side, although we did perform some numerical checks using 2048³ lattices. For all cases, the total energy of the system (when running Minkowski simulations) was conserved to the 10^{-9} level or better.

To provide a test of our lattice simulations and to facilitate comparison with previous studies, we also present the results of dimensionally reduced 1+1-dimensional simulations at several points in the subsequent sections. These simulations used a Fourier pseudospectral lattice discretization and a 10th order Gauss-Legendre time integrator. The basics of this approach are outlined in appendix B.

²As a check of this assumption, we also ran simulations with several other choices of initial fluctuations. One set included homogeneous bulk fluctuations with different initial spectra than the Minkowski vacuum. Another set involved initializing $\phi_{\text{init}} = \phi_{\text{bounce}}(r_E = |\mathbf{x}| + \delta r)$ with $\delta r = \sum_{\ell m} a_{\ell m} Y_{\ell m}$ a 2d random field obtained by realizing a collection of $a_{\ell m}$'s. Again, we used several different spectra for our $a_{\ell m}$'s to verify that our results were not sensitive to a particular choice. In every case we tried, the outcome of a collision of two bubbles was qualitatively the same as the results we present in the remainder of the paper.



Figure 4.4: Time evolution of a thin-walled bubble with $\delta = 0.1$ in the linear symmetry breaking potential. In the left panel we show the field value as a function of mt and position mx along a slice taken through the center of the bubble. This figure is from a three-dimensional lattice simulation. For reference, the location of the wall in the thinwall approximation is included as a dashed black line. In the right panel, we show field profiles at time mt = 0, 17.24, 34.48 on a slice through the center of the bubble. In order to demonstrate the accuracy of our numerics, we compare our lattice results to higherresolution one-dimensional simulations. For comparison, we also include the evolution of a bubble whose initial profile is given by the thin-wall approximation rather than an exact numerical calculation.

4.3 Evolution of a Single Bubble

First we consider the evolution of individual bubbles. Fluctuations in the angular dependence of the bubble radius for the case of individual bubbles in the thin-wall limit were studied previously and found to be stable [106, 108]. Our results will demonstrate that this is also true for the bulk fluctuations around single bubbles. We evolve a thin-wall bubble with $\delta = 0.1$ in the linear potential and a thick-wall bubble with $\delta = 0.99$ in the cubic potential. The corresponding instanton profiles can be found in the left panels of Fig. 4.3. For the thin-walled bubble, the field tunnels out very close to the true vacuum. As a result, we can think of this as a bubble with m^{-1} . For the single bubble, we can to a good approximation take this wall to be infinitely thin. The pressure differential between the interior and exterior of the bubble then causes the wall to accelerate and it follows a hyperbolic curve given by $r_{wall}(t) = \sqrt{r_{wall}(t=0)^2 + t^2}$ as shown in Fig. 4.4.

For the thick-walled bubble the evolution is somewhat different as seen in Fig. 4.5. The field now tunnels out far from the true vacuum. In the bubble interior it begins to oscillate around the minimum at $\phi = \phi_0$. In the standard slicing of Minkowski space used in the code, these oscillations appear as outward propagating spherical waves. The



Figure 4.5: Time evolution of a single thick-walled bubble with $\delta = 0.99$ in the cubic symmetry breaking potential. In the left panel, we plot the field as a function of mt and position mx along a slice through the center of the bubble. Regions where the field is near the false vacuum are blue, while regions where the field is near the true vacuum are red. In the middle panel, we plot the value of the field at the center of the bubble as a function of time. As expected the field undergoes damped harmonic oscillations. As a test of our numerical code, we have also included the result for two different choices of lattice spacing dx and also for a one-dimensional simulation using a pseudospectral discretization and 10th order Gauss-Legendre integrator. In the right panel we show the field profile at times mt = 0, 22.98 and 41.37, again for the two different lattice grid spacings and also the one-dimensional simulation.

leading edge of this spherical wave quickly develops into the bubble wall and propagates outward with a speed asymptotically approaching the speed of light. This is easiest to see by foliating the spacetime with hyperboloids centered on the nucleation center of the bubble. We label each of these hyperboloids by a new time coordinate \tilde{t} related to the time coordinate used in our simulations via $t = \tilde{t} \cosh \chi$ where χ is the radial coordinate along the hyperboloids. The line element in the new slicing is $ds^2 = -d\tilde{t}^2 + \tilde{t}^2 dH_3$ with H_3 the three-hyperboloid. For the exact instanton initial condition, the evolved field is a function of \tilde{t} only and satisfies

$$\frac{\partial^2 \phi}{\partial \tilde{t}^2} + \frac{3}{\tilde{t}} \frac{\partial \phi}{\partial \tilde{t}} + V'(\phi) = 0.$$
(4.8)

In this particular case the oscillating field in the interior of the bubble does not lead to a strong preheating instability because the potential seen by the field as it oscillates in the interior of the bubble is very nearly quadratic.³ To see this explicitly, define $\psi = \phi - \phi_0$ and reexpand the potential to obtain

$$U(\psi) \equiv V(\phi_0 + \psi) = (\lambda + \delta)\phi_0^2 \psi^2 + \frac{(3\lambda + \delta)}{3}\phi_0 \psi^3 + \frac{\lambda}{4}\psi^4.$$
 (4.9)

³For different choices of potential or couplings to additional fields the interior of the bubble may experience strong instabilities (see for e.g. [32]).



Figure 4.6: Floquet chart for $\ddot{f} + (A + 4B\sin(t) + 1.5B^2\sin^2(t)) f = 0$, corresponding to oscillations about the true vacuum minimum for $\delta = 1$. For oscillations in the interior of the thick-walled bubble, we have $A \ge 2$ and $B \le 0.5$.

The equation for linear perturbations around $\psi(\tilde{t})$ is then

$$\frac{\partial^2 (\tilde{t}^{3/2} \delta \psi)}{\partial (m_{eff} \tilde{t})^2} + \left(\frac{\kappa^2}{m_{eff}^2 \tilde{t}^2} + (1+\delta) + (3+\delta) \frac{\psi}{\phi_0} + \frac{3}{2} \frac{\psi^2}{\phi_0^2} - \frac{3}{4\tilde{t}^2} \right) (\tilde{t}^{3/2} \delta \psi) = 0.$$
(4.10)

where $m_{eff}^2 = \partial_{\psi\psi} U(\psi = 0) = 2\lambda\phi_0^2$ and κ^2 is an eigenvalue of the Laplacian on the unit three-hyperboloid. To gain some intuition about these instabilities, let's approximate the motion of ψ as it oscillates around the minimum by $\psi = \alpha\phi_0 t^{-3/2}\sin(m_{eff}\tilde{t} + \theta_0)$, with $\alpha < 1$ a numerical coefficient and θ_0 a phase. At the center of the bubble, the time coordinate \tilde{t} coincides with the time parameter t used in our simulations, so we can see from the middle panel of Fig. 4.5 that this is indeed a good approximation. Ignoring the time-dependence of all coefficients, the fluctuations then obey an equation of the form

$$\partial_{\overline{t}\overline{t}}f + \left(A + (3+\delta)B\sin(\overline{t}) + \frac{3}{2}B^2\sin^2(\overline{t})\right)f = 0.$$
(4.11)

For the continuum part of the spectrum, we have $\kappa^2 \ge 1$ so $A \ge (1+\delta)$ and from Fig. 4.5 we see that $B \le 0.5$ during the oscillations. In Fig. 4.6 we see that this puts us well into the weak resonance regime. Since the oscillations damp with time, a given mode will trace a line in the instability chart so that even in the long-time limit there is no exponential growth and the decay is perturbative.

4.4 3D Bubble Collisions Without Bulk Fluctuations

In this section we study collisions between pairs of nucleated vacuum bubbles working under the assumption of SO(2,1) symmetry. This allows us to compare results with previous treatments of this problem (where the SO(2,1) symmetry is assumed to hold exactly and the dynamics are reduced to one spatial dimension), while also providing a nontrivial test of our numerical approach. We consider both the case of the spectrally accurate numerical profiles for the bubbles, as well as initial profiles given by the thinwall approximation. Since the thin-wall profile is not an exact solution to the bounce equation, this latter choice is equivalent to a small breaking of the boost symmetry for the single bubbles. As we demonstrate below, even this minimal breaking of the symmetries of the bubble can have a dramatic effect of the full three-dimensional evolution, resulting in a complete breakdown of the boost symmetry shortly after the collision. This breaking of the boost symmetries is a new result that requires three-dimensional (more precisely higher than one-dimensional) simulations.

4.4.1 Evolution of SO(2,1) Background and Linear Evolution of Fluctuations

Before attacking the full problem, let's first consider linear stability analysis around the SO(2,1) symmetric solution. For the case of perturbations to a single bubble see [106, 107, 108]. Consider a pair of colliding bubbles possessing SO(2,1) symmetry. A convenient set of coordinates is given by

$$t = s \cosh \chi$$

$$x = x$$

$$y = s \sinh \chi \cos \theta$$

$$z = s \sinh \chi \sin \theta.$$
(4.12)

The SO(2,1) symmetry is now manifest for field configurations that depend only on s and x. Let's separate our field into a symmetric background piece and fluctuations $\phi = \phi_{bg}(s, x) + \delta \phi(s, x, \chi, \phi)$. For linearized fluctuations and ignoring backreaction we obtain

$$\frac{\partial^2 \phi_{bg}}{\partial s^2} + \frac{2}{s} \frac{\partial \phi_{bg}}{\partial s} - \frac{\partial^2 \phi_{bg}}{\partial x^2} + V'(\phi_{bg}) = 0$$
(4.13)

and

$$\frac{\partial^2 A_\ell}{\partial s^2} + \frac{2}{s} \frac{\partial A_\ell}{\partial s} - \frac{\partial^2 A_\ell}{\partial x^2} + \left(\frac{\ell^2}{s^2} + V''(\phi_{bg})\right) A_\ell = 0.$$
(4.14)

We have factored the perturbation into eigenmodes $\delta \phi = \sum_{\ell,n} A_{\ell}(s, x) C_{\ell,n}(\chi) e^{in\theta}$ with $n \in \mathbb{Z}$. The eigenfunctions $C_{\ell,n}$ and eigenvalues ℓ are determined by

$$\frac{1}{\sinh(\chi)}\frac{d}{d\chi}\left(\sinh(\chi)\frac{dC}{d\chi}\right) = \left(-\ell^2 + \frac{n^2}{\sinh^2(\chi)}\right)C.$$
(4.15)

Past studies that assume exact SO(2,1) symmetry restrict the treatment to a study of (4.13) with no consideration of the fluctuations described (initially) by (4.14). A sample collision between two bubbles in the linear symmetry breaking potential with $\delta = 0.1$ and the SO(2,1) symmetry imposed is shown in Fig. 4.7. The bubble walls



Figure 4.7: SO(2,1) symmetric collision between two bubbles in the linear symmetry breaking potential with $\delta = 0.1$. Red corresponds to regions where the field is near the true vacuum minimum and blue to regions where it is near the false vacuum minimum. The two bubble walls do not immediately annihilate each other in the collision, but instead undergo multiple bounces while slowly releasing energy into the bulk as radiation.

undergo multiple collisions, each time opening up a pocket where the field is localized near the false vacuum minimum. The bouncing behaviour we observe is characteristic of thin-wall bubble collisions in double-well potentials. This was first noted by Hawking, Moss and Stewart [73]. Due to the SO(2,1) symmetry, each pocket corresponds to an expanding torus in the full 3-dimensional evolution, where the field inside the torus is near the the false vacuum minimum. In chapter 3 we showed that long tubes with the field near the false vacuum in the interior (formed from the breakup of colliding domain walls) are unstable to collapse. Hence, we expect that when the cylindrical symmetry is not perfect the torii will fracture into localized blobs of field. In addition, small amplitude fluctuations localized near the axis connecting the bubble centers will see an oscillating background field as the two bubble walls bounce off each other. As a result, the fluctuations in these regions will be resonantly amplified, analogous to the situation with the planar domain walls. We will see in section 4.5 that both of these behaviours do indeed manifest themselves in the fully 3+1-dimensional problem.

4.4.2 3D Simulation of SO(2,1) Bubble Collisions

Before proceeding to the case with bulk fluctuations around the pair of nucleated bubbles, we first present results using initial conditions that preserve the SO(2,1) symmetry up to errors induced by using a superposition of single-instanton solutions. In the absence of numerical errors, the resulting time-evolution will preserve the full SO(2,1) symmetry. Therefore, this is a highly nontrivial test of our numerics and also provides a nice visualization of the three-dimensional field profile.

We consider our fiducial thin-walled instanton with $\delta = 0.1$ in the linear potential. The single-instanton profile can be seen in the top left panel of Fig. 4.3. In Fig. 4.8 we see that the lattice preserves both the rotational and boost symmetries quite well, especially considering that spatial discretization and discrete time-steps explicitly break both symmetries. For times $mt \gtrsim 80$ the surfaces of constant field are deformed slightly from being perfect hyperboloids near the edges of the collision, but they still exhibit the correct qualitative form. A late-time three-dimensional view of the same collision can be found in Fig. 4.9, where the preservation of the rotational symmetry about the collision access is clear. The total energy of the system is conserved to the 10^{-9} level, and the deviation from perfect boost symmetry persists when we halve the time step or change the order of the integrator. This indicates that the mild breaking of the boost symmetry is an artifact of the modified dispersion relationship of the finite-difference discretization rather than the time-stepping.

4.4.3 Sensitivity of Boost Symmetry to Initial Conditions

We now consider the sensitivity of our results to initial perturbations by breaking the boost symmetry while still preserving the rotational symmetry about the collision axis. Specifically, we perturb the instanton solution used to set initial conditions while assuming it is still only a function of r_E . The resulting function no longer satisfies (4.3) and therefore the resulting evolution of a single bubble will no longer be boost invariant, but will preserve the three rotational symmetries. To see this, suppose that we choose a single bubble initial field configuration $\phi_{init,sym}(\mathbf{x}, 0)$ such that the time-evolved field is a function of $|\mathbf{x}|^2 - t^2$ alone (ie. it is SO(3,1) invariant). Upon Wick rotation into Euclidean time, the SO(3,1) invariance translates into an SO(4) invariance so that the resulting Euclidean profile is a function of r_E alone. As well, since the time-evolved profile was



Figure 4.8: The evolution of ϕ along the line x = 0 = y where we have chosen our coordinates such that the two bubbles first collide at the origin and the collision occurs along the x-axis. For reference, we also include several lines of constant s, demonstrating that the boost symmetry is quite well preserved in the full three-dimensional simulation.



Figure 4.9: Left: Two slices of the field at mt = 92. The vertical slice is taken along the collision axis and cuts through the centers of the bubbles, while the horizontal slice is taken orthogonal to collision axis at the collision site. Right: Contours of the energy density $\rho/\lambda\phi_0^4$, showing the collection of expanding concentric torii whose interior are near the false vacuum and whose walls interpolate between the false vacuum interior and the true vacuum exterior. An animation for the evolving field profile can be found at www. cita.utoronto.ca/~jbraden/Movies/linear_del0.1_exactprofile_field.avi.

obtained by solving the Klein-Gordon equation in Lorentzian signature, the Euclidean profile must satisfy the Euclidean version of the Klein-Gordon equation. However, for a function of r_E alone, the Euclidean Klein-Gordon equation is simply the equation for the bounce (4.3).⁴ Since our perturbed initial condition does not satisfy (4.3), this means that the resulting evolved bubble cannot respect SO(3,1) symmetry. By construction the rotational symmetry is preserved, which means that this procedure must destroy the boost symmetries. An example of exactly this type of procedure is to use a thin-wall approximation rather than the exact instanton solution. Since the thin-wall approximation does not actually satisfy the instanton equation, this means it cannot respect the SO(3,1) symmetry. When we extend this to the two bubble initial conditions, the evolved fields will preserve the rotation symmetry about the collision axis, but break the two boost symmetries.

Here we consider the effect of using the thin-wall approximation combined with an approximate determination of ϕ_{false} and ϕ_{true} . Although this may seem somewhat contrived, this is the result if one writes the potential as $V_{sym} + \Delta V$ and solves for the profile of the wall by dropping the ΔV term (provided of course that the locations of the minima are perturbed by ΔV). Since we want the bubble profile to interpolate between the false and true vacua, we take the initial condition for a bubble centered at \mathbf{x}_0

$$\phi_{\text{init}} = \frac{(\phi_f - \phi_t)}{2} \tanh\left(\frac{|\mathbf{x} - \mathbf{x}_0| - R_{init}}{w}\right) + \frac{\phi_f + \phi_t}{2} \tag{4.16}$$

with $mR_{\text{init}} = \sqrt{2}\delta^{-1}$ and $mw = \sqrt{2}$. The notation $\phi_{f/t}$ is meant to distinguish the approximate locations of the false and true vacua from the exact values $\phi_{false/true}$. We further approximate the values of ϕ_f and ϕ_t to linear order in δ . We find $\phi_f = -1 + \delta/2 + \mathcal{O}(\delta^3)$ and $\phi_t = 1 + \delta/2 + \mathcal{O}(\delta^3)$. At this level of approximation, we then have

$$\phi_{\text{init}} \approx \phi_0 \tanh\left(\frac{\left(|\mathbf{x} - \mathbf{x}_0| - R_{init}\right)}{w}\right) + \frac{\delta}{2}.$$
 (4.17)

A comparison of approximation (4.17), the thin-wall approximation with an exact determination of $\phi_{false/true}$ and the numerical result are shown in Fig. 4.10. Clearly, the approximate solutions provide a very accurate description of the system initially, although the perturbation to the initial bubble radius is visible in the figure. A consequence of using approximate vacua, is that the equation of motion is violated over the entire domain of r_E .

 $^{^{4}}$ This argument does not rely on the boundary conditions for the bounce solutions being met, only that it satisfies the correct equation.



Figure 4.10: A comparison of the exact instanton solution to various thin-wall approximations for $\delta = 0.1$ in the linear potential (4.1). On the left we show the numerically generated instanton profile used to initialize the bubbles. For comparison, we also include the thin-wall approximation, both in the case when we determine the false and true vacua exactly (4.16), and also in the case when we approximate them to $\mathcal{O}(\delta^3)$ (4.17). In the center panel, we show the coefficients c_i appearing in front of each of our basis functions for the numerical profile. The exponential decrease of the coefficients as well as the presence of the roundoff plateau resulting from numerical roundoff errors are both clearly visible, again indicating that we have reached the limits of double precision arithmetic. Finally, in the right panel we plot the violation of the instanton equation (4.3) for both our numerical profile and the thin-wall approximations. When measured by the maximal violation of the equation of motion, our solution is ten orders of magnitude more accurate than the thin-wall result. The mapping parameters were d = 0.35 and $L = 1.6R_0$.



Figure 4.11: Field slice through the collision plane for a pair of colliding bubbles with initial conditions given by (4.17). The rotational symmetry about the collision axis is maintained to high-precision here, but one can now clearly see the loss of boost invariance as the system evolves. This develops on constant s slices exactly as one would expect for a resonantly amplified fluctuation with χ dependence.



Figure 4.12: Left: The same view of the field as Fig. 4.8, except for an initial bubble profile (4.17) rather than the numerical result. Right: Contours of energy density $-T_0^0$ for the same simulation. As can be seen, the collision results in the destruction of the boost symmetry. An animation of the field evolution is available at www.cita.utoronto. ca/~jbraden/Movies/linear_del0.1_thinwall_field.avi.

From Fig. 4.11, we see that the symmetry breaking fluctuations grow and become nonlinear on fixed s slices, exactly as expected for perturbations associated with the χ direction. As a check that this was not a numerical artifact, we ran the same simulation using two different choices for the finite-difference stencil. In all cases the resulting evolution was very similar as seen in Fig. 4.13. This, coupled with the preservation of the symmetry for the exact instanton initial condition and the explanation of the linear instability in terms of parametric resonance, strongly suggests that this is indeed a real effect and not an artifact of the breaking of the boost symmetry by the discrete lattice spacing and time steps. Note that the breaking of the boost symmetry is absent in the solution to (4.13). Thus, this is an effect which is only captured by intrinsically higher-dimensional simulations (in this case three-dimensional).

4.5 Bubble Collisions with Fluctuations

In section 4.4 we verified the accuracy of our lattice code and demonstrated the need for a full three-dimensional treatment even in a simple case where we break the boost symmetries but not the rotational symmetry. We now include a complete set of *bulk* fluctuations, including those that break the rotational symmetry as well as the boost


Figure 4.13: Direct comparison between our one-dimensional simulations using a 10th order Gauss-Legendre integrator and Fourier pseudospectral approximation for spatial derivates, and our full three-dimensional simulation. We plot the value of the field at the initial point of collision between the two bubbles as a function of time. If we use the correct initial instanton profile the two results agree extremely well, apart from a small shift in the time of the initial collision. However, once we break the boost symmetry by using approximate choices for the locations of the true and false vacuum large deviations between the solutions appear. This effect is *not* captured by a one-dimensional lattice that assumes SO(2,1) symmetry for the evolving field.

symmetries. As will be seen shortly, the dynamical amplification of these fluctuations can lead to a complete breakdown of all of the boost and rotational symmetries in the system. These results constitute the main part of our analysis. In this section we consider several classes of single-field potentials, finding that the extreme breaking of the spacetime symmetries is restricted to double-well potentials with only mildly broken Z_2 symmetry. Section 4.6 extends these results to the two-field case.

Our experience with the boost breaking fluctuations above, as well as the linear analysis performed in chapter 2, makes it clear that these fluctuations will undergo resonant amplification for certain types of bubble collisions. Eventually the amplified fluctuations begin to interact nonlinearly. At this point the split between background and fluctuations becomes blurred and we must study the full three-dimensional nonlinear field theory. For a planar symmetric wall-antiwall pair, we demonstrated in chapter 3 that the amplification of fluctuations transverse to the collision axis eventually lead to the dissolution of the walls and the creation of a population of oscillons distributed within a narrow slab around the site of the collision. Since we are considering bubbles here, the "transverse" wavenumbers can be split into a wavenumber associated with the radial hyperbolic direction χ and one associated with rotation about the collision axis θ . For definitions of χ and θ see (4.12). The fluctuations with χ dependence only were studied in the previous section when we looked at boost symmetry breaking. In the remainder of the paper we will explore the full evolution of three-dimensional bubble collisions with small initial fluctuations around the instanton profiles for a variety of potentials, including the final outcome of the nonlinear interactions of amplified linear fluctuations. For cases where the bubble walls bounce many times off of each other, we will see that the results match our intuition developed from the planar wall limit.

We take the initial bubble separation and the amplitude of the initial fluctuations to be independent free parameters. However, if we wish to study collisions between typical bubbles this will not be true. The action of the bounce, which determines the nucleation rate and therefore the typical bubble separation, scales as λ^{-1} . Meanwhile, the RMS amplitude of the fluctuations scales as $\sqrt{\lambda}$. Therefore, increasing the amplitude of the fluctuations has the effect of increasing the nucleation rate and thus decreasing the typical bubble separation. Our primary motivation to treat these parameters independently was numerical, as the finite simulation cube with periodic boundary conditions means that we have a finite amount of time before the bubbles begin to interact with their images and we can no longer trust our simulation. To study the effects of nonlinear interactions, the exponential growth of the fluctuations must push them into the nonlinear regime before this happens and we adjusted the initial amplitude to ensure this was the case.

4.5.1 Thin-Wall Double Well Case

Our first case is the collision of two thin-walled bubbles in the linear symmetry breaking potential with $\delta = 0.1$. Aside from the new scale associated with the radius of the bubbles, this case is qualitatively the same as the collision of a pair of planar walls in the same potential. We saw above that perturbing the bounce solution slightly can lead to a dramatic breaking of the two boost symmetries. From our previous study of planar walls in chapter 3, we expect that the full nonlinear evolution ultimately results in the dissolution of the bubble walls and the creation of a collection of oscillons in the collision region. In Fig. 4.14 we demonstrate that this is indeed the case. Near the center of the collision, the bouncing of the walls amplifies the initially small flucutations. Around $mt \sim 40$, these fluctuations become of similar amplitude to the oscillations of the SO(2,1) background. Shortly after this, the distinction between background and fluctuations in the collision regime breaks down and the field rapidly condenses into a population of oscillons. In our choice of time coordinate, the condensation occurs first near the center of the collision. Meanwhile, the outward propagating torii produced during the first few collisions of the walls develop ripples that eventually pinch off leading to the production of "rings" of oscillons near the outer edges of the collision region. The two mechanisms described above correspond to the two production mechanisms we anticipated in section 2.4: parametric amplification of fluctuations by the oscillating background field near the collision center, and the growth of fluctuations on the outward propagating torii.

4.5.2 Thin-Wall with Plateau

As our next example potential, we modify the region into which the field tunnels by inserting a long-flat plateau rather than a second well. This new potential is given by

$$V(\phi) = \begin{cases} \frac{1}{4} (\phi^2 - 1)^2 - \delta \phi + \tilde{V}_0 &: \phi < \phi_{true} \\ V_0 - \epsilon (\phi - \phi_{true}) &: \phi > \phi_{true} \end{cases}$$

where $V_0 = \frac{1}{4}(\phi_{true}^2 - 1)^2 - \delta \phi_{true} + \tilde{V}_0$ and (assuming $\delta > 0$) ϕ_{true} is the largest solution to $\phi_{true}^3 - \phi_{true} - \delta = 0$. This is meant as a toy example where the field tunnels out onto a flat inflationary plateau, with the slope determined by $\epsilon \ge 0$. The portion of the potential traversed by the instanton as it tunnels is unchanged from the double well case (4.1) and therefore the initial bubble profiles are the same. However, we expect the collision dynamics between this case and the thin-walled double-well to be radically different. For



Figure 4.14: Development of the instability for two colliding bubbles in Minkowski space including bulk fluctuations corresponding to $\lambda = 10^{-4}$ around the thin-wall approximation. In the top row we plot the field distribution on a 2-D slice through the plane where the bubbles collide (horizontal projection) and on a slice parallel to the collision axis through the centers of the bubbles (vertical projection). In the bottom row we show contours of the energy density. Our lattice has N = 1024 points per side with a box size of $mL = 13R_0 = 13\sqrt{2}\delta^{-1}$ and a time step dt = dx/12.5. The bubbles are nucleated with their centers separated by a distance $mR_{sep} = \frac{26}{5}R_0$. Videos corresponding to both sets of figures can be found at www.cita.utoronto.ca/~jbraden/Movies/linear_del0.1_wfluc_field.aviand www.cita.utoronto.ca/~jbraden/Movies/linear_del0.1_rhocontours.mp4.

a sufficiently energetic collision, the free passage approximation [101, 102, 160, 161] will hold shortly after collision. This tells us that the collision will displace the field a distance $\Delta \phi \sim 2\phi_0$ down the plateau. Unlike the double-well, this does not result in a restoring force pulling the field back towards the false vacuum. Therefore, there is no bouncing of the walls or other oscillations of the background field. Thus, there is no mechanism to pump fluctuations. Rather, the effect of the collision is to create a steep gradient in the field interpolating between the field at the tunnel out location and its displaced location down the plateau. This gradient then propagates away from the collision as seen in Fig. 4.15. In front of this gradient, the field is at $\phi \approx \phi_{true} \approx \phi_0$ while behind it the field has been displaced down the plateau to $\phi \approx \phi_{true} + 2\phi_0 \approx 3\phi_0$. Although this behaviour is very similar to that of a propagating domain wall, this gradient does *not* interpolate between local minima of the potential and thus there are no bound state fluctuations associated with it because V'' is never negative. Somewhat suprisingly, the field at the collision site does not remain stationary on the plateau. Rather, it begins to slowly retreat back to the tunnel out location, at least for the duration of our simulations. From Fig. 4.17, we see that the field profiles both with and without initial fluctuations are nearly the same, indicating that the fluctuations play a subdominant role in this case. As a result, this is a situation in which the SO(2,1) assumption provides an accurate description of the evolution.

Now consider the effect of adding a slope to the potential in the region where the field tunnels out, so that the field begins to roll down the plateau once it tunnels. Again, the collision effectively displaces the field further down the plateau and produces a steep gradient which again propagates into the bulk at nearly the speed of light. This gradient propagates on top of the previous evolution of the field down the potential. As with the flat plateau, Fig. 4.17 shows that the resulting evolution is insensitive to the choice of whether or not we include fluctuations, so that again we expect this situation to be well-described by 1+1-dimensional simulations.

4.5.3 Thick-Wall Double Well Case

As our final potential we consider a collision between two of our thick-walled bubbles in the cubic symmetry breaking potential with $\delta = 0.99$. In this case, the final outcome of the collision is somewhat unclear without running simulations. In Fig. 4.18 we show the result of one such collision, where we have made the assumption of SO(2,1) symmetry in order to run a very high resolution one-dimensional simulation. The collision still leads to large oscillations of the field around the minimum, but the field no longer becomes temporarily trapped in the false vacuum minimum between collisions. Instead of having repeated collisions between a pair of walls, we instead have oscillations of the field around the true vacuum. Within a few oscillations, the amplitude damps to $\leq 0.5\phi_0$. We already know from our study of single field bubbles that spatially homogeneous oscillations of the field with this amplitude do not lead to a strong instability. Since the oscillations here have additional spatial localization, it is thus clear that the fluctuations will not experience significant amplification in this case either. We see this directly in Fig. 4.19, where the rotational symmetry about the collision axis is preserved to a high degree during the collision. As well, Fig. 4.20 further shows that the boost symmetry is extremely well preserved.



Figure 4.15: A collision of two single-field bubbles with a flat plateau onto which the field tunnels rather than a second well. The horizontal panel shows the field on a slice orthogonal to the collision axis and through the center of the collision region. For orientation, the back panel is a slice parallel to the collision axis, illustrating the growth of the two bubbles. Blue corresponds to regions where the field is near the false vacuum, red regions where it is near the tunnel out point, and green regions down the plateau away from the tunnel out point. A illustrative video of the process can be found at www.cita.utoronto.ca/~jbraden/Movies/collision_plateau_del0.1_field.avi.

4.5.4 Bubble-Domain Wall Collision

Thus far we have only considered the collision of two bubbles in a reference frame in which they nucleate at the same time. We will now address the opposite regime where one of the bubbles is much larger than the other. To model this situation we will consider



Figure 4.16: A collision of two single-field bubbles with a tilted plateau $\epsilon = 0.01$. The field slices are the same as in Fig. 4.15. Inside the bubble, the field now rolls down the potential. However, the result of the collision is essentially the same as in the case $\epsilon = 0$. The field is displaced down the potential at the collision, and a steep field gradient then propagates into the interior of the bubble. Aside from this, the field rolls down the potential due to the constant force resulting from the slope.

the collision of a bubble with a planar domain wall, with the planar wall meant to be a substitute for the large bubble. In an actual collision between false vacuum bubbles, this would require the wall to have a very large Lorentz contraction factor $\gamma \gg 1$. This then creates a very large hierarchy between the size of the bubble and the thickness of the wall which we are unable to resolve with our fixed lattice. Thus, we will restrict ourselves to mildly relativistic walls. A physical situation where this could occur would be phase



Figure 4.17: Comparison of bubble profiles orthogonal to the collision axis for a simulation with initial fluctuations $\delta \phi \neq 0$ (black line) and without initial fluctuations $\delta \phi = 0$ (red triangles). The choice of potential and initial conditions are the same as Fig. 4.15 (left) and Fig. 4.16 (right). We have chosen four time times mt = 34.47, mt = 45.96, mt = 57.45 and mt = 80.43. The two profiles match extremely well at all times, indicating that the fluctuations are stable for this class of bubble collisions. The oscillations near the edge of collision region are numerical artifacts. In the left panel we show the result for a flat plateau $\epsilon = 0$, while on the right we have $\epsilon = 0.01$. For the purposes of plotting, we have downsampled our output grid by a factor of 2.

transitions occurring at finite-temperature, where interactions between the bubble wall and the surrounding medium lead to a maximal wall propagation speed. Of course the initial spectrum of fluctuations is different in this case, so a direct comparison with our results cannot be made, although the same qualitative behaviour should persist. As well, we will consider only bubbles in the linear symmetry breaking potential (4.1), again with $\delta = 0.1$. From our results for the collision of two bubbles nucleated at rest at the same time, it is clear that these types of collisions will again result in strong amplification of the fluctuations and the eventual production of a population of oscillons. The only new ingredient here is that the planar wall and bubble wall no longer carry an equal and opposite amount of field momentum $\phi \nabla \phi$. Therefore, if we consider a small box around any region where the planar wall and bubble are undergoing a collision, the energy within that box will carry a nonzero momentum. As well, the collision no longer occurs in a single spatial plane, but rather on a curved hypersurface defined by all of the instantaneous intersections between the bubble and the planar wall. As a result, the collision products will be produced along a nonplanar surface with a nonzero velocity (both relative to our simulation coordinates). Since we are not properly boosting an initial two-bubble field configuration to set our initial conditions, the above statement will be true in any coordinate system.



Figure 4.18: The evolution of a pair of colliding bubbles in the cubic potential with $\delta = 0.99$ under the assumption of exact SO(2,1) symmetry.

Figure 4.21 shows that this is indeed the case. As the wall accelerates into the false vacuum it partially engulfs the expanding bubble. As with the two bubble case, the field experiences a series of excursions back to the false vacuum. These drive an instability in linear (non rotationally invariant) fluctuations analogous to the instability in the pair of bubbles. The torii resulting from these excursions no longer distribute themselves in a single plane, but rather move in the same direction as the wall. As a result, the oscillons that are ultimately produced from the growth of these fluctuations are distributed in some narrow curved hypersurface rather than a plane.

4.6 Extension to Two Fields

By considering the simple case of a single-field potential, we discovered that the dynamics of bubble collisions can be considerably more intricate than expectations from assuming SO(2, 1) or SO(2) symmetry for the field profile. In particular, we found that for singlefield double-well potentials with moderately broken Z_2 symmetry the collision of a pair of bubbles ultimately leads to the production of a population of oscillons. This type of potential is often used to model first-order phase transitions, and thus our results may have some relevance there. However, when the bubbles begin to coalesce the problem will need to be treated as a many bubble problem, rather than just two, leading to a very large and explicit breaking of the SO(2,1) symmetry.





Figure 4.19: A collision of two-thick wall bubbles in the potential (4.2) with $\delta = 0.99$. The field slices and coloring are the same as in Fig. 4.14. A video of the field evolution is available at www.cita.utoronto.ca/~jbraden/Movies/thick_wall_collision.avi.

A natural place in which to consider collisions between isolated pairs of bubbles is in false-vacuum eternal inflation. Unfortunately, it seems exceedingly difficult to both realize inflation and produce oscillons in bubble collisions using a single-field potential with only two minima. The easiest way to have a viable open inflationary model produced by the nucleation of a bubble is to have the field tunnel out onto a plateau that can support $\Delta \ln a \sim 50$ efolds of inflation.⁵ Our mechanism for oscillon production relies on oscillations of the field around the true vacuum within the collision region. However, if

 $^{^5\}mathrm{Another}$ possibility is that collision displaces the field to a location where it begins to slow-roll, thus starting inflation.



Figure 4.20: Evolution on line orthogonal to collision axis showing the preservation of the boost symmetry. Fluctuations were taken to be those in the Minkowski false vacuum with $\lambda = 10^{-2}$.

the plateau is long enough to support a sufficient period of inflation, any rolling motion of the field along the plateau must experience many efolds worth of Hubble damping. Therefore, the requirement of an inflationary plateau makes it difficult for the field to rebound off of a "wall" in the potential and return to the false vacuum side of the barrier shortly after collision. Thus the basic mechanism by which initial fluctuations are amplified disappears.

However, embedding inflationary models based on bubble nucleation into a realistic high-energy theory will likely involve considering models with many scalar fields.⁶ With this in mind, consider a simple two-field potential

$$V(\sigma,\phi) = \frac{\lambda_{\sigma}}{4} \left(\sigma^2 - \sigma_0^2\right)^2 + \lambda_{\sigma} \delta\left(\frac{\sigma^3 \sigma_0}{3} - \sigma \sigma_0^3 + \frac{2\sigma_0^4}{3}\right) + \lambda_{\sigma} \frac{g^2}{2} \left(\sigma - \sigma_0\right)^2 \phi^2 + \lambda_{\sigma} \sigma_0^3 \epsilon \phi + V_0 \,.$$

$$\tag{4.18}$$

Schematically, the potential has the form $V_{tunnel}(\sigma) + V_{coupling}(\sigma, \phi) + V_{inflation}(\phi)$, where we have chosen the particular form $V_{inflation}(\phi) = \lambda_{\sigma}\sigma_0^3\epsilon\phi + V_0$. Our choice $V_{inflation} = \lambda_{\sigma}\sigma_0^3\epsilon\phi + V_0$ can be viewed as a linarization of the potential around the tunnel out region. However, by adjusting $V_{inflation}$, we can effectively reproduce any model of single-field inflation we wish. There is a local minimum at $\sigma \approx -\sigma_0$ and $\phi \approx 0$, and a long trough at $\sigma \approx \sigma_0$ along which σ is heavy and ϕ is light. Since we now have two mass scales at our disposal, we can accomodate the production of oscillons by exciting the σ field while

 $^{^{6}}$ Even this statement is probably too simplistic, as the correct high-energy theory may not even be describable in terms of a low-energy scalar field theory, but we will not concern ourselves with this.



Figure 4.21: Field and density evolution for a collision between a planar domain wall and a single vacuum bubble. In the top row we show two slices of the field configuration. The first slice (projected vertically) is perpendicular to the planar wall and cuts through the center of the bubble. The second slice (projected horizontally) is parallel to the domain wall and displaced slightly from the center of the bubble towards the initial location of the domain wall. As the wall sweeps past the bubble, the instantaneous collision location moves along the collision axis, as do the tubes of false vacuum and oscillons produced by the collision. In the bottom row, we instead show contours of energy density for the same three time slices.

simultaneously permitting slow-roll inflation along the ϕ direction.

We only consider parameter choices such that the tunnelling dynamics is dominated by σ , while the subsequent post-tunnelling evolution is dominated by ϕ . Although we have not performed an exhaustive study for different choices of g^2 , for $g^2 = 1$ this behaviour is generic for the thin-walled case $\delta \ll 1$. This is illustrated in Fig. 4.23, where we explicitly see that during tunnelling the field first moves almost exclusively in the σ direction, before making a sharp turn at the end so that it is moving along the slow-roll plateau. Effectively, the behaviour of the tunnelling field σ is nailed down and the inflaton field ϕ simply reacts to the presence of the domain wall in σ . In Fig. 4.24 we show the potential seen by the effective field $d\chi^2_{eff} = d\sigma^2 + d\phi^2$ as it tunnels through the barrier, as well as a comparison to an analogous single-field potential. The structure



Figure 4.22: The two-field potential (4.18) for $\epsilon = -0.01, \delta = 0.2, g^2 = 1$ and $V_0 = 0$. We have in mind a scenario where the field is initially trapped near the local minimum at $\sigma \approx -\sigma_0, \phi \approx 0$ and subsequently tunnels into the nearly flat trough at $\sigma = \sigma_0$. For clarity we have clipped the potential for $V > \lambda \sigma_0^4$.

of the potential as seen by the field while it tunnels and subsequently slow-rolls is thus very similar to the double-well with an appended slow-roll plateau above, with the only difference being the explicit form of the tunnelling portion. As well, notice that there is no steepening of the potential near the beginning of inflation as is often assumed in single-field models of open inflation from bubble nucleation.

Figure 4.25 and Fig. 4.26 illustrate the dynamics resulting from a collision between two of these bubbles. We do not perform an exhaustive study of possible behaviours as a function of initial bubble separation of model parameters, but simply choose a set of parameters and initial conditions to demonstrate that oscillons can be produced in collisions between bubbles in this model. For our choice of parameters, the collision causes a large excitation in σ and its subsequent evolution is very similar to the singlefield double well case. However, for a fixed $\sigma_{cur} \neq \sigma_0$ the potential $V(\sigma_{cur}, \phi)$ has a minimum at $\phi = -\epsilon \sigma_0^3/(\sigma - \sigma_0)^2 \approx 0$. As a result, the oscillations of σ pull the field ϕ back towards the origin, thus undoing the previous rolling down the plateau. The evolution of the σ field amplifies fluctuations and leads to the creation of oscillons in the σ direction. This is completely analogous to the single-field evolution. However, in the cores of the oscillons, the σ field makes large excursions away from σ_0 . As a result, within these cores ϕ remains trapped at the origin while the field outside of these cores once again begins to roll down the plateau. In the inflationary setting, there will still be a vacuum energy V_0 sufficient to drive a period of slow-roll inflation, so as the oscillons



Figure 4.23: The instanton solution used to set initial conditions for our numerical simulation. In the left panel we show the path of the instanton in field space superimposed on isocontours of the potential. In the middle panel we show the profile of the tunnelling field σ and the "inflaton" field ϕ as a function of the Euclidean radius r_E . In the right panel we plot the spectral coefficients for each of the fields, from which we can see the exponential convergence of the series and the roundoff plateau arising around $i \sim 80$. The model parameters were $\delta = 1/5$, $\epsilon = -0.01$ and $g^2 = 1$. Our mapping parameters were $L = 1.6R_0^{cubic}$ and d = 0.5 and we used 100 mode functions.



Figure 4.24: The potential seen by the field as it tunnels in the Euclidean radial direction (*blue line*). Here we have defined $d\chi^2_{eff} = d\sigma^2 + d\phi^2$ as the path length in field space. For comparison, we also include the potential $V(0, \chi_{eff})$ that would be seen by the field if we had instead tunnelled a distance χ_{eff} with $\phi = 0$ (*red dots*). In this latter case, the potential is then the same as our single-field cubic symmetry breaking potential.



Figure 4.25: Evolution of the "inflaton" field ϕ in our two-field model. The horizontal projection is a slice along the collision axis through the center of the bubbles and the vertical slice is orthogonal to the collision and centered on the collision site. Red corresponds to the ϕ value in the false vacuum and where the field originally tunnels. The pips where the field is pulled back up the potential are locations where the σ field is fracturing into oscillons. An animation corresponding to this evolution is available at www.cita.utoronto.ca/~jbraden/Movies/twofield_inflaton.avi.

dilute we expect inflation to eventually restart. A novel feature of this setup is that ϕ will be quite inhomogeneous (within the collision region) at the start of this new inflationary phase, and we expect this inhomogeneity to persist until the oscillons decay.

4.6.1 Model Building for Inflation

Let's now consider the consistency of our approach and application of this type of model to inflation. In our numerics, we have assumed that the instanton profile is well approximated by taking the background to be Minkowki. As well, for our lattice simulations we have further assumed that we can account for expansion of the universe by taking a uniform fixed Hubble constant.⁷ Relaxing this latter assumption is a significant numerical challenge and we therefore leave it to future work. This is required to obtain accurate predictions for the observational signatures that remain after inflation has occurred within the bubble. While this is certainly an interesting question, in this paper we are primarily concerned with the dynamics of the collision itself. The relevant time scale is then $(\lambda_{\sigma}\sigma_0^2)^{-1/2}$ and this must be much shorter than a Hubble time.

First consider the restrictions imposed by our computation of the instanton profiles.

⁷The figures presented in this paper were taken from simulations using a Minkowski background. However, we also did several runs with a fixed Hubble constant H over the entire simulation volume. The main effect of this expansion was to delay the time that the bubbles first collide, while the collision dynamics itself was unaffected.



Figure 4.26: Evolution of the "tunnelling" field σ in our two-field model with $\epsilon = -0.01$ and $\delta = 0.2$. As with previous figures, blue corresponds to $\sigma \sim -\sigma_0$ (ie. near the false vacuum) and red to $\sigma \sim \sigma_0$ (ie. in the "inflationary" trough). The vertical slice is parallel to the collision axis and the horizontal slice is orthogonal to the collision axis. An animation corresponding to this evolution is available at www.cita.utoronto.ca/~jbraden/Movies/twofield_tunnel.avi.

The least stringent requirement is that the CdL instanton exists. Roughly, this requires that the bubble wall fit within a Hubble radius determined at the local maximum of the potential H_{max} . For our setup, the thickness of the wall is determined by $V_{\sigma\sigma} \sim \lambda_{\sigma}\sigma_0^2$, and therefore we require $\lambda_{\sigma}\sigma_0^2 \gg H_{max}^2 \sim V_0/3M_P^2 + \lambda_{\sigma}\sigma_0^4/12M_P^2 \implies 12\frac{M_P^2}{\sigma_0^2} \gg 1 + \frac{4V_0}{\lambda_{\sigma}\sigma_0^4}$. A more stringent constraint comes from requiring that the initial radius of the bubble is much less than the Hubble scale in the ambient spacetime H_{fv} . We have $m_{eff}R_{init} \sim \delta^{-1}$ and $H_{fv}^2M_P^2 \sim \frac{4\delta\lambda\sigma_0^4}{9} + \frac{V_0}{3}$, so this gives $\frac{4\sigma_0^2}{3M_P^2}\left(1 + \frac{3V_0}{4\delta\lambda_{\sigma}\sigma_0^4}\right) \ll \delta$. In the limit that $V_0 \gg \delta\lambda_{\sigma}\sigma_0^4$, this gives $\frac{V_0}{\lambda_{\sigma}\sigma_0^2M_P^2} \ll \delta^2$. For the opposite limit we instead get $\frac{\sigma_0^2}{M_P^2} \ll \delta$.

Finally, consider the restriction imposed by assuming the approximation of a fixed homogeneous background Hubble. This constraint only needs to be fulfilled if we want an approximate description of the dynamics prior to the collision, or to track the long time evolution after the collision. The average expansion rate inside and outside the bubble must be much greater than the difference in expansion rates. As well, for the case when the field slow-rolls inside of the bubble, our approximation will only be valid when the difference between the Hubble parameter in the center of the bubble (where the field has rolled furthest) and the edge is again much less than the average. This then requires $V_0 \gg \delta \lambda_\sigma \sigma_0^4, |\lambda_\sigma \sigma_0^3 \epsilon \phi_{\mathbf{x}=0}(t)|$ where $\phi_{\mathbf{x}=0}$ is the value of ϕ at the center of the bubble at time t.

4.7 Observational Prospects

In the previous sections we demonstrated the presence of a previously ignored instability in the fluctuations around colliding vacuum bubbles in certain types of potentials. As a result of these instabilities, the near SO(2,1) symmetry of the initial configuration becomes badly broken during the course of the dynamical evolution, eventually resulting in the production of a population of oscillons in the collision region. In this section we briefly comment on some possible implications of these results, restricting ourselves to potential signals that rely of the breakdown of the SO(2,1) symmetry. This means we will not discuss possibilities such as a large scale modulation of coupling constants as could result from the single-field collisions with a plateau if the vev of the field forming the bubbles fixes an effective coupling constant in the theory. There are many scenarios in which one could imagine embedding our mechanism, and we will distinguish between two cases that we refer to as superhorizon and subhorizon collisions below. In the first we assume that our observable universe fits within one of the nucleated bubbles which is embedded within some parent false vacuum. The collision is then with a neighbouring bubble which has also nucleated within the parent vacuum. This is of course the standard scenario for testing open inflation resulting from bubble nucleation in an ambient false vacuum. Past studies of signatures from such collisions have been based on the assumption of SO(2,1) symmetry, so in this case in particular we would like to address any novel implications of the breakdown of the symmetry. One could also imagine scenarios where our observable universe forms in the future light cone of the collision, although we will not discuss this possibility here. In the second, we instead assume that the bubble nucleations are occurring on subhorizon scales within our Hubble volume. These nucleations could either have occurred in the past (such as an early first order phase transition within our Hubble volume) or in the present. The reader should keep in mind that we have not performed a detailed analysis for any of the possibilities listed in the section, so the magnitude of many of these effects may prove to be undetectably small. We plan to provide a more detailed study in a future publication, including the effects of gravitation as required to obtain the present day signal.

4.7.1 Superhorizon Bubble Collisions

First consider the case where our universe is contained within one of the bubbles. Our results demonstrate rather clearly that observational signatures of collisions do not necessarily posses an azimuthal symmetry about the collision center as is assumed in the existing literature. A remnant of the azimuthal symmetry will persist in that effects of the collision are confined to a circular disk of the sky, but the interior of this disk need not posses any additional symmetries.

Perhaps the most interesting possibility is the production of gravitational waves by the fracturing of the bubble walls at the onset of nonlinearity amongst the fluctuations. This effect is absent in an exact SO(2,1) collision. Initially the gravitational waves will be produced with subhorizon sized wavelengths, but as the oscillons dilute and inflation restarts they will be stretched outside the horizon. The resulting signal will have a characteristic wavelength as well as a directional dependence on the sky that may be detectable in polarization data. As well, the directional dependence of this signal will be highly correlated with other signatures such as a hot or cold spot produced by the collision. Since the amplitude of the waves will damp until they are stretched outside of the horizon, a detailed analysis is required in order to estimate the size of the signal.

Another effect is related to sign of the temperature perturbation induced by the collision. In the single-field case, the collision simply displaces the field down the plateau resulting in fewer efolds of inflation and thus $\delta \ln(a) < 0$ and we obtain a hot spot. This was verified in [145] for a specific choice of inflationary model. However, when considering our two-field inflationary model this simple intuituion no longer holds. In particular, prior to collision ϕ within each bubble has already begun to roll down the trough. The effect of the collision, at least for the parameters we used in this paper, is to pull ϕ back up the potential, as well as to excite oscillations in σ and eventually produce oscillons. If the effects of the oscillons and σ evolution were ignored, this would simply prolong the inflationary epoch leading to a positive $\delta \ln(a)$ and thus a cold spot. However, the full dynamics creates additional contributions to the energy density (the oscillons). This further perturb the expansion history and a detailed analysis is thus required to obtain the final perturbation.

Finally, when considering the two-field model, the inflationary epoch within the collision region will restart from a highly inhomogeneous state due to the population of oscillons. As long as these oscillons persist, they trap the inflaton field at the origin in regions of size m^{-1} . Whether or not any remnant of this (non-vacuum) initial state persists is a rather interesting question. In particular, one possibility is that that once the oscillons decay, the pips of inflaton field that were held near the origin will be stretched in physical size as the universe expands and could end up sourcing curvature fluctuations ζ .

All of these effects will be rather strongly constrained by data, so a certain amount of tuning will have to be applied in order to construct models that produce signatures at the right level. As well, additional tuning is needed in order for the underlying theory to predict a nonnegligible number of potentially observable collisions for a typical observer. Our purpose in this paper was simply to understand the dynamics of individual collisions so we will not touch on these issues here, although they must be addressed when performing a detailed study of the possible observational consequences of collisions.

4.7.2 Subhorizon Bubble Collisions

Now let's consider the case when the bubble collisions occur in some first-order phase transition within our horizon. Such a transition may have occurred in the past while the universe was extremely hot. However, we could also imagine a scenario where the phase transition is happenning around the present time. An interesting possibility would be for the field responsible for the dark energy to undergo a first order phase transition. As in the previous subsection, we restrict ourselves to signatures that result from collisions and the breaking of SO(2,1) symmetry, although there are many additional possibilities. The discussion here will in many ways mirror the discussion for the superhorizon collisions, with the possible signals being spatially homogeneous counterparts to the spatially localized effects discussed in the previous subsection.

First consider the production of gravitational waves during a first order phase transition. When the bubbles collide, gravitational waves are produced with wavelengths of order the typical size of the bubbles at collision. This source of gravitational radiation is well known and is often studied using the envelope approximation, which neglects all of the field dynamics associated with the collisions and treats the bubbles as if they were undergoing free expansion. However, if the phase transition is not too rapid (so that the bubbles have a chance to grow before colliding) then we expect that the fracturing process explored in this paper will occur in individual collisions. This will lead to an additional peak in the gravitational wave spectrum with a smaller wavelength associated with the size of the unstable modes produced during the collision. The two sources of gravitational waves mentioned above are a direct result of scalar field dynamics and will be present in vacuum. However, in a high temperature phase transition the subsequent (turbulent) dynamics of the plasma may source additional gravitational waves. In this case, the fracturing of the walls and long-lived oscillons may force turbulent motion and thus indirectly produce additional gravitational waves.

The production of oscillons during the phase transition also perturbs the expansion history of the universe by adding a component that behaves as collisionless dust to the ambient cosmological constant (vacuum transition) or radiation bath (high temperature transition). If the oscillons form a significant fraction of the post-transition energy density, this could lead to a temporary stage of matter domination in the early universe.

4.8 Applications to Other Scenarios

We have restricted ourselves in this paper to some very simple toy potentials that permit Coleman-deLuccia bubble nucleation. However, the basic mechanism–amplification of non SO(2,1) symmetric fluctuations–likely plays a key role in many other scenarios. For a small number of fields, the important feature of the potential appears to be the presence of a slightly asymmetric double-well structure in one of the field directions.

A direct application would be to the bubble baryogenesis scenario of [162]. This scenario takes a two-field model with slightly broken internal O(2) global symmetry. Baryon number is generated by the nucleation and expansion of bubbles, and also during their collisions. The collisions considered there very closely resemble our results for the thin-walled bubbles, except that SO(2,1) symmetry was enforced during the collisions. It would be interesting to see how the net baryon number production is influenced by breaking of the SO(2,1) demonstrated above.

Another possible application of these results is to boom and bust inflation [163] and other models of flux discharge cascades driven by bubble nucleation and expansion in a compact extra dimension [164, 165]. However, there are several caveats to a direct application of our results. First of all, the bubble walls in these models move at extremely relativistic speeds and are thus extremely Lorentz contracted in the center of mass frame for the collision. This means that the amount of time that the walls interact at each collision will be very short. As a result, at each collision the walls may simply pass through each other before the fracturing has time to occur as in our collisions. A second caveat is that the bubble walls in the flux cascade picture are meant to be branes instead of scalar field kinks. The nonlinear interactions that lead to the breakup of the walls are sensitive to the details of the high energy theory, so the nonlinear fracturing stage of the walls may be altered in this case.

A final scenario appearing in the literature where non-symmetric fluctuations are likely to play a key role but were not considered is in [166]. Here, the author's study latetime volation of the free-passage approximation by displacing the field to very near local maximum of the potential in the collision region. However, they make the assumption of planar symmetry and work in 1+1 dimensions. The nonplanar fluctuation modes will in this case experience a tachyonic instability [32, 31] that will quickly invalidate the planar symmetry approximation. In fact, the seeds of this are already in the author's work as their effect relies on precisely this tachyonic instability for the $k_{\perp} = 0$ mode. Beyond these obvious applications of the ideas presented in this paper, there are many other scenarios in which corrections to SO(2,1) symmetry may be expected. In this paper, we only considered collisions between bubbles formed from the same instanton, for which there is no topological constraint preventing the walls of the colliding bubbles from eventually annihilating each other. A natural extension is to consider potentials with additional minima, and collisions between bubbles formed from different instantons. In this case, the field value in the bubble interiors will enforce topological constraints on the resulting collision dynamics and domain walls will in general be formed in these collisions. However, these domain walls can have their own internal dynamics which can again lead to a breakdown of the SO(2,1) invariance. This is the generalization of the oscillations of the shape mode in the double well potential studied in chapter 3 to the case of bubble collisions. Since the internal dynamics generally emit radiation, and this radiation will not in general respect the SO(2,1) symmetry, any signatures that may result from it again differ from the results of an SO(2,1) simulation.

4.9 Conclusion

In this chapter we have performed full three-dimensional simulations of bubble collisions in scalar field theories. Our treatment is novel because we include for the first time the effects of quantum fluctuations (in the semiclassical wave limit) on the dynamics. Recent interest in this topic has been driven largely by false vacuum eternal inflation, in which the bubbles initially form through quantum nucleation. In this context, previous studies have assumed the resulting collision between two such bubbles possesses an exact SO(2,1) symmetry, which itself derives from the partial breaking of the SO(4) symmetry of the instanton solution. However, it is important to keep in mind that the instanton only describes the most likely bubble to nucleate. In reality, the actual shape of the nucleated bubble will be slightly deformed from the perfect instanton profile due to the quantum nature of the nucleation. Even ignoring these deformations to the initial bubble shape, the bulk fluctuations in the ambient spacetime and inside the bubble are present after the nucleation of a bubble. In other applications, such as high energy phase transitions, the nucleation of individual bubbles is again driven by coherent structures arising from random fluctuations. Hence in all these cases the exclusion of fluctuations, which is implicit in assuming exact SO(2,1) invariance, is *not* consistent with the process of nucleation. Since these fluctuations do not obey the assumed SO(2,1) symmetry (or SO(2) symmetry in the case of thermal nucleation), it is important to test this assumption.⁸

We have studied collisions between pairs of nucleated bubbles in a variety of singlefield and two-field potentials using highly accurate nonlinear three-dimensional lattice simulations. A novel aspect of this investigation was the development of a pseudospectral approach for finding SO(4) symmetric instanton solutions, rather than the ubiquitous overshoot-undershoot method. The accuracy of our instanton profiles are only limited by the roundoff errors associated with machine precision, and the procedure easily generalizes to multifield potentials. Although we only considered the case of instantons in Minkowski space, the method can be trivially generalized to the case of a fixed background geometry, and with a little extra effort to the case of a dynamical metric coupled to the scalar fields.

We studied two types of single-field potentials: double-well potentials with a broken Z_2 symmetry, and potentials with a single local minimum and the second minimum replaced by a linear plateau with an adjustable slope. Under the assumption of SO(2,1)symmetry, in double-well potentials with mildly broken Z_2 symmetry the bubble walls undergo repeat collisions, producing an outgoing pattern of toroidal waves centered on location of the initial collision and expanding in the plane orthogonal to the collision axis. Initially small symmetry breaking fluctuations experience a strong instability as a result of this motion in the symmetric background. These fluctuations are quickly driven into the nonlinear regime, at which point a split into an SO(2,1) symmetric background and nonsymmetric fluctuations loses its utility. A fully three-dimensional nonlinear description is then required to describe the resulting evolution. The nonlinear dynamics causes the bubble walls and expanding torii to fracture into a network of localized blobs of field known as oscillons. At this point, the SO(2,1) symmetry has been spoiled completely and any dimensional reduction based on it will be a very poor approximation to the collision dynamics. To the best of our knowledge, this is the first example where the three-dimensional nature of the problem plays an important role in the field dynamics for the collision between a pair of bubbles.⁹

Meanwhile, when we considered situations where the field tunnels out far away from a minimum of our potentials (either because the Z_2 is strongly broken or there is a long plateau), we found no evidence that the SO(2,1) breaking fluctuations experienced instabilities.¹⁰ As a result, these situations can be well-approximated by dimensionally

⁸As noted before, by this we mean that individual realization of the fluctuations do not preserve the symmetry, even if they do preserve the symmetry in a statistical sense.

⁹Easther et. al also performed three-dimensional bubble simulations [101], but they did not include fluctuations in their initial conditions and the effect that they discuss can be captured using 1+1-dimensional simulations.

¹⁰For the badly broken Z_2 double-well, this statement may be an artifact of our choice of potential,

reduced simulations.

The single field potentials we studied do not permit a situation where the bubble walls can fracture while the field inside the bubble simultaneously drives a period of slow-roll inflation. This essentially follows from the fact that inflation requires a long region of the potential whose characteristic mass scale is much less than the hubble constant. In contrast, the bouncing of the walls and subsequent production of oscillons only occur if the typical mass scale in the potential is instead much greater than the hubble constant. For the case of identical bubbles, within the collision regions the field explores a portion of the potential that is also supposed to be driving inflation. Since the conditions for inflation and oscillon production are mutually exclusive, this suggests that it is extremely difficult to both drive inflation and produce oscillons directly for a bubble collision in a single field potential. However, it is important to keep in mind that within the framework of inflation on a landscape there are many scalar degrees of freedom, not just one. We therefore considered a very simple two-field model in which we allowed one of the field directions to dominate the tunnelling and the other direction to drive slow-roll inflation. During a collision between two such bubbles, the dynamics of the tunnelling direction closely resembles that of the single-field collision in the double-well with mild Z_2 breaking. As a result, oscillons are able to form in this direction. The coupling between the tunnelling field and the inflaton direction then causes the "inflaton" field to be pulled back up the potential. This entire process again badly breaks the SO(2,1) symmetry and requires more than a one-dimensional simulation to capture.

Our simulations are not fully relativistic, so we are unable to track the subsequent evolution of the fields through the complete phase of inflation inside the bubble. This is certainly an interesting question, but it is also a very challenging numerical problem that is likely to be much more difficult than the corresponding symmetry reduced gravitational dynamics. However, since the relevant time scales for oscillon formation are much less than the Hubble time, we can view our results as setting the initial conditions for the subsequent inflationary epoch. Since the SO(2,1) symmetry is broken so strongly in some cases, this question cannot be addressed within the framework of a symmetry reduced 1+1-dimensional problem. In particular, previous signatures associated with bubble collisions have assumed an azimuthal symmetry will hold, which results from the assumption of SO(2,1) dynamics for the spacetime. To address this, one must determine how quickly the nonsymmetric part of the initial perturbations damp relative to the SO(2,1) preserving part. We plan to address these issues and use data to constrain the possible observables mentioned here in future work.

but for the plateau in the tunnel out region it should be generic.

Chapter 5

A Shock-in-Time: Post-Inflation Preheating

5.1 Introduction

Early inflation within our Hubble patch, if it occurred, was driven by the potential energy of an ultra-long wavelength coherent scalar effective field, which caused accelerated expansion of the Universe. This bosonic condensate would have been accompanied by shorter wavelength nearly Gaussian fluctuations of small amplitude. Such nearly Gaussian fluctuations, some possibly correlated with the inflaton, would also have been present in the graviton and any light scalar fields (which we refer to as isocons). Super-Hubble fluctuations led to a condensate of the long wavelength modes, from which the complexity of the Hubble patch that surrounds us must have arisen, ultimately producing a decelerating plasma of standard model particles in local thermal equilibrium. The condensate had low entropy associated with the sub-Hubble fluctuations, while the plasma had high effective thermal entropy ultimately stored in the photon and neutrino relics within our patch, with a comoving entropy density $S_{\gamma+\nu} \sim 10^{88}/(10\,{\rm Gpc})^3$, normalized to 10 Gpc.

The transition regime connecting the end-of-inflation (when the acceleration/deceleration boundary time-hypersurface was breached) to the hot primordial plasma in local thermal equilibrium is commonly called preheating. During preheating fluctuations experience strong instabilities as a result of the background motion of the long-wavelength condensate. When these fluctuations begin to probe the nonlinear regime, strong backreaction and rescattering effects cause the condensate to fracture leaving behind a highly inhomogeneous complex medium. Past studies have considered many interesting signatures from preheating, including baryogenesis, gravitational waves, topological defect production, and nongaussianities. This chapter will focus on a new potential signature – spatial and temporal variations in entropy production – with an eye to connecting them to nongaussianities.

Preheating is usually explored by following the evolution of classical field equations [26, 27, 28, 29] which, because of the highly nonlinear nature of mode-mode coupling, invariably requires simulations on a lattice [38, 39, 40, 41, 43]. Here we follow suit and study entropy production during preheating using high-resolution and highly accurate lattice simulations. During the nonlinear phases of preheating coherent inflaton oscillations must transform into a cascade of spatial modes. We will show using our simulations that the cascade's entropy tracks the transition from coherence to incoherence well.

Our simulations show that in a large class of models a sharp spike in entropy production accompanies the onset of nonlinearities. In an ordinary gas, the passage from supersonic to subsonic occurs through a spatial randomization front – a shock – where the entropy jumps dominantly over a mediation scale, with jump conditions on conserved variables holding. During preheating a coherent density (with small fluctuations) evolves into an incoherent mix of spatial modes over a relatively narrow mediation time $\Delta \ln t_s$ at a sharply defined $\ln t_s$ (and expansion factor $\ln a_s = \ln a(\mathbf{x}, t_s)$). Based on this similarity, we call the phenomenon a *shock-in-time*. If $\ln a_s$ varies spatially, a curvature imprint may remain. We apply this idea to a simple model of modulated preheating and find spatial modulations in the shock time that could produce observationally interesting curvature perturbations.

The entropy we use to track the cascade's evolution only strictly applies for an inherently stochastic system. The only randomness in our simulations comes from the choice of a particular realization of the initial field fluctuations. Once this choice is made, the subsequent evolution is unitary (up to numerical noise). If we had perfect knowlege of the states of all the system variables we would conclude that no entropy had been generated in any one realization. However, we do not have such perfect knowledge – from a coarsegrained view of the full "universe-in-a-box" \mathcal{U} , there is a system \mathcal{X} whose variables we are following and a reservoir \mathcal{R} of unobserved variables we marginalize over. Although the entropy of the universe $S_{\mathcal{U}}$ may be zero or nearly so, classical entanglement of the \mathcal{X} and \mathcal{R} variables leads to entropy generation as measured by $S_{\mathcal{X}}$.

We may define our field theory in terms of its n-point correlation functions (and potentially some additional information escaping the correlation hierarchy). Our choice of system variables is then a few low-order correlators, with the remaining higher-order correlators comprising the reservoir variables. In a non-linear theory, the hierarchy of evolution equations for the correlation functions couples the low-order correlators to higher order correlation functions. As a result, our system couples to the environmental degrees of freedom, and this interaction leads to the development of system-reservoir and reservoir-reservoir correlations. Therefore, information can be carried from the system variables into the reservoir. From the viewpoint of an observer with access to only the system variables, this will manifest as a change in the system entropy.

We formalize this intuition by defining our entropy via a maximization of the (differential) Shannon entropy subject to the constraints of a set of measurements made on the system. Here we will assume that we have made measurements of the covariance matrix for a collection of statistically homogeneous fields. For the case of single constrained field, the corresponding entropy is

$$S_{max} = \frac{1}{2} \sum_{\mathbf{k}} P(\mathbf{k}) + \frac{N_{lat}}{2} \ln 2\pi + \frac{N_{lat}}{2}$$
(5.1)

where $P(\mathbf{k})$ are the eigenvalues of the field's covariance matrix (ie. the power spectrum). Analogous expressions have appeared in several past studies [167, 168, 169, 170, 171, 172, 173, 174, 175]. However, in these works (5.1) was derived under the assumption that the fields were multivariate Gaussians. As a result, they restricted themselves to linear (or weakly nonlinear) field evolutions and generated entropy by discarding information about cross-correlations between different fields. Our use of (5.1) is instead motivated by restricted access to higher-point system correlation functions. As a result, it applies even if the fields are not Gaussian distributed, even though we are motivated to find approximately Gaussian variables. Therefore, we need not restrict ourselves to linear field dynamics and can instead study the stongly nonlinear fluctuation regime. One past study used a similar motivation of neglecting higher order correlations functions to study decoherence as a result of nonlinear interactions in scalar field theory [176].

The presence of additional constraints will modify the above result for the entropy. For example, in an inhomogeneous condensate – such as a collection of topological defects – there are correlations between various Fourier modes, whose entropy is therefore not properly determined by considering only the power spectrum. The condensate effectively acts as a background around which the field is constrained to fluctuate. Despite this qualification, the presence of a spike in the entropy production around the onset of nonlinearities should be robust even if a condensate forms, and the above caveat thus does not affect our main conclusions. In the particular examples considered here, the dominant contribution to the entropy comes from the largest k-modes (which are not part of some slowly varying condensate). More generally, if we were to consider a model in which topological defects are produced there would still be a rapid production of entropy at the moment the defects form, followed by additional production of entropy as the defects annihilate or decay.

The remainder of this chapter is organized as follows. In section 5.2 we discuss our general framework for non-equilibrium entropy as well as our coarse-graining procedure, and then introduce our models and numerical methods in section 5.3. Section 5.4 applies this approach to the variables $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$, where we demonstrate the existence of the shock-in-time. We also provide new evidence for the Gaussianity of the low-order statistics of these fields. In section 5.5 we extend the entropy and statistical calculations to the fundamental field variables, demonstrating that the shock is robust to this variable change but that the fields are noticably nongaussian. Section 5.6 reformulates the Shannon entropy for noncanonical choices of fields variables, allowing us to connect our results for $(\ln(\rho/\bar{\rho}), \partial_t \ln(\rho/\bar{\rho}))$ and the fundmental field variables. Explicit calculations of this noncanonical entropy are presented in section 5.7. We apply the shock-in-time concept to investigate the production of curvature fluctuations in section 5.8, then finally conclude.

5.2 Non-equilibrium Entropies from Constrained Collective Coordinates and Their Conjugate Forces

For a classical random field with N components q distributed according to a probability density functional (PDF) $f_f[q]$, we adopt the Shannon information entropy

$$S_{shannon}[f_f] = -\int d^N q f_f[q] \ln f_f[q] = -\langle \ln f_f \rangle_f$$
(5.2)

as our definition of the nonequilibrium entropy [177, 178]. We are using the notation $\langle \cdot \rangle_f$ to denote averaging with respect to $f_f[q]$. When we move to the continuum limit, we have $N \to \infty$ and the integration measure becomes a functional measure $d^N q \to \mathcal{D} q$. In the quantum theory, the field components q become operators \hat{q} and the Shannon entropy is replaced by the von Neumann entropy $S_{vN} = -\text{Tr} \hat{f}[\hat{q}] \ln \hat{f}[\hat{q}]$ involving the trace of the full density matrix \hat{f} . A significant issue with the Shannon entropy (5.2) is that for continuous variables it is *not* invariant under variables changes $q \to \tilde{q}(q)$. To solve this problem, the Kullback-Leibler (KL) divergence [179, 178] (also known as the relative entropy) is often introduced

$$S_{KL}(f_f||f_i) = \int d^N q f_f[q] \ln\left(\frac{f_f}{f_i}\right)$$
(5.3)

with the (normalized) reference probability distribution f_i absorbing the effects of the variable change. We do not explicitly consider the relative entropy in this paper, although we will explore a very similar approach in section 5.6.

Calculation of either the Shannon or von Neumann entropy requires full knowledge of the distribution of parameters, as encoded in either the PDF or density matrix. However, acquiring such detailed knowledge is overly ambitious. In a realistic scenario reduced information may come from empirical measurements of the probability distributions $P(\bar{\vartheta}) = \langle \delta(\vartheta(q) - \bar{\vartheta}) \rangle$ of a set of operators $\vartheta^A(q)$. Even more realistically, the measurements will be of low order ensemble-averaged correlations, in particular their means $\bar{\vartheta}^A \equiv \langle \vartheta^A \rangle$ and variances,

$$C^{AB}_{\vartheta\vartheta} \equiv \langle C^{AB}_{op,\vartheta\vartheta} \rangle$$

$$\delta\vartheta^{A}(q) \equiv \vartheta^{A} - \bar{\vartheta}^{A}, \quad C^{AB}_{op,\vartheta\vartheta}(q) \equiv \delta\vartheta^{A}\delta\vartheta^{A}.$$
(5.4)

Obtaining this set of reduced information about the full statistical properties provides a coarse-grained description of the fields.

This coarse-graining leads to a natural definition of the entropy associated with our limited knowledge of the system properties. We define the entropy to be equivalent to that of a field with distribution f_{ME} that maximizes the Shannon entropy subject to the constraints of various measurements. Our constraints are in the form of empirical statistical averages for a collection of operators ϑ^A , $\int f_{ME} \vartheta^A = \bar{\vartheta}^A$ and $\int f_{ME} = 1$, with the associated Lagrange multipliers denoted by κ_A and $\mathcal{F} \equiv \ln Z$ respectively. Throughout we will refer to this as the maximum entropy (MaxEnt) approach. When a solution to the maximization problem exists, the MaxEnt probability distribution is given by

$$f_{ME}(q) = \frac{e^{\kappa_A \vartheta^A(q)}}{\mathcal{Z}} \tag{5.5}$$

where we have defined the partition function

$$\mathcal{Z} = \langle e^{\kappa_A \vartheta^A} \rangle \,. \tag{5.6}$$

The resulting entropy is

$$S[f_{ME}] = \ln \mathcal{Z} - \kappa_A \bar{\vartheta}^A \tag{5.7}$$

with the lagrange multipliers κ_A chosen such that $\mathcal{Z}^{-1} \int e^{\kappa_A \mathcal{O}^A} \vartheta^A = \bar{\vartheta}^A$ for each A. This is similar to the Jaynesian viewpoint of statistical mechanics [180, 181], where the probability density (and entropy) of a system in statistical equilibrium are determined by maximizing the Shannon entropy (5.2) subject to the physical constraints on the system. These constraints often come in the form of values for a collection of conserved charges for the system in question. However, our viewpoint is slightly different as we are placing constraints based on measurements rather than physical considerations, therefore the importance of the observer making the measurements is explicit in our approach.

A familiar example occurs with just one operator, the total Hamiltonian energy of the system, $\vartheta = H(q)$. The standard textbook result for the entropy gives the thermodynamic relation $S = \beta \langle H(q) \rangle - \beta F$, where $F = -T \ln Z = -T \ln \text{Tr}e^{-\beta H}$ is the free energy and $\beta = T^{-1}$ is the inverse temperature, with the corresponding probability of obtaining a state given by the canonical ensemble $P_{can} = Z^{-1}e^{-\beta H}$. The inclusion of additional conserved charges and the associated Lagrange multipliers similarly leads to the grand canonical ensemble. More generally, in non-equilibrium thermodynamics spatial variations in locally conserved charges (such as the energy density) drive flows towards equilibrium. The charge operators therefore depend upon positions in the volume and are supplemented by additional operators describing the fluxes of these charges. In a relativistic theory, it is convenient to combine the charge and flux operators into 4-currents.

In the standard thermodynamics of canonical and grand canonical ensembles the mean of global variables such as energy and conserved charges are taken to be determined exactly. However, it is more realistic that the mean is an estimate with an error matrix associated with it. This error matrix is itself an estimate of $C_{op,\vartheta\vartheta}$; *i.e.*, $\int f_f \vartheta^A = \bar{\vartheta}^A + [C_{\vartheta\vartheta}^{\frac{1}{2}}]^{AB}\eta_B + \ldots$, with η_B a Gaussian random deviate $(\langle \eta_A \eta_B \rangle = \delta_{AB})$. Since the "noise" variance would itself only be an estimate, we could go to quartic order in ϑ correlators for the error in it, and so on. A nice aspect of closing off at quadratic order is that the exponential asymmetry associated with $\kappa_A \vartheta^A$ is regulated, and the required Gaussian integrals can be performed analytically. There is considerable interpretational elegance to, in effect, complete the square of the collective operator driving terms by allowing for the conjugate variable $G_{\vartheta\vartheta,AB}$ to the collective coordinate correlation function $C_{\vartheta\vartheta}^{AB} = \delta \vartheta^A \delta \vartheta^B$ in addition to the κ_A conjugate to $\bar{\vartheta}^A$ – from which $C_{\vartheta\vartheta}^{AB} = \langle C_{op,\vartheta\vartheta}^{AB} \rangle$ and other moments can be obtained by functional derivatives.

In this chapter we are interested in the special case of a set of collective operators ϑ^A with constrained means $\bar{\vartheta}^A$ and covariance matrix $C^{AB} = \langle \vartheta^A \vartheta^B \rangle$. Denoting the lagrange multipliers conjugate to $\bar{\vartheta}^A$ by λ_A and those conjugate to C^{AB} by G_{AB} , we find that the MaxEnt distribution has the form $f_{ME} \propto e^{-\frac{1}{2}G_{AB}\vartheta^A\vartheta^B + \lambda_A\vartheta^A}$. We can easily solve to find $G_{AB} = [C^{-1}]^{AB}$ and $\lambda_A = G_{AB}\bar{\vartheta}^B = C_{AB}^{-1}\bar{\vartheta}^B$. The resulting maximum entropy is given

by the Gaussian Shannon entropy

$$\frac{S_G}{N_{\vartheta}} = \frac{1}{2N_{\vartheta}} \ln \det C + \frac{1}{2} \ln 2\pi + \frac{1}{2} = \frac{1}{2N_{\vartheta}} \sum_{\mathbf{k}} \ln P(\mathbf{k}) + \frac{1}{2} \ln 2\pi + \frac{1}{2}$$
(5.8)

where $P(\mathbf{k})$ are the eigenvalues of the covariance matrix labelled by \mathbf{k} (our motivation for this notation will be clear shortly). Furthermore, we take our collective operators ϑ^A to be a collection of statistically homogeneous fields (which we denote φ_i^S), with the index S indicating the species and i denoting the lattice site. For this case, the index A on the ϑ collective variables includes information about the field species and the lattice site A = (S, i), and we have $N_{\vartheta} = N_{fld}N_{lat}$ with N_{fld} the number of collective field species and N_{lat} the number of lattice sites.

When we have a single (statistically homogeneous) collective field φ , the wavenumbers **k** label the eigenmodes of the covariance matrix $C_{\varphi\varphi}$. The corresponding eigenvectors are given by the power spectrum $P_{\varphi\varphi}(\mathbf{k})$ of the the field φ , obtained via Fourier transformation (with unitary normalization) of the covariance matrix in the relative spatial separation between two points. This is closely related to the Wigner function $W(\mathbf{X}, \mathbf{k})$ via

$$W_{\varphi\varphi}(\mathbf{X}, \mathbf{k}) \propto \int d^3 r e^{i\mathbf{k}\cdot\mathbf{r}} C_{\varphi\varphi}(X + r/2, X - r/2) \,.$$
 (5.9)

$$P_{\varphi\varphi}(\mathbf{k}) = \frac{\int d^3 X W(\mathbf{X}, \mathbf{k})}{\int d^3 X}$$
(5.10)

where W is independent of \mathbf{X} for a statistically homogeneous field. In this case, the Gaussian Shannon entropy becomes

$$\frac{S_G}{N_{lat}} = \frac{1}{N_{lat}} \sum_{\mathbf{k}} \ln P_{\varphi\varphi}(\mathbf{k}) + \frac{1}{2} \ln 2\pi + \frac{1}{2}.$$
 (5.11)

If the field is also isotropic, then $P_{\varphi\varphi}$ is a function of the wavenumber magnitude $|\mathbf{k}|$ only. If we have N_{fld} species φ^S , $S = 1, ..., N_{fld}$, then the Fourier transform can be used to block diagonalize the covariance matrix into $N_{fld} \times N_{fld}$ blocks labelled by \mathbf{k} , with components given by the auto- and cross-power spectra for that wavenumber. The power spectra in (5.11) is then replaced by the determinants of these full cross-power matrices.

Given the complexities of the nonlinear regime, it is not clear there will be any collective variables with a relatively simple distribution function. The hope is to find nearly Gaussian random fields φ^S whose distributions can be characterized by their mean $\bar{\varphi}^A$ and covariance $C^{ij}_{\varphi\varphi'}$. In the linear perturbation regime, all fluctuation variable combinations are related to a set of Gaussian random deviates describing normal modes of the fluctuations. These of course include the fundamental scalar fields ϕ and their momenta II. We do explore a maximum entropy Gaussian distribution function based upon our measurements of the full correlation function of these primitive field variables, but even at the visual level in simulations it is clear that these are not in fact nearly Gaussian in the nonlinear regime.

A fundamental collective variable combination of the underlying fields is the phonon associated with total comoving energy density fluctuations, a gauge and time hypersurface invariant quantity that in fact fully characterizes the inflaton. One cannot tell at the linear level whether it is the energy density or its logarithm which best characterizes the inflaton. However in [13, 182] it was shown that in the long wavelength limit of nonlinear stochastic inflation, $\ln a = \frac{1}{6} \operatorname{Tr} \ln g$ on uniform Hubble surfaces (essentially uniform comoving energy density surfaces) is a nonlinear generalization of any one of the gauge invariant variables that characterize curvature and are constant outside the horizon for single-field inflation. This has been much used subsequently [183]. One of the gauge invariant combinations is the relative comoving energy density fluctuation, which suggests that the log of the comoving energy density $\ln \rho_{com}$ might be of interest. By direct measurements in our simulations, we find that the energy density requires higher point correlations to characterize it, but the logarithm is found to be more nearly Gaussian. We refer to the quanta associated with the collective variable $\ln \rho_{com}$ as (energy) phonons. Obviously in linear theory these are equivalent to the ordinary idea of phonons. A nice aspect of the log is that the difference $\ln \rho / \rho_s$ is relatively insensitive to any smoothed large scale structure ρ_s . This is also the reason $\ln a/a_s$ is more relevant that a directly.

We find below that our basic conclusions about entropy generation rate are robust to variation in the specific choice of collective field variables. We even find the Gaussian entropy associated with the primitive field variables work. However, it is better to use a Gaussian entropy motived by our simulation measurements, involving the correlation function of $\varphi = \ln \rho / \rho_s$, where the reference ρ_s could have long wavelength structure in it — although in practice we use the density averaged over the simulation box of volume $V, \bar{\rho} = E/V$ for ρ_s , where E is the total energy in the box.

5.3 Models and Numerical Methods

We consider two models for the preheating phase following inflation, both of which exhibit broad band parametric resonance during the initial linear stages. First is a simple twofield model with potential

$$V(\phi,\chi) = \frac{m^2}{2}\phi^2 + \frac{g^2}{2}\phi^2\chi^2$$
(5.12)

where ϕ is the inflaton and χ is a field into which the inflaton will decay that we denote as the preheat field. We assume that the inflationary phase was driven by a large condensate value for ϕ , with the inflationary phase ending at $\bar{\phi}_{end} \sim \sqrt{2}M_P$. After this ϕ oscillates as a damped oscillator $\bar{\phi} \sim \cos(mt)/a^{3/2}$. We start our simulations at the point when $\epsilon = -\dot{H}/H^2 = 1$, as determined by an evolution of the homogeneous background equations. The initial mean value of χ is 0. Both the linear and nonlinear dynamics of this model with the initial conditions given above have been well-studied in the literature (see e.g. [27, 184, 39, 185]). During the homogeneous oscillations of ϕ , fluctuations in χ approximately satisfy the Mathieu equation resulting in the parametric amplification of a band of wavenumbers. Once the χ fluctuations become sufficiently large, they begin to excite fluctuations $\delta\phi$ leading to the creation of bubbly standing wave structures in the fields. Shortly after this these bubbly structures become strongly nonlinear, leading to a rapid cascade of fluctuation power to smaller scales, phase mixing and randomization of the fields.

Our second model is a single-field preheating model with potential

$$V(\phi) = \frac{\lambda \phi^4}{4} \tag{5.13}$$

with inflation ending at $\bar{\phi}_{end} \sim \sqrt{8}M_P$. The fluctuations in the field ϕ now experience an instability which is accurately modelled by the Lame equation [33]. As a result of the conformal nature of this model (at the classical level), this instability occurs at a fixed comoving wavenumber and thus ϕ resonantly excites its own fluctuations. Once again, nonlinear interactions lead to a cascade of fluctuation power to higher wavenumbers and the emergence of a slowly evolving state which is claimed to be a combination of weak wave turbulence and strong turbulence [186, 187, 185, 188, 189].

Due to the complexities of the scalar field dynamics as they enter the highly nonlinear regime, it is necessary to employ lattice simulations in order to properly study the dynamics. The necessary numerical techniques have been well-developed beginning with LATTICEEASY [38], and subsequently using more accurate time-integrations in DE-FROST and HLATTICE [39, 40], pseudospectral spatial discretizations in PSpectRe [41] and even a GPU-enabled version PyCOOL [43]. We use the lessons from these previous codes to develop a new MPI/OpenMP hybrid lattice code for the simulations in this chapter. For the 2-field model (5.12) we assume a metric of the form $ds^2 = -dt^2 + a(t)^2 dx^2$ and solve Hamilton's equations for the fields $\phi_i^A = \phi^A(x_i), y_a = a^{3/2}$ and their canonical momenta $\Pi_i^A \equiv a^3 \dot{\phi}^A(x_i), \Pi_a \equiv -8N_{lat} \dot{y}_a/3 = -4N_{lat} H a^{3/2}$

$$\frac{d\phi_i^A}{dt} = \frac{\Pi_i^A}{y_a^2} \tag{5.14}$$

$$\frac{d\Pi_i^A}{dt} = -y_a^2 \partial_{\phi^A} V + y_a^{2/3} \nabla^2 \phi_i^A \tag{5.15}$$

$$\frac{dy_a}{dt} = -\frac{3\Pi_a}{8N_{lat}} \tag{5.16}$$

$$\frac{d\Pi_a}{dt} = \sum_i \left(\frac{\Pi_i^T \Pi_i}{y_a^3} - 2y_a V - y_a^{-1/3} \frac{\nabla \phi_i^T \cdot \nabla \phi_i}{3} \right) \,. \tag{5.17}$$

For (5.13) it is instead convenient to work in conformal time τ with metric $ds^2 = a^2(\tau) (-d\tau^2 + dx^2)$, $\Pi_i = a^2 \partial_\tau \phi_i$, and $y_a = a$, $\Pi_a = -6N_{lat}\partial_\tau y_a = -6N_{lat}a^2H$ with resulting equations of motion

$$\frac{d\phi_i^A}{d\tau} = \frac{\Pi_i^A}{y_a^2} \tag{5.18}$$

$$\frac{d\Pi_i^A}{d\tau} = -y_a^4 \partial_{\phi^A} V + y_a^2 \nabla^2 \phi_i^A \tag{5.19}$$

$$\frac{dy_a}{d\tau} = -\frac{\Pi_a}{6N_{lat}} \tag{5.20}$$

$$\frac{d\Pi_a}{d\tau} = \sum_i \left(\frac{\Pi_i^T \Pi_i}{y_a^3} - 4y_a^4 V - y_a \nabla \phi_i^T \cdot \nabla \phi_i \right)$$
(5.21)

In the above we have defined the total number of lattice sites N_{lat} .

To numerically evolve the system, we employ a sixth-order Yoshida splitting method (c.f. [158, 40, 43]). For the spatial discretization we use a finite-difference stencil,

$$\nabla^2 \phi(x_i) = \sum_{\alpha} 2d_{\alpha}(\phi_{i+\alpha} - \phi_i) = \sum_{\alpha} 2d_{\alpha}\phi_{i+\alpha}$$
(5.22)

where we used $\sum_{\alpha \neq (0,0,0)} d_{\alpha} = -d_{(0,0,0)}$. Self-consistency requires the following definitions for the other relevant differential operators that will appear below

$$\nabla \phi^A(x_i) \cdot \nabla \phi^B(x_i) = \sum_{\alpha} d_{\alpha} \left(\phi^A_{i+\alpha} - \phi^A_i \right) \left(\phi^B_{i+\alpha} - \phi^B_i \right) \,. \tag{5.23}$$

For notational simplicity, we have defined $\phi_i = \phi(x_i)$ with superscripts in capital Roman letters indicating different field species and *i* labelling the grid sites. The coefficients d_{α} define the discretization scheme. We chose a second-order accurate and fourth-order isotropic stencil which uses the neighbouring lattice sites $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i = 0, \pm 1$ and d_{α} dependent only on $\sum |\alpha_i|$. The precise values of the coefficients are

$$dx^{2}d_{(1,0,0)} = \frac{7}{15} \qquad dx^{2}d_{(1,1,0)} = \frac{1}{10} \qquad dx^{2}d_{(1,1,1)} = \frac{1}{30} \qquad dx^{2}d_{(0,0,0)} = \frac{-64}{15}.$$
 (5.24)

To check for spatial convergence, in addition to varying the overall lattice size L and lattice spacing dx, we also checked part of the analysis using a pseudospectral approximation for the spatial derivatives.

For reference, the stress-energy tensor for the scalar fields is given by

$$T^{\mu}_{\nu} = \sum_{i} \partial^{\mu} \phi_{i} \partial_{\nu} \phi_{i} + \delta^{\mu}_{\nu} \left(-\sum_{i} \frac{\partial^{\alpha} \phi_{i} \partial_{\alpha} \phi_{i}}{2} - V(\phi) \right) .$$
 (5.25)

The local energy density measured observers comoving with the expansion is $\rho = -T_0^0$ and the local isotropic pressure is $P \equiv T_i^i/3$. Throughout $\langle \cdot \rangle$ will denote ensemble averages and $\overline{\cdot}$ spatial averages. For a statistically homogeneous field, these two averaging procedures may be interchanged, although it may take many realizations of the dynamics in order to properly sample the longest wavelength modes in the box.

5.4 Application to Preheating: the Phonon Energy Density Modes

5.4.1 Entropy in the Phonon Description

Now we apply the formalism developed in 5.2 to determine the production of entropy after inflation in the preheating model (5.12). Since our restriction to the measurement of two-point correlations results in an entropy that is the same as if the fields were multivariate Gaussians, it is desirable to choose collective variables that are at least approximately Gaussian.

Frolov [39] found that shortly after the onset of strong inhomogeneities in the fields, the one-point probability density of $\rho/\bar{\rho}$ quickly settled down into a nearly log-normal form in a variety of two-field preheating models. Motivated by this, we will compute the Gaussian Shannon entropy taking $\ln(\rho/\bar{\rho})$ as the underlying nearly Gaussian field. As well, it is desirable to have a second variable describing the instantaneous dynamical evolution. Thus, we introduce $\partial_t \ln(\rho/\bar{\rho})$ as a variable of interest into the preheating literature. Using the equations of motion, we have

$$\frac{\partial \ln(\rho/\bar{\rho})}{\partial t} = -3H\left(\frac{P}{\rho} - \frac{\bar{P}}{\bar{\rho}}\right) + \frac{\partial_i T_0^i}{\rho}$$
(5.26)

where

$$\partial_i T_0^i = \sum_I \frac{\dot{\phi}_I \nabla^2 \phi_I}{a^2} + \frac{\nabla \dot{\phi}_I \cdot \nabla \phi_I}{a^2} \,. \tag{5.27}$$

Here we have defined the local energy density $\rho \equiv -T_0^0$ and isotropic pressure $P \equiv T_i^i/3$ measured by observers comoving with the expansion of the background. The first term on the right hand side of (5.26) describes dilution of the energy density due to the expansion of the background spacetime, while the second arises from the transport of energy (heat currents) as measured by the comoving observers. Since we have multiple scalar fields, a full description of the system also includes the difference in ϕ and χ energies and its timederivative. Due to the coupling between the fields in the potential, a priori it is not clear what the simplest choice for the difference would be and we will restrict to consideration of $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$. There are two autocorrelations and one crosscorrelation that we can measure from this pair of variables. We will look at two different entropies, one assuming we have only measured the autocorrelations (equivalently power spectra) and one assuming we have also measured the cross-correlation (equivalently cross power)

$$S_{\ln\rho} = \frac{4\pi\Delta k}{2} \sum_{i}^{k_{cut}} k_i^2 \ln(P_{\ln\rho\ln\rho}P_{\partial_t\ln\rho\partial_t\ln\rho})$$
(5.28)

$$S_{\ln\rho}^{tot} = \frac{4\pi\Delta k}{2} \sum_{i}^{\kappa_{cut}} k_i^2 \ln(P_{\ln\rho\ln\rho}P_{\partial_t\ln\rho\partial_t\ln\rho} - |P_{\ln\rho\partial_t\ln\rho}|^2) \equiv \frac{4\pi\Delta k}{2} \sum_{i}^{\kappa_{cut}} k_i^2 \ln\Delta_{\ln\rho}^2(k)$$
(5.29)

where we have defined $P_{\alpha\beta}(\mathbf{k}_n) = \langle \tilde{f}_{\mathbf{k}_n}^{\alpha} \tilde{f}_{-\mathbf{k}_n}^{\beta} \rangle$ with $\tilde{f}_{\mathbf{k}_n} = \frac{1}{N_{lat}^{1/2}} \sum_i e^{i\mathbf{k}_n \cdot \mathbf{x}_i}$ the discrete Fourier transform of either $\ln(\rho/\bar{\rho})$ or $\partial_t \ln(\rho/\bar{\rho})$ with unitary normalization. With this convention, the eigenvalues of the covariance matrix are equal to $\langle |\tilde{f}_k|^2 \rangle$. Here we will mostly be concerned with entropy differences, so we have dropped the constant contribution $N_{lat} \ln 2\pi + N_{lat}$. In order to regulate the effect of the poorly resolved modes beyond the Nyquist frequency, we introduce a spatial frequency cutoff $k_{cut} < k_{nyq} = \pi N_{lat}^{1/3} L^{-1}$ with respect to which we define an effective number of degrees $\mathcal{N}_{eff}(k) = 4\pi \Delta k \sum_i^{k_{cut}} k_i^2 \approx 4\pi k_{cut}^3/3$. We will provide further evidence that these variables are approximately Gaussian in section 5.4.2.

Our main result is presented in Fig. 5.1 where we show the evolution of the entropy,

the effective Mach number $|\ln(\rho/\bar{\rho})|^{-1}$ (see below) and an indicator of the production of curvature perturbations $3(1 + \langle w \rangle_t) \ln a + \ln \rho$. This final quantity is constant for epochs when the equation of state w is a constant. Therefore, during these epochs we can easily compare the difference in total expansion between different Hubble patches from the end of inflation to a fixed energy density ρ_{comp} . We've used $\langle \cdot \rangle_t$ to denote a time-average over a few oscillations of the background. As well, the rate of entropy production dS/dt (not to scale) is included as a green line with the red band in the background indicating its amplitude. We see that there is a short regime of rapid entropy production at $\ln a \sim 2.9$ (or mt = 120) which lasts for $\delta \ln a \sim 0.1$. This is preceded by a stage of linear parametric resonance during which the entropy decreases slowly (as $-2 \ln a$) and succeeded by a state of highly inhomogeneous nonlinear dynamics where the entropy production is very small. The somewhat slow decrease before the onset of nonlinearities is due to the damping of



Figure 5.1: Various illustrations of the shock in time. Top: Evolution of the entropy for the energy phonons around the onset of strong nonlinearities amongst the fluctuations. Middle: The effective Mach number $|\ln(\rho/\bar{\rho})|^{-1}$. Bottom: The quantity $3(1+w)\ln a + \ln \rho$, which is useful for studying the production of adiabatic density perturbations. In all cases, the time-derivative of the entropy dS/dt (with arbitrary normalization) is shown as the green curve, with the red band indicating the location of the shock-in-time.
the linear fluctuations from the expansion and can be accounted for using the methods of section 5.6.

The transition regime therefore connects a highly coherent low entropy state at early times to a much higher entropy incoherent state at late times. A similar phenomena occurs at a hydrodynamic shock, which acts as a randomization front as it passes through the medium, transforming an unstable supersonic coherent flow into a subsonic incoherent flow. This randomization leads to a jump in the entropy ΔS as the shock passes, possibly with an additional relaxation phase after the shock has passed in which additional entropy is produced. In addition to the entropy, other hyperbolically conserved quantites also undergo rapid changes as the shock passes, with the matching conditions in the limit of an infinitely thin shock known as jump conditions. Given these similarities, we will refer to the entropy production event at the onset of strong nonlinearity in the system as the *shock-in-time*. In the hydrodynamic case, the production of entropy is mediated by viscous effects and collisionless dynamics, while for preheating the mixing is due to strong field gradients and nonlinearities. The Mach number provides a quantitative measure of the unstable nature of the background, with an instability occuring whenever the speed of the coherent bulk flow exceeds the sound speed of the medium, $c_{bulk}^2 >$ c_{sound}^2 . For our shock, it is the unstable nature of the coherent energy density (in the context of the oscillating background fields) that leads to the instability, so that we take $|\ln(\rho/\bar{\rho})|^{-1}$ as an analogue to the Mach number. When the inhomogeneities are small, we have $|\ln(\rho/\bar{\rho})|^{-1} \approx \langle \delta^2 \rangle^{-1} \gg 1$, while in fluctuation dominated case it becomes of order one. From the center panel of Fig. 5.1, we see that the shock-in-time indeed tracks the transition to inhomogeneity as measured by our analogue Mach number. The hydrodnamic shock front is a spacelike hypersurface for any instant in time, so the jump conditions relate the values of various quantities at two points in space on either side of the shock at a fixed moment of time. The shock-in-time, on the other hand, occurs at a fixed moment in time, possibly with some modulation in this time as a function of spatial position. Therefore, conserved quantities such as $T^{\mu 0}$ experience rapid changes in *time*, leading to jump conditions connecting two moments in time rather than two spatial positions.

Now that we have presented our main result, let's consider the nature of the transition in more detail. Initially, $S_{\ln\rho}$ and $S_{\ln\rho}^{tot}$ oscillate in time with the overall evelope of the amplitude decaying as $-2\ln a$. This corresponds to linear evolution of field inhomogeneities, so we can approximate $\ln \rho$ and its time derivative to linear order in the fluctuations

$$\ln(\rho/\bar{\rho}) \approx \frac{1}{\bar{\rho}} \left(\frac{\bar{\Pi}_{\phi} \delta \Pi_{\phi}}{a^6} + m^2 \bar{\phi} \delta \phi \right)$$
(5.30)

$$\partial_t \ln(\rho/\bar{\rho}) \approx \frac{3H}{\bar{\rho}} \left(\frac{(w-1)\bar{\Pi}_{\phi}}{a^6} \delta \Pi_{\phi} + (1+w)V_{\phi}(\bar{\phi})\delta\phi \right) + \frac{\bar{\Pi}_{\phi}\nabla^2\delta\phi}{a^5\bar{\rho}}$$
(5.31)

where we have set $\bar{\chi} = 0 = \dot{\chi}$ and defined $w \equiv \bar{P}/\bar{\rho}$. The homogeneous background ϕ oscillates in a quadratic potential, so $\bar{\phi}, \dot{\bar{\phi}} \sim a^{-3/2}$ and $\bar{\rho} \sim a^{-3}$. As for the fluctuations, for $k \leq ma$ the modes behave as a massive scalar with $\delta\phi_k, \delta\dot{\phi}_k \sim a^{-3/2}$. Meanwhile, for $k \gtrsim ma$ the modes instead behave as a massless scalar with $\delta\phi_k \sim a^{-1}$ and $\delta\dot{\phi}_k \sim a^{-2}$. Outside of the resonant bands, similar considerations hold for the $\delta\chi$ modes with the transition between massive and massless behaviour instead set by $g\langle|\bar{\phi}|\rangle_t$ where $\langle\cdot\rangle_t$ indicates a time-average over a few oscillations of the background ϕ . This behaviour is illustrated in Fig. 5.2 and Fig. 5.3. Using these scalings we see that $\delta \ln(\rho/\bar{\rho})_k \sim a^0(a^{-1/2})$ and $\partial_t \delta \ln(\rho/\bar{\rho})_k \sim a^{-2}(a^{-3/2})$ for $k \leq ma$ $(k \gtrsim ma)$ as seen in Fig. 5.3. In either case $\sqrt{P_{\ln\rho}P_{\partial_t \ln\rho}} \sim a^{-4}$. We will return to the underlying origin of the oscillations in the entropy in section 5.6.



Figure 5.2: Evolution of individual spectral amplitudes for the fields ϕ (*left*), Π_{ϕ} (*center*) and χ (*right*). We have rescaled the amplitudes to remove the overall damping of the modes with $k \gg ma$. Initially, only χ fluctuations with $k \lesssim 10m$ grow from linear parametric resonance. At $a \sim 10$, second order effects lead to the growth of ϕ fluctuations in the same wavenumbers. Finally, this growth saturates at the shock and there is a rapid growth of modes outside the resonant band.

In Fig. 5.4 we show how the fluctuations distribute themselves in Fourier space. During the early stages only χ fluctuations experience parametric resonance, and since $\bar{\chi}, \dot{\chi} \approx 0$ no corresponding amplification occurs in the adiabatic energy phonons. At $a \sim 10$ or $mt \sim 60$, the ϕ fluctuations begin to grow due to second-order effects resulting in the growth of phonon fluctuations at $k \leq 10m$. Accompanying this growth in fluctuation power of $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$ individually, cross-correlations also develop between the two fields as seen in Fig. 5.5. This continues until the shock-in-time, when the growth



Figure 5.3: Evolution of individual spectral amplitudes for $\ln(\rho/\bar{\rho})$, $\partial_t \ln(\rho/\bar{\rho})$ and $\Delta_{\ln\rho}^2$, rescaled to remove the overal damping of the modes with $k \gg m$. Fluctuations in these degrees of freedom are oblivious to the initial linear resonance experienced by χ . However, once the ϕ fluctuations begin to grow due to second order effects, fluctuations in the energy density at $k \leq 10m$ also begin to grow. These modes continue to grow until saturating at the shock-in-time leading to the rapid growth of modes outside the resonant band.

of fluctuations in the instability band saturates and nonlinearities lead to a rapid cascade of power to higher k-modes. This is then followed by a much slower cascade of energy with higher comoving wavenumbers becoming excited very gradually or not at all. Since the box itself is expanding, this does not necessarily lead to a development of power at smaller spatial scales as in the normal description of turbulence.

5.4.2 Statistics of the Density Phonons

We now justify our choice of $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$ as appropriate variables for our Max-Ent description by considering the one-point statistics of these fields both in real space and in Fourier space. This is an important validation of our MaxEnt procedure, since only including information about the two-point correlation function results in an inferred multivariate Gaussian probability distribution functional. If the true field distribution is highly nongaussian, then this approach will overestimate the entropy.

Let's first consider the one-point PDF for our two phonon variables in real space. For a Gaussian field, these PDFs must also be Gaussian, and thus provide a first (albeit weak) test of the field Gaussianity. In fact, given that we constrain our fields by measured two-point statistics, it is entirely reasonable to assume that we should also include measurements of one-point PDF statistics when doing MaxEnt as well. In Fig. 5.6 and Fig. 5.7 we show the one-point PDFs of $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$. Prior to the shock the fluctuations evolve linearly, with the $k \gtrsim ma$ modes dominating the overall one-point statistics. Therefore, RMS fluctuation amplitudes of $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$ decay as $a^{-1/2}$



Figure 5.4: Left: Fluctuation determinants associated with various measurements of two-point correlation functions of our phonon modes. In all cases, we have normalized the determinants to the their values at the beginning of the simulation (when $\epsilon = 1$). Shown are curves at mt = 50, 75, 87.5, 100, 112.5, 125, 137.5, 150, 200, 300, 400, with solid blue lines corresponding to measurements of only the diagonal of the covariance matrix $P_{\ln\rho,\ln\rho}P_{\partial_t \ln\rho,\partial_t \ln\rho}$ and red dashed lines corresponding to measurements of the full covariance matrix $\Delta_{\ln\rho}^2 \equiv P_{\ln\rho,\ln\rho}P_{\partial_t \ln\rho\partial_t \ln\rho} - |P_{\ln\rho\partial_t \ln\rho}|^2$. Right: Evolution of $\ln(\Delta_{\ln\rho}^2)$. At early times the distribution undergoes large oscillations. This is due to the background oscillations of the scalar field condensates and the noncanonical nature of the variables being used here. We will explore this further in a later section.

and $a^{-3/2}$ respectively. This results in an overall damping of the width of the one-point PDF by the same factor of a. During the shock, the fluctuations interact nonlinearly and modes are excited in a much broader range of wavenumbers, resulting in the creation of large amplitude fluctuations and a rapid growth in the width of the one-point distributions. After the shock, these distributions then evolve very slowly for the remaining duration of our simulations.

In Fig. 5.7 we examine the shape of the PDFs in more detail by plotting them for several different time-slices. For comparison, a Gaussian fit is also included. Although we don't include it here, the distribution of $3H\left(\frac{P}{\rho}-\frac{\bar{p}}{\bar{\rho}}\right)$ is very narrow compared to that of $\partial_t \ln(\rho/\bar{\rho})$. Therefore, $\partial_t \ln(\rho/\bar{\rho}) \approx \frac{\partial_i T_0^i}{\rho}$. In the rest frame of observers comoving with the background, local changes to $\ln(\rho/\bar{\rho})$ are predominantly driven by the currents transporting energy around the medium rather than dilution from the expansion. Throughout the linear evolution prior to the shock, the distribution of $\ln(\rho/\bar{\rho})$ remains very nearly Gaussian, as it is well-approximated by the contribution linear in the field fluctuations and their derivatives. Since the (linear) field fluctuations are themselves Gaussian, $\ln(\rho/\bar{\rho})$ inherits this property. The same holds true for $\partial_t \ln(\rho/\bar{\rho})$ with one additional caveat. When $\bar{\phi}^2 \leq \langle \delta \dot{\phi}^2 \rangle$, the nonlinear terms in $\partial_t \ln(\rho/\bar{\rho})$ become as important as the linear



Figure 5.5: Modulus of the normalized cross correlation $|C_{\ln\rho\partial_t \ln\rho}| = |P_{\ln\rho\partial_t \ln\rho}|/\sqrt{P_{\ln\rho\ln\rho}P_{\partial_t \ln\rho\partial_t \ln\rho}}$ between $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$. For $mt \leq 60$, it undergoes oscillations induced by the oscillations of $\bar{\phi}$. Once second-order effects start to build fluctuations in $\delta\phi$ correlations appear at $k \leq 10$. These then spread to larger wavenumbers during the shock-in-time at $mt \sim 120$ before rapidly dissipating in the post-shock state.

terms resulting in a significantly nongaussian one-point distribution with extended tails. From (5.27) we see that if the $\bar{\phi}\nabla^2\phi$ term dominates, then $\partial_t \ln(\rho/\bar{\rho})$ will be linearly related to the fields and thus nearly Gaussian.

We can obtain more information by further decomposing $\partial_i T^{i0}$ into various components. One possibility is to simply consider each of the individual pieces separately as in Fig. 5.8. By themselves, each of the individual terms has a highly nongaussian onepoint PDF which is sharply spiked near the origin. However, at mt = 120 and mt = 122.5the more Gaussian shape associated with $\bar{\Pi}_{\phi} \nabla^2 \phi / a^5 \bar{\rho}$ is present. At mt = 110 this contribution is clearly subdominant (since $\bar{\Pi}_{\phi} \approx 0$) resulting in the extended tails in the full PDF of $\partial_i T^{i0}$ seen in Fig. 5.7. By comparing with the remaining figures in Fig. 5.7 we see that although each individual term has an extremely spiky structure with long tails, when summed to obtain $\partial_t \ln(\rho/\bar{\rho})$ they produce a distribution that is much closer to Gaussian, albeit with somewhat extended tails.

An alternative decomposition is to split $\partial_i T^{i0}/\rho$ into a piece arising from the evolution of the energy defined locally at each lattice site (the kinetic and potential energy) and the piece arising from the energy due to couplings between lattice sites (the gradient



Figure 5.6: Time-evolution of the 1-point probability density functions of $\ln(\rho/\bar{\rho})$ (*left*) and $\partial_t \ln(\rho/\bar{\rho})$ (*right*). Initially, only linear fluctuations are present due to our initial conditions approximating vacuum fluctuations. Before the shock these fluctuations evolve linearly and decay in amplitude as $a^{-1/2}$ and $a^{-3/2}$ respectively. At the shock there is a rapid growth in the typical size of the fluctuations, with the PDFs quickly settling into a nearly constant form shortly after.

energy). Denoting these two contributions $\partial_i T_{loc}^{i0}$ and $\partial_i T_{grad}^{i0}$ respectively, we have

$$\partial_i T_{loc}^{i0} = -\frac{\dot{\phi}\nabla^2 \phi + \dot{\chi}\nabla^2 \chi}{a^2} \tag{5.32}$$

$$\partial_i T^{i0}_{grad} = -\frac{\nabla \dot{\phi} \cdot \nabla \phi + \nabla \dot{\chi} \cdot \nabla \chi}{a^2} \,. \tag{5.33}$$

The results of this decomposion are shown in Fig. 5.9, where we also include $\partial_i T_{\phi}^{i0} = -\nabla(\dot{\phi}\nabla\phi)/a^2$ with an analogous definition for χ . Both pre-shock and post-shock, the distributions of the *differences* for either the ϕ/χ , or local/gradient split appear quite Gaussian, especially compared to the individual components.

Although the simplicity of the one-point distributions given above is rather remarkable, we can provide further evidence that the phonons are approximately Gaussian by looking at Fourier mode statistics. Specifically, we consider the one-point distributions of individual bands of Fourier modes for $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$. One way to quantify these distributions is to look at a few low order moments for the real and imaginary parts of the modes. In Fig. 5.10 we plot the excess kurtosis κ_4 , which we define for a homogeneous and isotropic field f(x) with Fourier transform $\tilde{f}_{\mathbf{k}}$ as

$$\kappa_4(k) \equiv \frac{2\langle Re(\tilde{f}_k)^4 + Im(\tilde{f}_k)^4 \rangle}{\langle Re(\tilde{f}_k)^2 + Im(\tilde{f}_k)^2 \rangle^2} - 3.$$
(5.34)



Figure 5.7: 1-point probability distributions of $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$ for several times corresponding to the pre-shock state (mt = 110), during the shock (mt = 120, 122.5, 125) and late-time post-shock state (mt = 150, 300). The circles are the numerically computed values of the PDF, while the solid lines are fits of Gaussians to the distribution. Before the shock, $\ln \rho$ has a nearly Gaussian distribution, while $\partial_t \ln \rho$ does provided ϕ is not too small. At mt = 110, we have $\phi \approx 0$ so that the nonlinear terms are important leading to the extended tails relative to the Gaussian.



Figure 5.8: Split of $\partial_i T^{i0}/\rho$ into components based on individual terms appearing in the expansion. At all the times illustrated, the distributions are significantly nongaussian, with long tails and a very peak structure. However, at mt = 120 and 122.5 the contribution of the linear (and nearly Gaussian) $\bar{\Pi}_{\phi} \nabla^2 \phi/a^2 \bar{\rho}$ term is visible in the PDF of $\dot{\phi} \nabla^2 \phi/a^2 \rho$.



Figure 5.9: 1-point PDFs of the currents associated with each field ϕ, χ and with the local $\partial_i T_{loc}^{i0}$ and gradient $\partial_i T_{grad}^{i0}$ energies (see (5.33) for a definition).

For a Gaussian distribution $\kappa_4 = 0$, so this provides a measure of the nongaussianity of the Fourier modes, as well as delivering information about localization in scale. At small wavenumbers, we have fewer modes to sample so there is a correspondingly larger uncertainty in the estimate of κ_4 . Aside from this scatter at low-k, the excess kurtosis remains small at all wavenumbers throughout the pre-shock evolution, with a small nongaussianity developing at values $k \sim 300m$ near the Nyquist. However, at the shock we see the rapid development of large nongaussianities at the scales associated with the linear instabilities due to the onset of strong nonlinearities induced by the buildup of fluctuations from parametric resonance. This nongaussianity then spreads to larger wavenumbers as nonlinearities excite the higher k-modes, before dissipating in the post-shock state.

We can further probe the nongaussianity of the modes by considering the probability density function for the real and imaginary parts of the Fourier modes. These are shown in Fig. 5.11 and Fig. 5.12, where we plot the empirical PDF's for $\widetilde{\ln \rho_k} / \sqrt{\langle | \widehat{\ln \rho_k} |^2 \rangle}$ and $\partial_t \widetilde{\ln \rho_k} / \sqrt{\langle |\partial_t \widetilde{\ln \rho_k}|^2 \rangle}$. The PDFs are obtained by first estimating the power spectrum $\langle |\tilde{f}_k|^2 \rangle$ in bins of width $\Delta k \equiv 2\pi L^{-1}$. We then normalize each individual Fourier mode by prewhitening $\tilde{f}_k/\sqrt{\langle |\tilde{f}_k|^2 \rangle}$ and compute the resulting PDF in bins of width $k_{nyq}/5$. We do not include the PDF for modes with wavenumbers near the Nyquist frequency $k_{nyq} = \pi N_{lat}^{1/3} L^{-1}$, since these modes are sensitive to the effects of the lattice cutoff. As with the kurtosis we combine the real and imaginary parts of the Fourier modes to create a single PDF. From the top left and bottom right panels we see that prior to the shock and after the shock, the distributions are very well approximated by a Gaussian. Meanwhile, as we move through the shock, a large nongaussianity develops first in the low-k modes then spreading to higher momenta as seen in the mt = 120, 122.5 and 125 panels respectively. Finally, shortly after the shock (at mt = 150) a small residual nongaussianity remains in the third bin, in agreement with Fig. 5.10. By mt = 300this has dissipated and there is no indication of deviations from Gaussianity in the sub-Nyquist Fourier modes.

Through a suite of measurements taken from lattice simulations, we have demonstrated the post-shock statistics of $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$ are remarkably simple (ie. Gaussian) provided we restrict ourselves to one-point distributions of individual Fourier modes. Of course, the full-field statistics are probably quite nonGaussian with the full phase space distribution wound into an extremely complicated pattern. However, this information is stored in the higher n-point correlators and is inaccessible when making coarse-grained measurements on the system. From the point of view of an observer with restricted access to only two-point correlations and one-point PDFs, the fields are thus effectively multivariate Gaussians. There still remains a question of the joint statistics that we have not addressed, as well as issues of correlations between Fourier modes with different wavenumbers. However given the inhomogeneous and complex nature of the post-shock state, it is rather remarkable that such a simple description can be found even at the level of one-point statistics.



Figure 5.10: Excess kurtosis κ_4 (defined in (5.34)) for $\ln \rho_k$ and $\partial_t \ln \rho_k$ as a function of comoving wavenumber k/m for several times mt before, during and after the shock. The small sample sizes at small k lead to a large scatter in the measured value. During the shock, a large postive excess kurtosis develops in the wavenumbers resonantly excited by the oscillating background. The nongaussianity in the Fourier modes then rapidly propagates to larger wavenumbers as higher k-modes are excited by nonlinear interactions. Shortly after the shock, the kurtosis returns to 0, which is the expected value for a Gaussian distribution, providing further evidence for the Gaussianity of the phonon modes after the shock.



Figure 5.11: PDFs of Fourier components $\widetilde{\ln \rho_k}/\langle |\widetilde{\ln \rho_k}|^2 \rangle$ in various bins for the same times as those in Fig. 5.10. We have combined the distributions for the real and imaginary parts of the Fourier modes (which each have variance $\sigma_k^2 = \langle |\widetilde{\ln \rho_k}|^2 \rangle/2$).



Figure 5.12: PDFs of Fourier components of $\partial_t \widetilde{\ln \rho_k} / \sqrt{\langle |\partial_t \widetilde{\ln \rho_k}|^2 \rangle}$ in various bins. The notation and analysis procedure are the same as Fig. 5.11.

5.5 Application to Preheating: the Field Description

5.5.1 Entropy in the Field Description Constrained by a Measured Two-Point Correlation Function

In the previous section, we studied the Gaussian Shannon entropy of a collection of scalar fields during preheating assuming a measured two-point correlation function. We argued that the energy density phonons $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$ provide a set of collective excitations whose statistics are well-approximated as Gaussian. A more conventional approach would have been to instead treat the fundamental fields (ϕ, χ) and their canonical momenta $(\Pi_{\phi}, \Pi_{\chi}) = (a^3 \dot{\phi}, a^3 \dot{\chi})$ as the variables being measured. Indeed, since the field variables are a set of canonical coordinates while our energy phonon modes are not, we may expect that the fields (along with any other set of canonical variables) to hold a privileged position with respect to entropy. However, we will demonstrate shortly that the field variables (in particular ϕ) display visible nongaussianity after the shock. Therefore, the canonical nature of the field variables is somewhat offset by the nearly Gaussian nature of the phonon variables, making $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$ more natural with respect to our MaxEnt procedure. We address these issues in section 5.6 where we derive the connection between the two choices and show the special role taken by canonical variables. For now, we demonstrate that the shock-in-time also occurs in the field variables, although the strength and duration vary in detail compared to the phonon description.

Since we have two field variables and two momenta, there are a total of ten two-point correlation functions we can measure. Our inferred entropy will then depend on the exact combination we assume we have measured. Once again, define $P_{\alpha\beta}(k) \equiv \langle \tilde{q}^{\alpha}_{\mathbf{k}} \tilde{q}^{\beta}_{-\mathbf{k}} \rangle$, where q^{α} represents any one of the fields or their canonical momenta and we again take the unitary normalization for the Fourier transform. This is simply the Fourier transform of the full covariance matrix for the system, and at the level of two-point correlations gives full information about the system. As well, we define the (normalized) cross-correlation as $C_{\alpha\beta} \equiv P_{\alpha\beta}/\sqrt{P_{\alpha\alpha}P_{\beta\beta}}$. We can now consider several different entropies, each defined by the components of $P_{\alpha\beta}$ that we assume we can access. In particular, we will consider the following five entropies

$$S_{n_{\phi}} \equiv \frac{4\pi\Delta k}{2} \sum_{k} k^2 \ln(P_{\phi\phi}, P_{\Pi_{\phi}\Pi_{\phi}})$$
(5.35)

$$S_{n_{\chi}} \equiv \frac{4\pi\Delta k}{2} \sum_{k} k^2 \ln(P_{\chi\chi} P_{\Pi_{\chi}\Pi_{\chi}})$$
(5.36)

$$S_{\phi} \equiv \frac{4\pi\Delta k}{2} \sum_{k} k^2 \ln \det \langle (\phi, \Pi_{\phi})^{\dagger} (\phi, \Pi_{\phi}) \rangle \equiv \frac{4\pi\Delta k}{2} \sum_{k} k^2 \ln(\Delta_{\phi}^2)$$
(5.37)

$$S_{\chi} \equiv \frac{4\pi\Delta k}{2} \sum_{k} k^2 \ln \det \langle (\chi, \Pi_{\chi})^{\dagger} (\chi, \Pi_{\chi}) \rangle \equiv \frac{4\pi\Delta k}{2} \sum_{k} k^2 \ln(\Delta_{\chi}^2)$$
(5.38)

$$S_{tot} \equiv \frac{4\pi\Delta k}{2} \sum_{k} k^2 \ln \det \langle (\phi, \Pi_{\phi}, \chi, \Pi_{\chi})^{\dagger} (\phi, \Pi_{\phi}, \chi, \Pi_{\chi}) \rangle \equiv \frac{4\pi\Delta k}{2} \sum_{k} k^2 \ln(\Delta_{tot}^2) \quad (5.39)$$

where we have dropped the $N_{fld} \frac{N_{lat}}{2} (\ln 2\pi + 1)$ constant piece for notational simplic-

ity. $P_{\phi\phi}P_{\Pi_{\phi}\Pi_{\phi}}$ corresponds to measurements of the spectrum of ϕ and its canonical momentum Π_{ϕ} with no additional cross-correlation information, and similarly for χ . $\Delta_{\phi}^2 = P_{\phi\phi}P_{\Pi_{\phi}\Pi_{\phi}} - |P_{\phi\Pi_{\phi}}|^2$ includes information on the correlations between the field and its canonical momentum. Finally Δ_{tot}^2 is the result assuming we have measured all 10 two-point correlation functions.

The determinants appearing in (5.35) and (5.36) are closely related to the notion of particle number that usually appears in the preheating literature, where the ϕ particle occupation number is defined as

$$n_{k}^{\phi} + \frac{1}{2} = \frac{1}{2\omega_{k}^{\phi}} \left(\langle |\dot{\phi}_{k}|^{2} \rangle + (\omega_{k}^{\phi})^{2} \langle |\phi_{k}|^{2} \rangle \right)$$
(5.40)

and a similar definition holding for the number of χ particles. The effective frequency is typically defined as $(\omega_k^{\phi})^2 = \frac{k^2}{a^2} + \langle V_{\phi\phi} \rangle$. More generally, this definition should allow for mixing between the various fields and consider the (time-dependent) eigenvectors of $k^2/a^2 + \langle V_{\phi_i,\phi_j} \rangle$ with the resulting effective frequencies defined by the corresponding eigenvalues. To understand the relationship to our expression, note that this is simply the expression for the occupation number of a simple harmonic oscillator with frequency (or equivalently energy per mode) ω_k . Therefore, in the limit that (5.40) is valid each Fourier mode can be considered a harmonic oscillator and we have $\omega_k^2 = \langle |\dot{\phi}_k|^2 \rangle / \langle |\phi_k|^2 \rangle$. When this holds we have $n_k^{\phi_i} + \frac{1}{2} = a^{-3} \sqrt{P_{\phi_i \phi_i} P_{\Pi_{\phi_i} \Pi_{\phi_i}}}$. We see that the factor a^{-3} has appeared to produce the physical (rather than comoving) particle density. Our definition is somewhat more general as no explicit reference is made to the effective frequency ω_k . Δ_{ϕ}^2 and Δ_{χ}^2 provide further generalization by including the correlations between the fields and their momenta. Thus these two quantities can properly account for the squeezing that occurs as the modes are excited by parametric resonance, which (5.40) is blind to. Furthermore, Δ_{ϕ} and Δ_{χ} are invariant under canonical transformations between only (ϕ, Π_{ϕ}) or (χ, Π_{χ}) respectively. Finally, Δ_{tot}^2 accounts for all correlations in the system and is invariant under arbitrary canonical transformations of the fields.

In Fig. 5.13 and Fig. 5.14 we show the fluctuation determinants associated with each of these choices of measurements. We also show the evolution of the magnitude of each of the cross-correlations in Fig. 5.15. During the linear evolution, fluctuations in χ experience broad band parametric resonance, leading to the production of fluctuations in the χ determinants with no corresponding growth of the ϕ determinants. There is a significant correlation between χ and Π_{χ} during this stage and as a result $P_{\chi\chi}P_{\Pi_{\chi}\Pi_{\chi}} > \Delta_{\chi}^2$ in the resonant band. This is expected since no entropy is produced for a quadratic field theory with an external time-dependent mass due to a complete cancellation be-

tween the growing amplitudes and cross-spectra. However, in our simulations there are (small) nonlinearities, numerical noise, and uncertainties associated with estimation of the covariance matrix. The combination of these leads to a partial (rather than full) cancellation during the early nearly linear stages of evolution. As we move towards the shock, second-order effects lead to the growth of ϕ fluctuations, again with significant cross-correlations between the various fields. Finally, at the shock nonlinearities lead to the rapid growth of fluctuations in an extended region of momentum space along with additional cross-correlations. This is followed by a much slower cascade of fluctuations to larger (comoving) wavenumbers.

Now consider the effect of this dynamical evolution on entropy production as illustrated in Fig. 5.16. As with the case for the energy phonons, we cutoff the sum at some wavenumber k_{cut} less than the Nyquist frequency $k_{nyq} = \pi N_{lat}^{1/3} L^{-1}$ and define the effective number of degrees of freedom as $\mathcal{N}_{eff} \equiv 4\pi\Delta k \sum_{k=0}^{k_{cut}} k_i^2 \approx 4\pi k_{cut}^3/3$, where we have sampled our spectra at the discrete frequencies k_i spaced at intervals $\Delta k = 2\pi L^{-1}$. In all cases we consider, the last approximate equality holds to better than the half-percent level. This definition of \mathcal{N}_{eff} does not include the number of fields N_{fld} used to compute the entropy. This factor is $N_{fld} = 2$ for $S_{n_{\phi}}, S_{n_{\chi}}, S_{\phi}, S_{\chi}$ and $N_{fld} = 4$ for S_{tot} . Note that N_{fld} is closely related to the effective number of relativistic species g_* familiar from thermal field theory, with $N_{fld} = 2g_*$ for the scalar fields considered here. As with the energy phonon modes, there is once again a sharp increase of the entropy at $mt \sim 120$, indicating that the shock is robust to our choice of variables. A nice feature of this set of variables is that prior to the shock there are no visible oscillations in the entropy and no overall damping as a multiple of $\ln a$. However, the entropy does not saturate as quickly post-shock and there is nonnegligible production right up to the end of our simulation. This is further illustrated in the right panel of Fig. 5.16 where a long tail in the entropy production rate dS/dt is visible well past the shock. As we will see shortly, the fields remain nongaussian after the shock, so it is unclear if this increase in entropy could be accounted for by including additional information about the distribution of Fourier modes in our definition of the entropy.

To assess the impact of correlations between the various fields, it is useful to define

$$\Delta S_{\alpha\beta} = \frac{4\pi\Delta k}{2} \sum_{k} k_i^2 \ln\left(1 - |C_{\alpha\beta}|^2\right) = \frac{4\pi\Delta k}{2} \sum_{k} k_i^2 \ln\left(1 - \frac{|P_{\alpha\beta}|^2}{P_{\alpha\alpha}P_{\beta\beta}}\right) \tag{5.41}$$

which measures the change in entropy if we assume that we have measured the diagonal of the full covariance matrix (ie. all of the power spectra) and the one additional crossspectra $P_{\alpha\beta}$. The evolution of these six quantities appear in the right panel of Fig. 5.17,



Figure 5.13: Determinants associated with measurements of various two-point correlation functions of the fundamental field and their canonical momenta. The definitions of the various quantities are given in the main text. In order to emphasize the changes induced by the evolution of the system, we have normalized each determinant to its value at the start of the simulation.

with the corresponding cross-correlations in Fig. 5.15. Prior to the shock, only correlations between the fields and their own canonical momenta appear. This is due to the squeezing nature of the parametric resonance process, which leads to the production of standing wave-like patterns during the linear regime rather than an incoherent super-



Figure 5.14: Contribution to entropy per k-bin $\Delta kk^2 \ln \Delta^2/\Delta^2(t=0)$ for Δ_{ϕ}^2 (top left), Δ_{χ}^2 (top right) and Δ_{tot}^2 (bottom). The oscillations present in the phonon degrees of freedom are absent. However, once again the rapid production of entropy distributed through a range of wavenumbers is evident at the shock-in-time.



Figure 5.15: Cross-correlations between the various fundamental fields (ϕ, χ) and their corresponding canonical momenta (Π_{ϕ}, Π_{χ}) for several times through the development of the shock. The cross-corelations are defined as $|C_{\alpha\beta}(k)| \equiv |P_{\alpha\beta}(k)|/\sqrt{P_{\alpha\alpha}P_{\beta\beta}}$.

position of travelling waves. During the shock, correlations develop between all of the variables. After the shock these then damp away except for the correlations between each field and its own canonical momenta.



Figure 5.16: Left: Evolution of the entropy per effective degree of freedom for S_{ϕ}, S_{χ} and S_{tot} (see main text for definitions). The solid, dashed, dot-dashed and dotted lines correspond to the choice of cutoff wavenumber $\frac{k_{cut}}{m} = 320, 240, 160, 80$ respectively. Right: Time derivative for each of the five entropies definied in (5.35)- (5.39). The shock-in-time is still a prominent feature in all five quantities, although the subsequent relaxation back to zero entropy production is much slower than for the energy phonon description.

5.5.2 Statistics of the Field Variables

Now consider the validity of the assumption of Gaussian statistics for the fundamental scalar fields, and by extension the accuracy of our MaxEnt prescription in determining the actual entropy of the fields. Previously, we demonstrated that the one-point Fourier statistics of $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$ are remarkably Gaussian at all times except for a short interval around the shock-in-time. We repeat the analysis of section 5.4.2 using $(\phi, \chi, \Pi_{\phi}, \Pi_{\chi})$ as our collection of fields instead of $(\ln(\rho/\bar{\rho}), \partial_t \ln(\rho/\bar{\rho}))$. For definitions of the relevant quantities as well as the procedure used to obtain them, please see (5.34) and the subsequent text.

As with the energy phonons, when the large inhomogeneities and nonlinear interactions between the fluctuations develop at the shock, there is a corresponding broadening of the PDFs in each of the field variables and their momenta as seen in Fig. 5.18. While the system passes through the shock-in-time, the one-point PDFs are multimodal and highly nonGaussian, just as with the $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$. However, the post-shock one-point distributions of ϕ and $\dot{\phi}$ remain nongaussian even long after the shock as seen in Fig. 5.19. As well, the χ one-point distribution acquires extended nonGaussian tails. Therefore, already at the level of spatial one-point distributions it is clear that the field variables are less suitable for our MaxEnt prescription than the energy phonons.

As with the phonons, we again consider the Gaussianity of individual Fourier modes of the fields through both the excess kurtosis in Fig. 5.20 and binned one-point PDFs



Figure 5.17: *Left:* Evolution of entropy differences illustrating the importance of various cross-correlations to the total entropy. *Right:* Contribution to the entropy associated with individual measurements of cross-correlations between various phase-space variables as defined in (5.41).

in Fig. 5.21 and Fig. 5.22. Exactly as for the phonons, the nonlinear interactions between the resonantly excited modes lead to a large buildup of kurtosis at $k \lesssim 50m$ at the beginning of the shock. In the individual Fourier amplitude PDFs, this manifests as a peaking of the distribution relative to a Gaussian. As the cascade proceeds (both during and after the shock), the nonGaussianity (as measured by the kurtosis) moves to larger wavenumbers. However, unlike the phonon fields, the kurtosis of ϕ does not completely dissipate after the shock, but instead a nonGaussian component persists for wavenumbers $k \sim 150m$ well after the shock. The χ field does not maintain such a localized (in scale) set of nonGaussian modes, but a slight excess kurtosis remains for the larger k-modes near the Nyquist. This is (at least partially) due to the finite grid spacing. Thus, unlike the one-point PDF, there are no obvious signs of nonGaussianity in the χ modes when considering the distribution of Fourier modes. From our analysis, it is thus unclear whether the nonGaussianity of the spatial 1-point PDF of χ arises from mode couplings in Fourier space or from the nonGaussianity of poorly resolved superNyquist modes. We intend to return to this question in the future by considering the evolution of the 1-point field PDF as we apply various smoothing kernels to the field to remove the modes near the Nyquist.

From the results of this subsection, it is clear that the fundamental field variables remain significantly nonGaussian after the shock. This is in contrast to the phonon modes (in particular $\ln(\rho/\bar{\rho})$), whose one-point statistics are remarkably Gaussian shortly after the shock-in-time has occurred. Thus, our MaxEnt prescription is less well motivated when considering the field variables compared to the phonon variables, although the



Figure 5.18: Evolution of the one-point probability density functions for each of the fields and their time-derivatives over the course of the simulation. In order to counter the damping due to the expansion, we plot $a^{3/2}\delta\phi$ and $a^{3/2}\delta\dot{\phi}$, with a similar scaling applied to χ .

qualitative features of the entropy are insensitive to the particular choice.

5.6 Maximum Entropy for Noncanonical Variables

Thus far we have presented two entropies based on different choices for collective variables that we have measured for a collection of scalar fields undergoing a resonant preheating instability. In both cases, there is a short well-defined period of rapid entropy production – the shock-in-time – connecting a regime of approximately linear fluctuation evolution with a regime of complex nonlinear evolution of the fluctuations. Although the qualitative behaviour is the same in both cases, they differ in quantitative details. This is not unexpected, since after the shock $(\delta \phi_i, \delta \Pi_i)$ and $(\delta \ln(\rho), \delta \partial_t \ln(\rho))$ are nonlinearly related to each other so that knowledge of the two-point correlations of one set of variables is inequivalent to knowledge of the two-point correlators of the other set of variables. Since this was the basic assumption built into our construction of the entropy, in the post-shock



Figure 5.19: Normalized PDFs of the fields and their time-derivatives at various times during the evolution. The markers and dashed style lines are empirical measurements taken from simulations. The solid lines are Gaussian fits to the data for $a^{3/2}\phi$ and $a^{3/2}\chi$.

state we expect quantitative differences between the two definitions. Furthermore, the phonon variables do not constitute a complete description of the system (except for the case of a single-field system), and additional information can be stored in appropriate energy differences which we did not account for.

However, during the preshock evolution the fields are well described by a set of homogeneous field condensates interacting with a collection of *linear* fluctuations. Similarly,



Figure 5.20: Excess kurtosis κ_4 of the field variables ϕ and χ .



Figure 5.21: Probability density function of Fourier moments for the field ϕ normalized to the power spectrum (ie. RMS fluctuation amplitude).

the fluctuations in the energy density are also linear to a good approximation. Therefore, there is a linear transformation between the two variable sets (assuming we also include $N_{fld} - 1$ energy differences and their time derivatives) and one might therefore expect that knowledge of the correlators in either coordinate system should be equivalent. However, even at early times, we see that the two definitions used above are inequivalent. In particular, the entropy in the phonon variables undergoes oscillations while also ex-



Figure 5.22: Probability density function of Fourier moments for the field χ normalized to the power spectrum (ie. RMS fluctuation amplitude).

periencing an overall damping. In contrast, prior to the shock the entropy in the field variables is very nearly constant. Since the preshock dynamics of the fluctuations is linear, this property of the entropy in the field variable description is desirable. We will now reconcile this apparent contradiction, which will also shed further light on the special role that is played by the fundamental fields ϕ_i .

As we alluded to earlier, the origin of this discrepancy is that the field variables and their canonical momenta constitute a set of canonical coordinates, while the $(\ln(\rho), \partial_t \ln(\rho))$ variables do not. Let φ^A denote a collection of (possibly noncanonical) fields that we are using to describe our system, with A labelling the particular field. In the lattice case, the values of the fields at each lattice site φ_i^A coordinatize the phase-space of the system, so we will refer to them as coordinates. For notational simplicity, we suppress the index A in the following discussion and define the functional measure $\mathcal{D}\varphi \equiv \prod_{i,A} d\varphi_i^A$ throughout. The entropy functional introduced earlier was based on averaging defined as $\langle F(\varphi) \rangle_c = \int \mathcal{D}\varphi P[\varphi] F(\varphi)$ so $S = -\int \mathcal{D}\varphi P \ln P = \langle -\ln P \rangle_c$. In the remaining discussion we refer to this as *canonical* averaging. As is well known, this entropy is not invariant under coordinate changes $\varphi \to \tilde{\varphi}$ since the probability density in the new coordinates acquires a factor of the Jacobian $\tilde{P}[\tilde{\varphi}] \left| \frac{\partial \tilde{\varphi}}{\partial \varphi} \right| = P[\varphi]$. We propose to instead compute entropy using a *noncanonical* definition of ensemble averaging

$$\langle F(\varphi) \rangle_{nc} \equiv \int \mathcal{D}\varphi \sqrt{\mathcal{G}} Q[\varphi] F(\varphi) \qquad \mathcal{J} \equiv \mathcal{G}^2 \equiv \left| \frac{\partial \varphi_c}{\partial \varphi} \right|$$
(5.42)

with φ_c some collection of *canonical* field coordinates. Whenever φ are a canonical set of fields, $\mathcal{J} = 1$ and this definition reduces to our previous one.¹ Generally, we have $\mathcal{J} = \sqrt{\mathcal{G}}$ where \mathcal{G} is the determinant of a metric on field space. The invariant functional measure $\mathcal{D}_{nc}\varphi = \sqrt{\mathcal{G}}\Pi_i\varphi_i$ then takes the same form as the invariant measure familiar from general relativity $\sqrt{|g|}d^dx$. We thus define a noncanonical entropy functional

$$S_{nc} \equiv -\langle \ln Q[\varphi] \rangle_{nc} = -\int \mathcal{D}\varphi \sqrt{\mathcal{J}} Q \ln Q \,. \tag{5.43}$$

With ensemble averaging defined via (5.42), the PDFs $Q[\varphi]$ are invariant under coordinate changes, and thus so is the entropy. Equivalently, we can use the canonical averaging procedure (where the PDFs do transform)

$$S_{nc} = -\langle \ln(P/\mathcal{J}) \rangle_c = -\int \mathcal{D}\varphi P[\varphi] \ln\left(\frac{P[\varphi]}{\mathcal{J}}\right) \,. \tag{5.44}$$

The transformation of the PDF is now absorbed by the Jacobian, so in either case S_{nc} is invariant under arbitrary changes of the variables φ , and thus is a suitable generalization of the differential entropy to the noncanonical case. We can move between the two definitions (5.43) and (5.44) through the identification $P[\varphi] = \mathcal{J}Q[\varphi]$.

The alert read will undoubtedly notice that this last definition of the entropy is very similar to the (negative of) the Kullback-Leibler (KL) divergence [179], if we were to replace \mathcal{J} with a reference probability distribution. Indeed, we are using \mathcal{J} to absorb the transformation properties of the PDF in exactly the same manner as the reference distribution absorbs the transformation in the KL divergence. However, we do not require that \mathcal{J} be properly normalized so that the usual theorem about the positivity of the KL divergence (which would imply $S_{nc} < 0$) does not apply.

As an alternative derivation of the noncanonical entropy consider the relation of the (continuous) differential Shannon entropy to the discrete version $S_{discrete} = -\sum_i p_i \ln(p_i)$, where p_i are the probabilities for a discrete set of outcomes labellel by i. For simplicity, we will consider only a single variable, which we denote x, with probability *density* $\mu(x)$. The generalization to the case of many variables and discretized fields is straightforward. From the probability density, form a discrete set of probabilities by partitioning x into a collection of subintervals Δx_i centered on x_i . We then associate a probability $P_i \equiv \mu(x)\Delta x_i$ with each of these intervals, resulting in the Shannon entropy $S_{discrete} = -\sum_i P_i \log P_i = -\sum_i \mu(x_i)\Delta x_i \log(\mu(x_i)) - \sum_i \mu(x_i)\Delta x_i \log(\Delta x_i)$. Taking

¹Equivalently, we could absorb \mathcal{J} into a definition of the noncanonical functional measure.

the length of the intervals $\Delta x_i \to 0$ we obtain

$$S_{discrete} = -\int dx \mu(x) \log(\mu(x)) - \lim_{\Delta x_i \to 0} \sum_i P_i \log(\Delta x_i).$$
 (5.45)

The final term is an infinite constant dependent on the choice of discretization of the interval that must be subtracted to obtain the differential entropy. Now instead suppose we choose a new variable y = y(x) with corresponding probability density $\nu(y)$. Let's once again slice the interval up into segments Δy_i and demand that $\Delta y_i\nu(y_i) = \Delta x_i\mu(x_i)$ and $y_i = y(x_i)$. $S_{discrete}$ is the same in both cases and we have $S[x] - S[y] = -\lim_{\Delta x_i \to 0} \sum_i P_i \log(\Delta x_i/\Delta y_i) \to -\langle \log(|\partial x/\partial y|) \rangle^2$ Letting x be our canonical variable, this gives precisely the additional term in (5.44).

Given a set of constraints $\langle \mathcal{O}_i(z) \rangle_{nc} = \alpha_i$,³ one can easily show that the maximum entropy distribution (if it exists) is

$$Q_{nc}^{MaxEnt}[\varphi] = \frac{e^{-\sum_{i}\lambda_{i}\mathcal{O}_{i}(\varphi)}}{Z_{nc}}.$$
(5.46)

where we have explicitly solved for the Lagrange multiplier $\lambda_{norm} = \ln Z_{nc}$ associated with the overall normalization of the probability

$$Z_{nc} = \int \mathcal{D}\varphi \mathcal{J} e^{-\sum_i \lambda_i \mathcal{O}_i(\varphi)} \,. \tag{5.47}$$

The remaining Lagrange multipliers λ_i are determined by the constraints

$$\langle \mathcal{O}_i \rangle_{nc} = \frac{1}{Z_{nc}} \int \mathcal{D}\varphi \mathcal{J} e^{-\sum_i \lambda_i \mathcal{O}_i} \mathcal{O}_i = \alpha_i$$
 (5.48)

with the corresponding constrained entropy

$$S_{nc} = \ln Z_{nc} + \sum_{i} \lambda_i \alpha_i = \ln Z_{nc} - \sum_{i} \frac{\partial \ln Z_{nc}}{\partial \ln \lambda_i}$$
(5.49)

For the special case of a constant \mathcal{J} and a measured covariance matric C, we find

$$S_G^{nc} = \frac{1}{2} \ln \left(\frac{\det C}{\mathcal{A}_{eff}^2} \right) + \frac{N}{2} \ln 2\pi + \frac{1}{2} Tr \mathbb{I} \qquad \mathcal{A}_{eff} \equiv \mathcal{J}^{-1} \,. \tag{5.50}$$

²This of course assumes that $\nu(y)$ is absolutely continuous with respect to $\mu(x)$ so that no singularities appear in the limit.

³All of the following results can equivalently be obtained using the canonical ensemble averaging and (5.44).

The final result (5.50) has a very simple interpretation. $\sqrt{\det(C)}$ is a measure of the volume in phase space occupied by the fluctuations in the variables φ , while \mathcal{A}_{eff} is a measure of the phase space volume in the transformed variables φ occupied by a single unit of phase space volume in the original canonical variables φ_C . Consideration of the von Neumann entropy (see section 5.6.1) dictates that the canonical variables are the correct choice in which to partition phase space. Therefore, the state-dependent contribution to the entropy is simply $\ln(n_{PV})$ where n_{PV} is a measure of the number of fundamental units of phase volume occupied by the fluctuations.

Before continuing, we also note the amusing fact that (in the semiclassical limit) the Jacobian determinant is also stored in the two-point correlation function. To see this, consider a collection of (possibly noncanonical) observables $\hat{z}_{\alpha}(\hat{q})$ labelled by α , where \hat{q} represent a set of canonical observables for the system. For the case of latticized fields, $\alpha = (S, i)$ where S denotes the particular field and *i* the lattice site. The complete set of two-point correlation information is then stored in

$$W_{\alpha\beta} = \langle \hat{z}_{\alpha} \hat{z}_{\beta} \rangle \tag{5.51}$$

where $\langle \cdot \rangle$ denotes a quantum expectation value. W naturally splits into a symmetric piece and an antisymmetric piece

$$W_{\alpha\beta} = \frac{1}{2} \langle \{\hat{z}_{\alpha}, \hat{z}_{\beta}\} \rangle + \frac{1}{2} \langle [\hat{z}_{\alpha}, \hat{z}_{\beta}] \rangle$$
(5.52)

with $\{\cdot, \cdot\}$ the anticommutator and $[\cdot, \cdot]$ the commutator. The second term is the "quantum" part of the two-point function that provides information on the discretization of phase space. In the semiclassical limit this becomes clear since $[\cdot, \cdot] \rightarrow i\hbar[\cdot, \cdot]_{PB} + \mathcal{O}(\hbar^2)$ with $[F, G]_{PB} = \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x_i}$ the canonical Poisson bracket. Arranging our canonical variables $q^T = (\vec{x}, \vec{p})$ so that the position coordinates appear first and the momentum coordinates second,

$$[z_i, z_j]_{PB} = \frac{\partial z_i}{\partial q_m} J_{mn} \frac{\partial z_j}{\partial q_n}$$
(5.53)

with $J = \begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{bmatrix}$. From this we see that $|\det[\hat{z}_{\alpha}, \hat{z}_{\beta}]| = \hbar^2 \left|\frac{\partial z}{\partial q}\right|^2$ is simply the square of the Jacobian determinant for the transformation from the canonical variables q to our new coordinates z. This suggests a possible formulation in which the Jacobian factor can be obtained by maximizing entropy with repect to the full quantum two-point information, although we don't pursue this here.

5.6.1 Von Neumann Entropy of a Gaussian Theory

In this subsection we explicitly consider the connection between our (classical) Shannon entropy and the corresponding (quantum) von Neumann entropy. This will establish canonical variables as the fundamental description in which no additional Jacobian is needed. It will also explicitly demonstrate the origin of the $\ln 2\pi$ contribution to the entropy as a fundamental phase space volume.

There are many ways to establish the connection between the classical and quantum entropies. We will proceed by interpreting the Gaussian probability density functional for the fields $P[\phi_i, \Pi_i]$ as the Wigner function corresponding to some density matrix $\hat{\rho}$. In order for our lattice simulations to accurately approximate the full quantum dynamics, the fluctuations $\delta\phi$ and $\delta\Pi$ must initially be weakly coupled. Therefore, our initial density matrix is Gaussian to a very good approximation.

For simplicity, consider the case of a single pair of canonical field variables (x, p) with Wigner function

$$W(x,p) \propto \exp\left(-\frac{1}{2}\left(ax^2 + bp^2 + 2cxp\right)\right)$$
(5.54)

The corresponding density matrix (in the position basis) is

$$\langle x|\hat{\rho}|x'\rangle = \rho(x,x') = \int dp e^{ip(x-x')} W\left(\frac{x+x'}{2},p\right)$$

$$= \sqrt{\frac{ab-c^2}{2\pi b}} \exp\left(\frac{-(ab-c^2)}{2b}\left(\frac{x+x'}{2}\right)^2 - \frac{(x-x')^2}{2b} - i\frac{c(x^2-x'^2)}{2b}\right).$$

$$(5.55)$$

We also have the relations

$$\frac{ab-c^2}{b} = \frac{1}{\langle x^2 \rangle} \qquad \frac{1}{b} = \frac{\langle x^2 \rangle \langle p^2 \rangle - \frac{1}{2} \langle \{x, p\} \rangle^2}{\langle x^2 \rangle} \qquad \frac{c}{b} = -\frac{\langle \frac{1}{2} \{x, p\} \rangle}{\langle x^2 \rangle} \tag{5.57}$$

which allow us to reexpress the density matrix in terms of expectation values of combinations of the operators \hat{x} and \hat{p} . The resulting (Gaussian) von Neumann entropy is

$$S_{vN} = -\text{Tr}(\bar{\rho}\ln(\bar{\rho})) = (n_{PV} + 1)\ln(n_{PV} + 1) - n_{PV}\ln n_{PV}$$
(5.58)
$$= \left(\Delta^2 + \frac{1}{2}\right)\ln\left(\Delta^2 + \frac{1}{2}\right) - \left(\Delta^2 - \frac{1}{2}\right)\ln\left(\Delta^2 - \frac{1}{2}\right)$$

(5.56)

where we have defined

$$\Delta^2 \equiv n_{PV} + \frac{1}{2} = \langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle - \frac{1}{4} |\langle \{ \hat{x}, \hat{p} \} \rangle|^2$$
(5.59)

which is the analogue of our previously defined fluctuations determinants $\Delta_{\ln \rho}^2, \Delta_{\phi}^2, \Delta_{\chi}^2, etc.$ Taking the limit $\Delta^2 \to \infty$, we then obtain

$$S_{vN} \approx \ln \Delta^2 + 1 + \mathcal{O}(\Delta^{-2}) = S_{shannon} - \ln 2\pi.$$
(5.60)

5.7 Entropy Production in Single-Field $\lambda \phi^4$ Preheating

Now that we have the relation between entropy in different choices of field coordinates, we show that during the linear stages the field and energy phonon descriptions give the same result, provided we use the noncanonical definition of the entropy. In order to avoid unnecessary technical complications, in this section we will study a single-field preheating model with potential $V(\phi) = \lambda \phi^4/4$. A brief synopsis of the preheating instability in this model can be found in section 5.3. Choosing ϕ as our field variable, the corresponding canonical momentum is then $\Pi \equiv a^2 \partial_\tau \phi = a^3 \dot{\phi}$. There are no longer any entropy modes associated with differences between two different fields and there are the same number of energy phonon fields as fundamental scalar fields.

5.7.1 Canonical Entropy in $\lambda \phi^4$

Fig. 5.23 shows the canonical entropy in both the field and the energy phonon description. In both cases the shock is present, although it is much stronger for the phonons. Unlike the $m^2\phi^2 + g^2\phi^2\chi^2$ model we studied above, the shock now has additional structure with dS/dt possessing two peaks. However, despite this difference in the details, the shock is still very well-localized in the time for the phonons, while possessing a much longer tail for the field description. The qualitative behaviour of the one-point distributions and breaking of nongaussianity is very similar to the $m^2\phi^2 + g^2\phi^2\chi^2$ model, as will be evident from the following brief summary. Prior to the shock the flucutations evolve linearly, leading to Gaussian distributions for both the fields and the energy phonons.⁴ During the shock significant nongaussianity develops in all of the fields due to the nonlinear

⁴Once again, there is the additional caveat that in the short time intervals when $\dot{\phi}^2 \leq \langle \delta \dot{\phi}^2 \rangle$, nonlinear terms in $\partial_t \ln(\rho/\bar{\rho})$ are important and the one-point distribution becomes temporarily nongaussian.



Figure 5.23: Left: The entropy per mode for $\lambda \phi^4$ preheating using the energy phonon $(\ln(\rho/\bar{\rho}), \partial_t \ln(\rho/\bar{\rho}))$ description as well as the field description. In order to remove the short-time scale oscillations associated with evolution of the homogeneous components of the fields, we have filtered the signal with a Kaiser filter. Middle: The effective Mach number $|\ln(\rho/\bar{\rho})|^{-1}$ for $\lambda \phi^4$ preheating, again showing a rapid decline around the shock-in-time. Right: Entropy production rate per effective degree of freedom for the same entropies as the left plot. Again, a Kaiser filter has been applied to remove the high frequency oscillations. For all figures we used a box with $N = 512^3$ lattice sites with side length $\sqrt{\lambda}M_PL = 20$ when $\epsilon = 1$ at the start of the simulation. The cutoff on the Fourier modes to compute the entropy was $k_{cut} = k_{nyq} = \pi N_{lat}^{1/3}L^{-1}$.

interactions of the fluctuations. At first the nongaussian contributions are confined to $k \leq 10m$, corresponding to the modes experiencing parametric resonance. However, this quickly spreads to higher wavenumbers as a rapid cascade transfers power to smaller scales at the shock as illustrated by the excess kurtosis in Fig. 5.24 and Fig. 5.25.

5.7.2 Noncanonical Entropy in $\lambda \phi^4$

Now consider the evolution of the noncanonical entropy for the energy phonons $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$. During the early stages of the evolution we expand to linear order in the field fluctuations to obtain

$$\delta \ln \rho \approx \frac{V'(\bar{\phi})}{\bar{\rho}} \delta \phi + \frac{\bar{\Pi}}{a^6 \bar{\rho}} \delta \Pi$$
$$\delta \partial_t \ln \rho \approx \frac{3H(w-1)\bar{\Pi}}{a^6 \bar{\rho}} \delta \Pi + \left(\frac{3H(1+w)V'(\bar{\phi})}{\bar{\rho}} + \frac{\bar{\Pi}}{a^5 \bar{\rho}} \nabla^2\right) \delta \phi \,. \tag{5.61}$$

Since this is a linear transformation, the Jacobian is field independent and we can use (5.50). By Fourier transforming we can simultaneously diagonalize both the covariance matrix and the Jacobian and perform the variable change for each Fourier mode



Figure 5.24: Kurtosis κ_4 as defined in (5.34) for the energy phonons $\ln(\rho/\bar{\rho})$ and $\partial_t \ln(\rho/\bar{\rho})$ in single-field $\lambda \phi^4$ preheating.

independently. This gives the (linear) Jacobian

$$\mathcal{J}^{-1} = \Pi_{\mathbf{k}} \mathcal{J}_{\mathbf{k}}^{-1} \qquad \mathcal{J}_{\mathbf{k}}^{-1} = \left| \frac{\partial (\delta \ln \rho_k, \partial_t \delta \ln \rho_k)}{\partial (\delta \phi_k, \delta \Pi_k)} \right| = \left| \frac{\bar{\phi}^2 k^2}{a^5 \bar{\rho}^2} - 6H \frac{\bar{\phi} V'(\bar{\phi})}{a^3 \bar{\rho}^2} \right|.$$
(5.62)

In (5.62), we have assumed that we can replace $-\nabla^2 \delta \phi \rightarrow k^2 \delta \phi$ upon Fourier transforming. However, our numerical simulations use a finite-difference stencil for the Laplacian so that this relationship will be distorted for values of k too near the Nyquist frequency. Specifically, the coupling of neighbouring lattice sites via the stencil produces off-diagonal terms in the full Jacobian matrix. These then lead to trigonometric corrections to the dispersion relationship.

After the shock the fluctuations interact nonlinearly and the required transformations between the variables become significantly more complicated. Unfortunately, the resulting Jacobian is nondiagonal in both real space (due to the lattice couplings induced by the derivative operators) and in Fourier space (due to the nonlinearity). This makes computing the required determinant a rather nontrivial task, and once the Jacobian is known the MaxEnt computation itself is much more involved than in the case of linear transformations. Our primary purpose in this paper is to demonstrate the existence of the shock-in-time, which exists for both the field and energy phonon variables. Since a proper computation of the Jacobian and resulting noncanonical entropy will not change this conclusion, we content ourselves here with the much simpler task of comparing the two coordinate systems assuming that the transformation is linear throughout the evo-



Figure 5.25: Excess kurtosis κ_4 (5.34) for the field variables ϕ and $\partial_{\tau}\phi$ in single field $\lambda \phi^4$ preheating. As the shock is approached, nongaussianities develop in the modes $k \leq 10\sqrt{\lambda}M_P$ corresponding to the linear resonant instability. As the fields move through the shock, additional bands of nongaussianity appear which then spread to higher wavenumbers. At late times a nonnegligible amount of nongaussianity persists form $k \sim 60\sqrt{\lambda}M_P$.

lution. This will be an accurate approximation prior to the shock, but will presumably be quantitatively quite poor during the post-shock evolution.

In Fig. 5.26 we plot the fluctuation determinants for both the fields and the phonons rescaled by the linear Jacobian. The distortion of the dispersion relationship for k near the Nyquist frequency is visible for $k \gtrsim 30m$. However, these modes remain unexcited during the linear evolution so this simply contributes a constant offset between the entropies prior to the shock. The resulting entropies are shown in Fig. 5.27. During the stages of linear evolution of the fluctuations prior to the shock, we see that the noncanonical entropies in either description match to very high accuracy, modulo an overall constant associated with our use of a finite-difference stencil to approximate the derivatives. Meanwhile, comparing the canonical entropies leads to large oscillating variations. After the shock this equivalence breaks down as a result of our incorrect determination of the Jacobian in this regime as well as the nongaussian nature of the field variables.

Despite the remarkable agreement throughout most of the preshock evolution, there are short intervals of time when the cancellation between $S_{\ln\rho}^{nc}$ and $\ln \mathcal{A}_{eff}^2$ is less precise and blips appear between the two noncanonical entropies. This arises because the *linear* transformation (5.61) is singular for $k^2 = 6a^2 H \lambda \bar{\phi}^3 / \bar{\phi}$ (if a solution exists). Furthermore, whenever $\dot{\phi} = 0$ the transformation is singular for all wavenumbers. Around these points,



Figure 5.26: Spectral decomposition of the determinants that enter into the calculation of noncanonical entropies in the field and energy phonon descriptions. In the left panel we show slices at $\sqrt{\lambda}M_P\tau = 0, 25, 50, ..., 150$ showing the excellent agreement between the two descriptions prior to the shock. In the right panel we instead show $\sqrt{\lambda}M_P\tau =$ 162.5, 175, ..., 237.5 demonstrating the breakdown of the equivalence between the phonon and field description if we incorrectly use the linear Jacobian.



Figure 5.27: Comparison of noncanonical entropies in the field (S_{nc}^{ϕ}) and energy phonon $(S_{nc}^{\ln\rho})$ descriptions. For comparision, we also include the phonon entropy computed using the canonical prescription $S_c^{\ln\rho}$ and the correction from the Jacobian $S_{\mathcal{J}}$. By definition $S_{nc}^{\ln\rho} = S_c^{\ln\rho} + S_{\mathcal{J}}$. In the left panel we plot the preshock evolution to demonstrate the excellent agreement between the two variable choices. In the right panel we show the post-shock evolution as well, where we continue to (incorrectly) use the linear Jacobian to estimate the noncanonical corrections to the energy phonons.



Figure 5.28: A comparison of the phonon entropy determinant, the field entropy determinant and the Jacobian for three times illustrating *left*: The excellent agreement between the two descriptions when the linear transformation possesses no singularies, *center*: a blip in the agreement due to a small band of wavenumbers where the linear transformation is singular and nonlinear terms and numerical noise becomes important and *right*: a time with $\overline{\Pi} \approx 0$ so that the entire linear transformation is nearly singular and nonlinear effects are important in a wide range of wavenumbers.

additional nonlinear terms in the transformation must be taken into account. As well, in the former case the Jacobian changes sign at the singular point. When numerically estimating power spectral densities, we must bin wavenumbers into various bands leading to a fuzziness in wavenumber space. This means that we cannot precisely resolve the cross-over point leading additional errors. These phenomena are illustrated in Fig. 5.28, where we give an example where the linear transformation has no singularities, an isolated singularity at a single wavenumber k and a near singularity for a range of wavenumbers.

5.8 Modulated Preheating from the Shock-in-Time

We've demonstrated the existence of fairly sharply defined hypersurface (the shock-intime) on which the universe transitions from a state of high spatial coherence and low entropy to an incoherent state with high entropy. Prior to the shock, the evolution of the universe is well described by a scale factor coupled to a collection of *homogeneous* fields oscillating in a potential. Post shock, the expansion is instead driven by a highly nonlinear medium whose collective variables seem to be $\ln(\rho)$ and its time derivative. Except for special choices of the field Lagrangian, the (time-averaged) equation of state $w = \bar{P}/\bar{\rho}$ will be different before and after the shock. In fact, for most examples of preheating that have be considered in the literature, w(t) is a time-dependent quantity after the shock [190]. For our two-field preheating model (5.12) this can be seen in the bottom panel of Fig. 5.1. If w = const we have $e^{3(1+w)\ln a}\rho = const$, while we clearly see that the logarithm of this quantity is evolving. Therefore, given a physical mechanism to modulate the time that the shock occurs between Hubble-sized patches at the end of inflation, we can create models in which different regions of the observable universe underwent a different expansion history.⁵ This allows for the generation of adiabatic density perturbations, which we call modulated preheating (for an example of another mechanism by which preheating can generate adiabatic perturbations see [55, 191, 57, 58]).

A specific example of such modulation occurs when the coupling between the inflaton and preheat fields is itself a function of some other light isocurvature mode σ . For (5.12) we would have $g^2 \rightarrow g^2(\sigma)$. This is similar in spirit to the usual mechanism of modulated reheating [192, 193, 194, 195, 196]. In these studies, the decay rate is assumed to be a simple function of $g^2(\sigma)$ and the universe is assumed to instantaneously transition between matter and radiation domination. However, in our example the universe does not immediately transition to a radiation bath and the timing of the transition can be an extremely complicated function of the initial value of the modulating field.

As an explicit example of such a model we take $g^2(\sigma) = \delta^2 \sigma^2$, so that during the preheating dynamics we have

$$V(\phi, \chi, \sigma) = \frac{m^2}{2}\phi^2 + \frac{\delta^2}{2}\sigma^2\phi^2\chi^2.$$
 (5.63)

We further assume that the inflationary dynamics has lead to the creation of large scale inhomogeneities in σ but not χ . Indeed, if $\bar{\chi} = 0$ then $m_{eff,\sigma}^2 = 0$ and σ will indeed be light, although we have in mind a case where the effective potential (5.63) is only valid near the end of inflation and the potential in the inflationary regime could be of a much different form. If the dynamics of σ we frozen at a fixed value σ_0 , we then have an effective coupling $g_{eff}^2 = \delta^2 \sigma_0^2$ between ϕ and χ of the form $g_{eff}^2 \phi^2 \chi^2/2$. Other theoretically well-motivated couplings include $g_{eff}^2 \sim e^{\alpha\sigma}$ or $g_{eff}^2 \sim e^{\beta\sigma^2}$.

In Fig. 5.29 we show $dS_{\ln\rho}/dt$ (normalized to it's maximum value over the sampled values of g_{eff}^2 and $\ln a$) as we vary the effective coupling g_{eff}^2 . In the left panel we take g^2 to be a fixed external parameter. In the right panel we take $g_{eff}^2 = \delta^2 \sigma_0^2$ with σ a dynamical field with nonzero vev at the beginning of the simulation and $\delta M_P^2/m = 100$. This choice does not provide a realistic model for the modulation, since σ must acquire Planck scale fluctuations to scan the range of g_{eff}^2 values plotted in Fig. 5.29. Rather, this model is meant as an illustration that the mechanism can continue to operate even with dynamical modulating fields. For either case there is a strong modulation of the shock

⁵Here we are assuming that the preheating model is not one of the special cases where the pre- and post-shock equations of state are the same.


Figure 5.29: Dependence of shock hypersurface of coupling constant in the Lagrangian. In the left panel we show the resulting taking g^2 to be a fixed constant parameter in the simulations. In the right panel we instead take the coupling to be a dynamical field with potential (5.63) whose initial condition σ_0 is varied between the simulations. In either case, there is a strong modulation of the shock-in-time hypersurface as a function of the effective coupling, and comparison of the two plots demonstrates that for this particular model the approximation of a fixed coupling g^2 gives and accurate estimate of the timing of the shock.

hypersurface as g_{eff} is varied. Comparing the left and right figures (accounting for the different linear scales), we see that the approximation of a fixed rather than dynamical g^2 reproduces the details of the hypersurface remarkably well.

The comoving curvature perturbation generated by this mechanism is determined by the differences in the overall expansion histories between different Hubble patches (simulation volumes) from the end of inflation (at $\epsilon = 1$) to a fixed energy density ρ , $\zeta_{preheat} = \delta \ln a|_{\rho}(\sigma_0)$. Since the equation of state changes abruptly at the shock, the modulation of $\ln(a_{shock})$ allows for the production of $\zeta_{preheat}$. These curvature perturbations $\zeta_{preheat}$ will then add to the perturbations generated by inflation ζ_{inf} . Since the preheating process is local, $\zeta_{preheat}$ is simply a pointwise mapping of some other (Gaussian) random field σ . Thus, it should be considered a local form of nonGaussianity which may be strongly nonGaussian. However, $\zeta_{preheat}$ and ζ_{inf} are uncorrelated so this form of nonGaussianity is very different from the typical local f_{NL} parameterization $\zeta = \zeta_G + \frac{3}{5} f_{NL} (\zeta_G^2 - \langle \zeta_G^2 \rangle)$ where ζ_G is a Gaussian random field. As a result, current constraints on this model are much weaker than those on f_{NL} , although $\zeta_{preheat}$ must still be subdominant to avoid spoiling the near Gaussianity of the CMB.

There are two different regimes in which this modulation could occur. In the first, we could imagine that the spread of g_{eff}^2 within our observable patch is much smaller than the typical scale of the structure in the shock, but the value of σ smoothed over our present Hubble patch is drawn from some distribution. The statistics of the generated ζ can then be determined using standard methods based on Taylor expansion, but the required derivatives will depend on the specific super-Hubble value of σ that is realized in our observable Hubble patch. For the second case, g^2 instead realizes values probing some of the structure in the shock, leading to the generation of a strongly nonGaussian component to the density perturbations that is uncorrelated with those produced by the inflaton and poorly parameterized by a Taylor expansion. We then have a combined curvature perturbation $\zeta(x) = \zeta_{inf}(x) + F_{NL}(\sigma(x))$, exactly as found in the massless preheating model [55].

5.9 Conclusion

In this chapter we studied the production of entropy during the preheating phase following inflation. If we broadly define preheating as the cosmological epoch between the end of inflation, when $\epsilon = 1$, and the establishment of a dense plasma in local thermal equilibrium with some temperature T_{rh} then *all* of the entropy of the primordial plasma must be generated during this transition. To explore this regime of tremendous entropy production, we introduced the Shannon entropy as our definition of the system entropy. A full calculation of the Shannon entropy would require full knowledge of the probability density functional for the fields. Obtaining such a large amount of information about the fields is unrealistic. Therefore, we obtained a coarse-grained entropy by assuming we had access to only the two point-correlations of the fields and maximizing the Shannon entropy subject to this constraint.

Based on this procedure, we found that for simple models of preheating based on broad parametric resonance there is a rapid production of entropy over a short timeinterval around the onset of strong nonlinearity in the system. This rapid change in the entropy is very similar to the jump in entropy across a hydrodynamic shock. For this reason, we have coined the phrase *shock-in-time* for the transition from a state of coherent oscillating scalars to a state of inhomogeneous nonlinearly interacting fluctuations. We demonstrated that the existence of the shock was robust to our choice of coarse-grained fields by explicitly computing the entropy in energy phonons $(\ln(\rho/\bar{\rho}), \partial_t \ln(\rho/\bar{\rho}))$ and in the fundamental fields and momenta (ϕ_i, Π_{ϕ_i}) .

Further investigation into the evolution of the fields and the post-shock cascade revealed that the low-order statistics of the energy phonons were remarkably Gaussian except for a brief time interval around the shock-in-time. Meanwhile the fields remained significantly nonGaussian well after the shock, with the (subNyquist) nonGaussianity concentrated in comoving wavenumbers $k \sim 150m$. We also investigated the development of cross-correlations between the various fields, including their impact on entropies based on measurements of the full covariance matrix.

One disturbing feature of the Shannon entropy was that the entropy in the field and energy phonon descriptions were not equal. While this is to be expected in the complex post-shock state, prior to the shock the energy phonons can be expressed linearly in terms of the field fluctuations. Therefore, there should be a linear transformation between the two variable sets. Since no information is lost in making such a transformation, we would like our definition of entropy to give the same result in either basis prior to the shock. To address this we introduced a noncanonical version of the Shannon entropy and related it to the von Neumann entropy. Using single-field massless preheating as an example, we demonstrated that during the linear stages our noncanonical entropy was the same using either the fundamental fields or the energy phonons as our collective coordinates.

Finally, as an application of the shock-in-time concept, we studied the production of adiabatic curvature perturbations mediated by the large-scale variation of coupling constants in the potential. The shock accurately tracks the transition from the lowentropy coherent condensate to a high-entropy fluctuation dominated medium. Except for very special models, the pre- and post-shock states will have different equations of state. Therefore, modulations in the time of the shock between different Hubble patches at the end of inflation produce a corresponding adiabatic curvature perturbation. These perturbations are nonGaussian, but have a very different form than is usually assumed for primordial nonGaussianities.

Chapter 6

Conclusions

In this thesis we studied the nonequilibrium nonlinear dynamics of inhomogeneous scalar fields in several cosmological contexts. In particular, we included the semiclassical effects of initially small quantum fluctuations around initial configurations possessing a high degree of spatial symmetry. This included consideration of domain wall collisions possessing both planar and SO(2,1) symmetry (in the absence of fluctuations), as well as the preheating dynamics at the end of inflation, when the background configuration is spatially homogeneous. For all cases considered, the fluctuations – which have been completely neglected in previous of the domain wall collisions – played a critical role in the full system dynamics. In many cases, the resulting dynamics also displayed either temporal or spatial intermittency.

We first studied collisions between planar symmetric domain walls and SO(2,1) symmetric vacuum bubbles. These types of collisions are common in braneworld cosmologies and first order phase transitions. Our first step, presented in chapter 2, was to conduct a linearised analysis of nonplanar fluctuations around planar symmetric collision backgrounds. By extending well-known techniques from the Floquet theory of ODEs to the case of PDEs, we were able to demonstrate the existence of exponential instabilities in the fluctuations. This included generalizations of broad parameteric resonance and narrow parametric resonance to the case of a wave equation with a time- *and* space-dependent effective frequency. Through a study of the corresponding mode functions and simple physical intuition, we also argued that these instabilities will be a generic feature in collisions between extended membrane-like objects.

The presence of the linear instabilities motivated a full nonlinear study of both planar wall and vacuum bubble collisions, presented in chapters 3 and 4 respectively. In contrast to previous investigations, we included small initial fluctuations that did not respect the symmetry of the background. To obtain the evolution of the initial field configuration we made use of high resolution parallel three-dimensional lattice simulations. This required significantly more numerical sophistication than past investigations, where the assumption of exact symmetries and exclusion of fluctuations meant the dynamics were captured by 1+1 dimensional field equations. We found that the inclusion of fluctuations can radically alter the resulting field dynamics. Rescattering effects and nonlinear interactions between the amplified fluctuations lead to a complete breakdown of the initial symmetry assumption. This breakdown of the planar or SO(2,1) symmetry manifested itself as an inhomogeneous dissolution of the walls. As a result of this, a collection of oscillons were produced from the energy stored in the colliding walls. For the case of vacuum bubble collisions, an interesting consequence of the symmetry breaking is the production of gravitational waves during the collision. When considering potential observables in the false vacuum eternal inflation scenario, these gravitational waves are one example of a novel phenomenon that has been missed by past analyses based on SO(2,1) symmetry. In light of this, it is worthwhile to re-evaluate how generic conclusions based on imposing exact SO(2,1) symmetry really are.

Finally, we considered the process of preheating after inflation. Our focus was on the dynamical production of entropy as the low-entropy coherent inflaton condensate fractured into a high-entropy incoherent state characterized by strongly interacting inhomogeneous field fluctuations. We found that the entropy was generated predominantly in a short time-interval around the onset of nonlinearities in the field fluctuations, which we denoted a *shock-in-time*. In addition to this study of entropy, we also considered the field statistics in some detail. Despite the complexity of the post-shock state, fluctuations in $\ln \rho$ and its time derivative were found to highly Gaussian at the level of one-point statistics. This was in contrast to the fundamental field variables, which displayed nonnegligible nongaussianity even long after the onset of strong inhomogeneities in the system. Since the shock connects two phases with different equations of state, a modulation of the timing of the shock between different Hubble patches will create (nongaussian) adiabatic density perturbations. We identified one possible mechanism to produce such a modulation – the variation of the coupling constant between the inflaton and preheat fields in different Hubble patches.

In summary, we found that quantum fluctuations around highly symmetric but dynamical backgrounds can experience strong instabilities. The eventual nonlinear interactions of these fluctuations can dramatically alter the behaviour of the system relative to expectations derived under the assumption that the symmetry holds exactly. This means that analysis of the scenarios considered in this thesis requires more sophisticated tools than those usually employed in the literature. Treating these problems properly is especially important because the additional complexity resulting from the full field dynamics may provide a new observational handle on currently unconstrained cosmological scenarios. For example, in this thesis we demonstrated that novel and previously unidentified observational signatures may be produced by the collision with another universe in the false vacuum eternal inflation scenario. We also provided a new mechanism by which preheating may generate observable density perturbations. Since many of these signatures rely on the nonlinear interactions in the underlying field theory, they are likely to also display more sensitivity to model parameters and initial conditions. From the viewpoint of differentiating models this lack of universality is a huge plus, even if it means the required theoretical analysis is more difficult.

The work presented in this thesis naturally lends itself to several future research directions, as well as interesting connections to other work in progress. Specifically, the results in chapter 5 naturally tie in with current work in collaboration with J. Richard Bond, Andrei Frolov and Zhiqi Huang on the production of density perturbations during preheating from caustics in the ballistic motion of trajectories in the equivalent homo*geneous* field dynamics. The shock-in-time provides a natural separation between the ballistic description and subsequent full field dynamics (*i.e.* including all gradient couplings in the equation of motion) which must be treated on the lattice. The perturbations produced by the caustic mechanism have the same structural form as those presented in chapter 5, and we have results demonstrating that they can lead to observable patterns on the sky. As mentioned previously in the thesis, the novel dynamics presented here could also produce a variety of interesting observational signatures. These include, but are not limited to: production of gravitational waves from collision with an external bubble in false vacuum eternal inflation, and production of nonGaussian but spatially intermittent density perturbations at preheating. An obvious avenue of investigation is to develop theoretical tools to constrain such signals using observation, and then perform the necessary analysis using publically available data. Among the interesting challenges is the development of numerical simulations that include the effect of inhomogeneous gravity on the lattice.

Appendices

Appendix A

Collective Coordinate Approximation for Double Well Collisions

In this appendix we will present a brief derivation of our collective coordinate approximation for the repeated collisions of two walls. The key step is to make a drastic reduction in the number of degrees of freedom of the system. We will ignore all radiation as well as the shape mode and assume that the field profile takes the form of an interacting kink-antikink pair given by

$$\phi_{bg} = \tanh\left(\gamma \frac{x + r(t)}{\sqrt{2}}\right) - \tanh\left(\gamma \frac{x - r(t)}{\sqrt{2}}\right) - 1 \tag{A.1}$$

with $\gamma^2 = (1 - \dot{r}^2)^{-1}$. Our goal is now to obtain an effective Lagrangian for the locations of the kink and antikink r. The reader should note that this approximation is terribly naive as we are ignoring the effects of the shape mode and the production of radiation, as well as any distortion of the kink shapes while they are near each other. However, as emphasized in the main text, we wish to separate out the effects of the actual collision from the subsequent evolution and the approximation (A.1) does exactly this.

Before proceeding, it should be clear that our final Lagrangian should be that of a pair of relativistic point particles interacting through a potential, along with some corrections induced by the finite thickness of the kinks. For simplicity, we will assume $\dot{\gamma} = 0$. These terms only arise from the kinetic term for the fields, and we are effectively dropping a finite thickness correction for the individual kinks, as well as an interaction piece. Substituting this into the lagrangian for the field, we find

$$L = \omega \dot{r}^{2} \left[S_{2}(0) + S_{2}(\omega r) \right]$$

$$- \omega \left[S_{2}(0) - S_{2}(\omega r) \right]$$

$$- \omega^{-1} \left[\sinh^{2}(2\omega r)S_{2}(\omega r) - \sinh^{3}(2\omega r)S_{3}(\omega r) + \frac{\sinh^{4}(2\omega r)}{4}S_{4}(\omega r) \right]$$
(A.2)

where we have defined $\omega = \gamma/\sqrt{2}$ and $S_n(\alpha) \equiv \int \operatorname{sech}^n(x+\alpha)\operatorname{sech}^n(x-\alpha)dx$. The required integrals can be easily obtained by considering $\int_{\mathcal{C}} f(z)dz$ with $f(z) = z\operatorname{sech}^n(z+r)\operatorname{sech}^n(z-r)$ and the contour \mathcal{C} given by $(-\infty,\infty) \cup [\infty,\infty+i\pi] \cup (\infty+i\pi,-\infty+i\pi) \cup [-\infty+i\pi,-\infty]$. For completeness, the required results are

$$S_{2}(\alpha) = \frac{4}{\sinh^{2}(2\alpha)} \left(\frac{2\alpha}{\tanh(2\alpha)} - 1\right)$$

$$S_{3}(\alpha) = \frac{2}{\sinh^{5}(2\alpha)} \left[4\alpha(2 + \cosh(4\alpha)) - 3\sinh(4\alpha)\right]$$

$$S_{4}(\alpha) = \frac{-4}{\sinh^{4}(2\alpha)} \left[\alpha \coth(2\alpha)(12 - 20 \coth^{2}(2\alpha)) - \frac{8}{3} + 10 \coth^{2}(2\alpha)\right].$$
(A.3)

Notice that the interactions between the kink and antikink depend not only on their separations, but also on their relative speeds. This is because these interactions are generated by integrals of the overlap between the kink and antikink. As their speeds increase, they Lorentz contract which changes the amount of overlap. When the kink and antikink are far apart, this overlap is exponentially small. Therefore, we will make the following additional approximation. When performing the integrals, we will keep overall γ multipliers only on the overall constant pieces.

This gives us our final effective Lagrangian

$$L[r(t)] = -\frac{4\sqrt{2}}{3}\sqrt{1-\dot{r}^2} - V_{eff}(r) + \gamma K(r)\frac{\dot{r}^2}{2}$$
(A.4)

where we have defined the effective potential

$$-V_{eff} \equiv \omega \left[4 \left(1 + \gamma^{-2} \right) - 16\omega r \gamma^{-2} + 8 \left(-\omega r - 3\gamma^{-2} + \omega r \gamma^{-2} \right) \coth(2\omega r) + 4 \left(-1 + +5\gamma^{-2} + 12\omega r \gamma^{-2} \right) \coth^2(2\omega r) + 8\omega r \left(1 - 5\gamma^{-2} \right) \coth^3(2\omega r) \right]$$
(A.5)

and

$$K(r) = \frac{S_2(\omega r)}{S_2(0)}$$
 (A.6)



Figure A.1: The effective potential $V_{eff}(r)$ and noncanonical contribution to the kinetic term K(r) for our effective single-particle Lagrangian describing the separation of the kink and antikink pair. Also included are the individual contributions from the gradient energy (V_{grad}) and potential energy (V_{pot}) in the original scalar field Lagrangian. For comparison, we have also included the asymptotic potential $-8\sqrt{2}e^{-2\sqrt{2}r}$ for $\omega r \gg 1$.

It is easy to see that this potential vanishes exponentially fast for $\omega r \gg 1$. We only wish to consider bound motions in this paper, so we must have $E - \frac{4\sqrt{2}}{3} < 0$. Therefore, at large r the walls will move nonrelativistically and we can set $\gamma \approx 1$. This is of course incorrect for $\omega r \leq 1$, but in this regime the kink and antikink are close to each other and interacting strongly. In this regime, the kink profiles are likely to be deformed and our ansatz will be a poor description of the full field configuration. Making this further approximation we obtain our final form for the effective potential

$$2^{-1/2}V_{eff} \equiv -4 + 8\omega r + 12\coth(2\omega r) - (24\omega r + 8)\coth^2(2\omega r) + 16\omega r \coth^3(2\omega r).$$
(A.7)

In Fig. A.1 we show this effective potential as well as the noncanonical part of the kinetic coupling.

Now that we have a rather simple effective action, we will construct an analytic approximation to the background motion. During most of the motion the walls will be well-separated with $\omega r \ll 1$, so we can approximate the motion as occurring in the potential $V_{eff}(r) \approx -8\sqrt{2}e^{-2\sqrt{2}r}$. The noncanonical contribution to the kinetic term vanishes exponentially in this limit as well, so we will set it to zero. Finally, for bound motions we also have $\gamma \approx 1$ so we can approximate the relativistic kinetic term by it's nonrelativistic limit.

$$t = \sqrt{\frac{M}{2}} \int_{r_{max}}^{r} \frac{d\tilde{r}}{\sqrt{V(r_{max}) - V(\tilde{r})}} \implies T = \sqrt{2M} \int_{r_{max}}^{r_{min}} \frac{dr}{\sqrt{V(r_{max}) - V(r)}}$$
(A.8)

Approximating the full motion by the $\omega r \gg 1$ potential, we find

$$r(t) = r_{max} + \frac{1}{2\sqrt{2}} \log\left(\cos^2\left(\frac{\pi t}{T}\right)\right) \qquad T = \frac{\pi\sqrt{M}}{2\sqrt{\Delta V}} = \frac{\pi}{2\sqrt{6}} e^{\sqrt{2}r}$$
(A.9)

A major problem with this result is that $r \to -\infty$, when in reality energy conservation will enforce a minimum value of r. The easiest way to cure this is to simply cutoff the logarithmic divergence as

$$r(t) = r_{max} + \frac{1}{2\sqrt{2}} \log\left(\cos^2\left(\frac{\pi t}{T}\right) + e^{-2\sqrt{2}(r_{max} - r_{min})}\right).$$
 (A.10)

which enforces the condition $r(T/2) = r_{min}$ where $V_{eff}(r_{min}) = V_{eff}(r_{max})$ and $r_{min} < 0$. In Fig. A.2 we compare the accuracy of this analytic approximation to a full solution of the equations for the Lagrangian (A.4). We have approximated $\gamma = 1$ in K(r) and $V_{eff}(r)$ but have otherwise included both the noncanonical kinetic correction and relativistic corrections. Our approximation (A.10) is very accurate for most of the evolution.

The above procedure can easily (but tediously) be generalized to the asymmetric well. However, the main effect of this term is to break the Z_2 symmetry so that the two potential minima are no longer degenerate. Therefore, the main effect is to introduce a contribution to the effective potential of the form $[V(\phi_f) - V(\phi_t)]r$ for $\omega r \gg 1$. In chapter 3 we will consider a full (3+1)-dimensional nonlinear solution of the field equations for a planar kink-antikink initial state with small initial fluctuations. There we will explicitly see that the parametric pumping of fluctuations occurs thus confirming the intuition developed in this paper.



Figure A.2: Comparison of our analytic approximation to the evolution of r(t) to numerical simulation of the effective equations for the background. The solid lines are our analytic approximation (A.10) and the triangles are a numerical solution to the equations of motion for the Lagrangian (A.4). Aside from a small lengthening of the period in the full solution, we see that our approximation is very accurate, with the accuracy improving as we increase the initial separation.

Appendix B

Numerical Approach and Convergence Tests

In this appendix we will briefly summarize the numerical techniques used in chapter 2. First we describe our strategy for implementing time-evolution in our numerical codes. We used different approaches when solving for the nonlinear background (2.11) and the linear fluctuations (2.12). For the one-dimensional nonlinear wave equation (2.11), we used a 10th order accurate Gauss-Legendre quadrature based method. Explicitly, this amounts to a specific choice of implicit Runge-Kutta process. Given an initial condition \mathbf{y}_t to $d\mathbf{y}/dt = \mathbf{H}(\mathbf{y})$, the solution at time dt is obtained by solving

$$\mathbf{f}^{(i)} = \mathbf{H}\left(\mathbf{y}_t + dt \sum_{j=1}^{\nu} a_{ij} \mathbf{f}^{(j)}\right)$$
(B.1)

$$\mathbf{y}_{t+dt} = \mathbf{y}_t + dt \sum_{i=1}^{\nu} b_i \mathbf{f}^{(i)}$$
(B.2)

where a_{ij} , b_i and c_i are numerical constants defining the process. For the Gauss-Legendre methods, the c_i 's are the roots of $P_{\nu}(2c-1)$ where $P_{\nu}(x)$ is the Legendre polynomial of degree ν . The a_{ij} 's and b_i 's are then solutions to

$$\sum_{j=1}^{\nu} a_{ij} c_j^{l-1} = \frac{c_i^l}{l} \qquad l = 1, \dots, \nu$$
(B.3)

$$\sum_{j=1}^{\nu} b_j c_j^{l-1} = l^{-1} \,. \tag{B.4}$$

This amounts to approximating the integrals required to perform the time-evolution using Gauss-Legendre quadrature. Due to the excellent convergence properties of these quadrature approximations, the result is an order 2ν integrator.¹ If the reader would like explicit formulae for ν up to 5 please see Table 2 of Butcher [197], although it is far easier in practice to simply solve (B.3) numerically.

As for the linear fluctuation equation (2.12), we instead employed Yoshida's [158] operator-splitting technique that was introduced into the preheating community by Frolov and Huang. For further details on this method see for example [40, 43, 158]. For this set of integrators, the solution to df/dt = H(f) is first written as $f(t + dt) = e^{\mathbf{H}dt}f$, where **H** should now be interpreted as an operator acting on f. We decompose $\mathbf{H} = \sum_i \mathbf{H}_i$, where the action of each individual \mathbf{H}_i on f is simple to compute. Finally, we reexpress the time evolution operator $U \equiv e^{\mathbf{H}dt}$ as a product of exponentials for the individual \mathbf{H}_i operators, $e^{\mathbf{H}dt} = U(w_M)U(w_{M-1})\ldots U(w_0)U(w_1)\ldots U(w_M) + \mathcal{O}(dt^{n+1})$, where $U(w_i) \equiv e^{w_i\mathbf{H}_1dt/2}e^{w_i\mathbf{H}_1dt/2}$ is a second-order accurate time-evolution operator for time-step w_idt and we have specialized to the case of an operator split into only two parts. Via clever choices of the number and value of the numerical coefficients w_i , integrators of various orders n may be constructed. For this paper, we have chosen coefficients that produce an $\mathcal{O}(dt^6)$ evolution given by

$$w_{1} = -1.17767998417887100694641568096431573$$

$$w_{2} = 0.235573213359358133684793182978534602$$
(B.5)
$$w_{3} = 0.784513610477557263819497633866349876$$

$$w_{0} = 1 - 2(w_{1} + w_{2} + w_{3}) = 1.31518632068391121888424972823886251.$$

Both of these approaches have the added benefit that for Hamiltonian systems they are symplectic integrators. For this reason, in this paper we have found it convenient to use Hamilton's form for the evolution equations. With the exception of the collective coordinate location for the bouncing walls in the double well, all of our Hamiltonians can be split into two exactly solvable pieces so that $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ and we will provide these even for the nonlinear wave equations (even though the splitting is not required for the Gauss-Legendre method). For the planar walls is (up to an overall normalization)

$$\mathcal{H}_{planar,1} = \sum_{i} \frac{\pi_{\phi,i}^2}{2} \qquad \mathcal{H}_{planar,2} = \sum_{i} \frac{G[\phi_i]}{2dx^2} + V(\phi_i) \,. \tag{B.6}$$

¹Here we take the order of the integrator (denoted by n) to be the highest power in dt for which the approximate solution is exact. This means the leading order error term is $\sim dt^{n+1}$.

Meanwhile, the linearized fluctuations evolve in the Hamiltonian

$$\mathcal{H}_{fluc,1} = \sum_{i} \frac{\pi_{\delta\phi,i}^2}{2} \qquad \mathcal{H}_{fluc,2} = \sum_{i} \frac{G[\delta\phi_i]}{2dx^2} + \frac{1}{2}V''(\phi_{bg}(x,t))\delta\phi_i^2.$$
(B.7)

Finally, the Hamiltonian for the SO(2, 1) invariant bubbles is

$$\mathcal{H}_{bubbles,1} = \sum_{i} \frac{\pi_{\phi,i}^2}{2s^2} \qquad \mathcal{H}_{bubbles,2} = \sum_{i} s^2 \left(\frac{G[\phi_i]}{2} + V(\phi_i) \right) \,. \tag{B.8}$$

In all three cases, $\pi_{f,i}$ represents the canonical momentum for field f at lattice site i. The operator $G[\phi_i]/2$ is a discrete approximation to $(\partial_x \phi(x_i))^2/2$.

Finally we describe our approach to the spatial discretization of the system. For all production runs we used a Fourier pseudospectral approximation for the field derivatives. The only derivative appearing in the various equations of motion is the one-dimensional laplacian along the collision axis ∂_{xx} . Therefore, in practice the system was evolved in real space, with the Laplacian evaluated in Fourier space through the use of the FFT. Although the resulting FFT and inverse FFT are numerically more expensive than a finite-difference approximation, the continuum limit is approached much more rapidly as seen in Fig. B.2. This is especially important when computing Floquet exponents, as our approach requires solving $4N^2$ individual PDE's in order to form the fundamental matrix where N is the number of grid points. As well, in order to maintain a fixed accuracy in the time-integration, the ratio dx/dt should be kept constant meaning that the total work required scales as N^3 for a finite-difference approximation and $N^3 \log(N)$ for a pseudospectral approach.² As a result, the spectral approach ended up requiring less CPU time than the finite-difference approach, while simultaneously being (orders of magnitude) more accurate.

In order to provide independent verification of our results, we also performed several runs using finite-difference discretizations of $G[\phi_i]$. The Hamiltonian was discretized directly, thus ensuring a consistent discretization of $\nabla^2 \phi$ and $(\nabla \phi)^2$. We tested with both a second-order accurate and fourth-order accurate stencil given by

$$(\nabla \phi)^2 dx^2 \approx G[\phi_i]_2 = \frac{1}{2} \left[(\phi_{i+1} - \phi_i)^2 + (\phi_{i-1} - \phi_i)^2 \right]$$
(B.9)

²When comparing run times, the reader should keep in mind that the limiting factor on modern computing architecture is often the speed at which data can be obtained from memory, not necessarily the number of floating point operations. However, for the problems we were concerned with, the spectral approach proved to be much faster if an accuracy better than the tenth of a percent level was desired.

and

$$(\nabla\phi)^2 dx^2 \approx G[\phi_i]_4 = \frac{-1}{24} \left[(\phi_{i+2} - \phi_i)^2 + (\phi_{i-2} - \phi_i)^2 \right] + \frac{2}{3} \left[(\phi_{i-1} - \phi_i)^2 + (\phi_{i+1} - \phi_i)^2 \right]$$
(B.10)

respectively, where dx is the lattice grid spacing. The corresponding laplacian stencils $L[\phi_i]/dx^2$ satisfying $\sum_i G[\phi_i] + \phi_i L[\phi_i] = 0$ (on periodic grids) are then the familiar second-order accurate

$$\frac{\partial^2 \phi}{\partial x^2} dx^2 \approx L[\phi_i]_2 = (\phi_{i+1} - \phi_{i-1}) - 2\phi_i \tag{B.11}$$

and fourth-order accurate centered difference

$$\frac{\partial^2 \phi}{\partial x^2} dx^2 \approx L[\phi_i]_4 = \frac{-1}{12} (\phi_{i-2} + \phi_{i+2}) + \frac{4}{3} (\phi_{i-1} + \phi_{i+1}) + \frac{-5}{2} \phi_i \,. \tag{B.12}$$

B.0.1 Convergence Tests

Here we present several tests of the convergence properties of our numerical codes. The combination of high-order time-integrations and spectrally accurate derivative approximations leads to a rapid convergence of both the nonlinear field evolution used to study the background dynamics and the floquet exponents determined by solving the perturbation equations.

Several measures of this convergence for the case of the nonlinear wave equation and initial conditions the same as the bottom right panel of Fig. 2.3 are shown in Fig. B.1. In the top row we show the pointwise convergence of the solution as we vary the number of grid points N (or equivalently the grid spacing) and the time-step dt, thus independently checking our Fourier spatial discretization and our Gauss-Legendre integrator. We consider two closely related measures,

$$\|\Delta\phi^{(p)}\|_{L1} \equiv N_{base}^{-1} \sum_{\{x_i\}} |\phi^{(p+1)}(x_i) - \phi^{(p)}(x_i)| \quad \text{and} \quad \|\Delta\phi^{(p)}\|_{max} \equiv \max_{\{x_i\}} |\phi^{(p+1)}(x_i) - \phi^{(p)}_i(x_i)|$$
(B.13)

where $\phi^{(p)}$ denotes the numerical solution for the *pth* approximation (here either the number of grid points or the time step), and we take $N^{(p+1)}/N^{(p)} = 2 = dt^{(p)}/dt^{(p+1)}$. In order to properly compare solutions with different spatial resolutions, we took all sums over the grid from the solution with N = 2048 points. From the top left panel, we see that the solution very rapidly converges (to the level of machine precision) as we increase the spatial resolution, exactly as we expect for a properly resolved spectral code. Looking also at the top right panel, we see that the growing error at late times is due to errors

in the time-stepping rather than the spatial discretization. One may worry about this apparent issue with the time-stepping, but this is really just a demonstration that making a pointwise comparison of the fields is not necessarily the best measure of convergence. In particular, the apparent errors that accumulate at late time occur because we have an oscillating localized blob of field. Small errors in the oscillation frequency then lead to errors in the instanteous value of the field which manifest themselves as what appears to be accumulating errors at late times. As a further test of our time-stepping procedure we check the conservation of the energy density $\rho = \langle T^{00} \rangle$ and field momentum $P^x = \langle T^{0x} \rangle$, where $T^{\mu\nu}$ is the energy-momentum tensor of ϕ and $\langle \cdot \rangle$ denotes a spatial average, for a range of time steps dt. For all choices of time step, we see that the field momentum is conserved to machine precision, while for $dt \geq dt/5$, the energy is also conserved to machine precision.

To understand the accuracy with which our Floquet exponents are computed (and to demonstrate the great gains in accuracy obtained via a spectral approach), we now provide some convergence plots for the largest Floquet exponent in Fig. B.2. Here we can directly compare the the individual Floquet exponents, so we plot

$$\Delta \mu^{(p)} \equiv |\mu_{max}^{(p+1)} - \mu_{max}^{(p)}|.$$
(B.14)

For orientation, the top left panel shows $\mu_{max}T_{breather}$ for the choice v = 0.5 and a range of k_{\perp} values. The top right panel we show the rate of convergence as the timestep is varied. As expected for a sixth-order accurate integrator, the error decreases rapidly, although not quite as quickly as for the Gauss-Legendre integrator. Also of note is that the error decreases uniformly for all values of k_{\perp} (except for those values that are already at machine roundoff levels) indicating that important features such as the locations of stability bands where $\mu_{max} = 0$ are not shifting around as the timestep is varied. In the bottom row we show similar convergence plots as the number of grid points are varied for the spectral approximation and for a second-order and fourthorder accurate finite-difference scheme. As promised, the spectral method converges very rapidly compared to the finite-differencing methods. However, equally important is the fact that the convergence is again uniform for all k_{\perp} , while it is not for the finitedifference methods. Taking the far right 4th order chart as an example, there are several extreme spikes in the region $k_{\perp}^2(1+v^{-2})$ for which the difference between the N=128and N = 256 approximation is of order machine precision, but then rises to 10^{-3} level when comparing to the N = 512 solution. The ultimate source of these appearing and disappearing spikes is a slight shifting of the edges of the stability bands as the resolution



Figure B.1: Convergence of our one-dimensional lattice code for the double-well potential with $\delta = 1/30$, mL = 1024 and various choices of grid spacing dx and time step dt. We plot the two norms defined in (B.13). The apparent accumulating errors at late times are due to small errors in the oscillation frequency and initial phase of the oscillon that has formed at the origin. Decreasing the time step past dx/5 does not lead to a decrease in this error, suggesting that it arises due to machine roundoff error. Finally, in the bottom two we demonstrate the convergence of both energy and momentum of the system for the same choices of dt as the top right plot, demonstrating that our time-stepping procedure has indeed converged (to the extend that conserved quantities are conserved).

is varied.



Figure B.2: Convergence plots for the largest Lyapanov exponent around a sine-Gordon breather with v = 0.5 for a range of k_{\perp}^2 values. The lattice was chosen to have length L =58. For reference, the top left panel shows $\mu_{\max}T_{\text{breather}}$ for the case N = 64, dt = dx/20, corresponding to the parameters used in the instability chart in the main text. In the top right panel, we show the difference in the numerically determined values of $\mu_{\max}T_{\text{breather}}$ holding the number of grid points fixed (at N = 64) while varying the discrete time step dt using a spectral derivative approximation. Finally, in the bottom row we show the convergence properties as the grid spacing is decreased, using a spectral (bottom left), second-order finite-difference (bottom center) and fourth-order finite-difference (bottom right) approximation for the Laplacian. In all three graphs, we took dx/dt = 20 and used a sixth-order accurate Yoshida scheme. Because of the rapid convergence, the instability charts (with the exception of the N = 32 case) are visually nearly identical to that displayed in the left panel above.

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