Energy Devices – The wind (Lecture 4)

Eric R. Switzer
(http://kicp.uchicago.edu/∼switzer/)

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Lecture outline:

- What is the wind? The embodied power and force of the wind.
- Lift and drag
- Power extracted from the wind and an upper bound on the efficiency.
- Some considerations in real turbines.

These notes also fill in a few mathematical details that are not in the lecture, including a derivation of the Betz limit.

1 The Betz Limit – Derivation

There is a fundamental limit on the efficiency with which power can be extracted from the wind. This is known as the *Betz limit* (due to Betz, Lanchester, and Zhukovsky). As shown in Fig. 4, for a given mass flow rate, $\dot{m}$, the power in the wind is $P = (1/2)\dot{m}v^2$. The total power extracted by the turbine is the difference between the embodied power in and the embodied power out:

$$P_{\rm extracted} = P_{\rm in} - P_{\rm out} = \frac{1}{2}\dot{m}(v_{\rm in}^2 - v_{\rm out}^2). \quad (1)$$

We know from Fig. 1 that the mass flow rate $\dot{m}$ is given by $\dot{m} = \rho v A$ (where ρ is the air density, v is the wind speed, and A is the swept area). What are the known quantities? We know the area of the turbine $A_{\rm turbine}$, and we know the velocity of incoming and outgoing wind. If we knew the wind speed at the turbine, we would know the mass flow rate and

Image sources and resources:

Fig. 2 (bottom left)	http://en.wikipedia.org/wiki/File:Campo_de_Criptana_Molinos_de_Viento_1.jpg.
Resource: typical drag coef.	http://en.wikipedia.org/wiki/Drag_coefficient.
Resource: notion of lift.	http://en.wikipedia.org/wiki/Lift_%28force%29.
Derivation of the Betz limit following	*Wind Turbine Fundamentals, Technologies, Application, Economics* by E. Hau.

the total power extracted. It is tempting to say that the velocity at the turbine is just the incoming wind speed, but the wind actually slows significantly even before it hits the turbine! Likewise, we know the velocity before and after the turbine, but we *do not know the area swept out by that flow* – the only swept area we know is the turbine itself. (See Fig. 5 for the geometry.) Thus, we have one unknown that must be eliminated by a physical constraint.

In Fig. 4, we also reasoned that the force exerted by the wind is $F = \dot{m}v$. The turbine feels a force from the air blowing on it, but it also feels a force in the opposite direction from the air blowing out of it. Thus,

$$F_{\text{turbine}} = \dot{m}(v_{\text{in}} - v_{\text{out}}). \tag{2}$$

Now recall that mechanical work[1] is the force times the distance, $W = Fd$, and that the power applied is the time rate of change of that work, or the force times the velocity, $P = Fv$. Call the wind velocity at the turbine v_{turbine}. Then the power that must be sapped out of the flow by the turbine is

$$P_{\text{extracted}} = Fv_{\text{turbine}} = v_{\text{turbine}}\dot{m}(v_{\text{in}} - v_{\text{out}}). \tag{3}$$

Thus, on the one hand, we have used an energy argument to find that the power extracted is $\frac{1}{2}\dot{m}(v_{\text{in}}^2 - v_{\text{out}}^2)$. On the other hand, we have used a force/momentum argument to show that the power extracted is $v_{\text{turbine}}\dot{m}(v_{\text{in}} - v_{\text{out}})$, so we can simply equate these and solve for the *unknown* v_{turbine}.

$$\frac{1}{2}\dot{m}(v_{\text{in}}^2 - v_{\text{out}}^2) = v_{\text{turbine}}\dot{m}(v_{\text{in}} - v_{\text{out}}) \Rightarrow v_{\text{turbine}} = \frac{1}{2}(v_{\text{in}} + v_{\text{out}}). \tag{4}$$

Thus, the velocity at the turbine is the geometric mean of the velocity in front of and behind the turbine. We now know the turbine area and the velocity of air passing through it, so we know the mass flow in Eq. 1, namely $\dot{m} = \rho A_{\text{turbine}} v_{\text{turbine}}$, and we can write the power extracted in terms of the incoming and outgoing wind velocities,

$$P_{\text{extracted}} = \frac{1}{4}\rho A_{\text{turbine}}(v_{\text{in}}^2 - v_{\text{out}}^2)(v_{\text{in}} + v_{\text{out}}). \tag{5}$$

We can now compare the power extracted to the total power embodied in the wind,

$$P_{\text{wind}} = \frac{1}{2}\rho A_{\text{turbine}} v_{\text{in}}^3. \tag{6}$$

Note that this is somewhat subtle: this is the power embodied in wind passing through the same area as the turbine, but with a velocity as if the turbine weren't there, namely the incoming wind speed v_{in}. The ratio of the power extracted to the total power embodied in the wind through that area is then the efficiency

$$\begin{aligned} \epsilon_{\text{Betz}} &= \frac{P_{\text{extracted}}}{P_{\text{wind}}} = \frac{\frac{1}{4}\rho A_{\text{turbine}}(v_{\text{in}}^2 - v_{\text{out}}^2)(v_{\text{in}} + v_{\text{out}})}{\frac{1}{2}\rho A_{\text{turbine}} v_{\text{in}}^3} && (7) \\ &= \frac{1}{2}\frac{(v_{\text{in}}^2 - v_{\text{out}}^2)(v_{\text{in}} + v_{\text{out}})}{v_{\text{in}}^3} && (8) \\ &= \frac{1}{2}\left[1 - \left(\frac{v_{\text{out}}}{v_{\text{in}}}\right)^2\right]\left[1 + \left(\frac{v_{\text{out}}}{v_{\text{in}}}\right)\right]. && (9) \end{aligned}$$

[1] Here assuming constant, parallel forces.

That is, the efficiency only depends on the ratio of the wind speed exiting the turbine to the wind speed into the turbine, $v_{\mathrm{out}}/v_{\mathrm{in}}$. We now want to optimize this efficiency by finding the best wind speed ratio $v_{\mathrm{out}}/v_{\mathrm{in}}$. Note that the first term prefers the case where the output speed goes to zero, extracting all the power from the flow. The second term prefers some output velocity. This reflects the fact that to extract power, there must be a mass flow. This reaches a maximum efficiency when $v_{\mathrm{out}}/v_{\mathrm{in}} = 1/3$. The maximum efficiency is then

$$\epsilon_{\mathrm{Betz}}\Big|_{\mathrm{max}} = \frac{16}{27} = 59.3\%. \tag{10}$$

This is the Betz limit on mechanical efficiency for extracting power from the wind. Note that it has nothing to do with thermodynamic efficiency, and is instead a mechanical limit. Well-designed, real turbines will reach efficiencies of $\sim 44\%$.

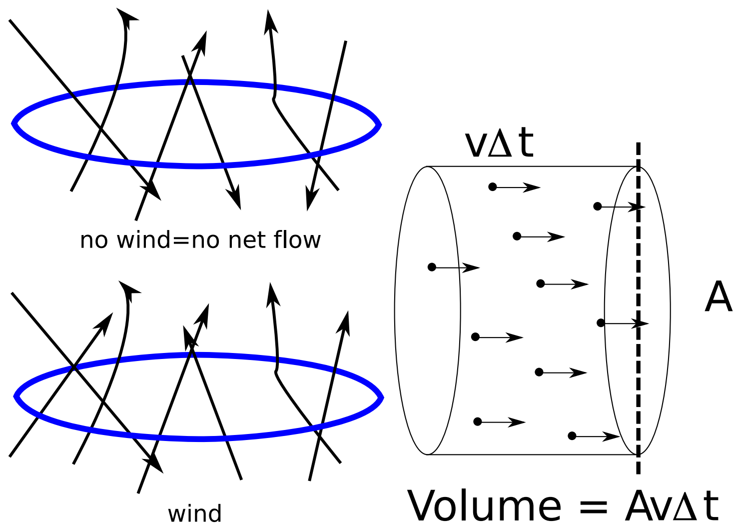

Figure 1: **Left:** Air molecules are constantly in motion, with a typical velocity of ~ 1000 mph, but they collide at a rate of several GHz (10^9 times per second). In a still room, there is no net motion of molecules through the ring – wind is the net motion of air molecules through some area. Wind produces some net flow of mass through the area swept by the ring. **Right:** A cylinder containing air moving at velocity v that will pass through the area A in a time Δt. Its volume is $Av(\Delta t)$; multiplying by the air's density ρ gives the rate that mass flows through the area, $\dot{m} = \rho A v$. For example, a 10 mph wind through an area the size of a door frame is 18 lb/sec.

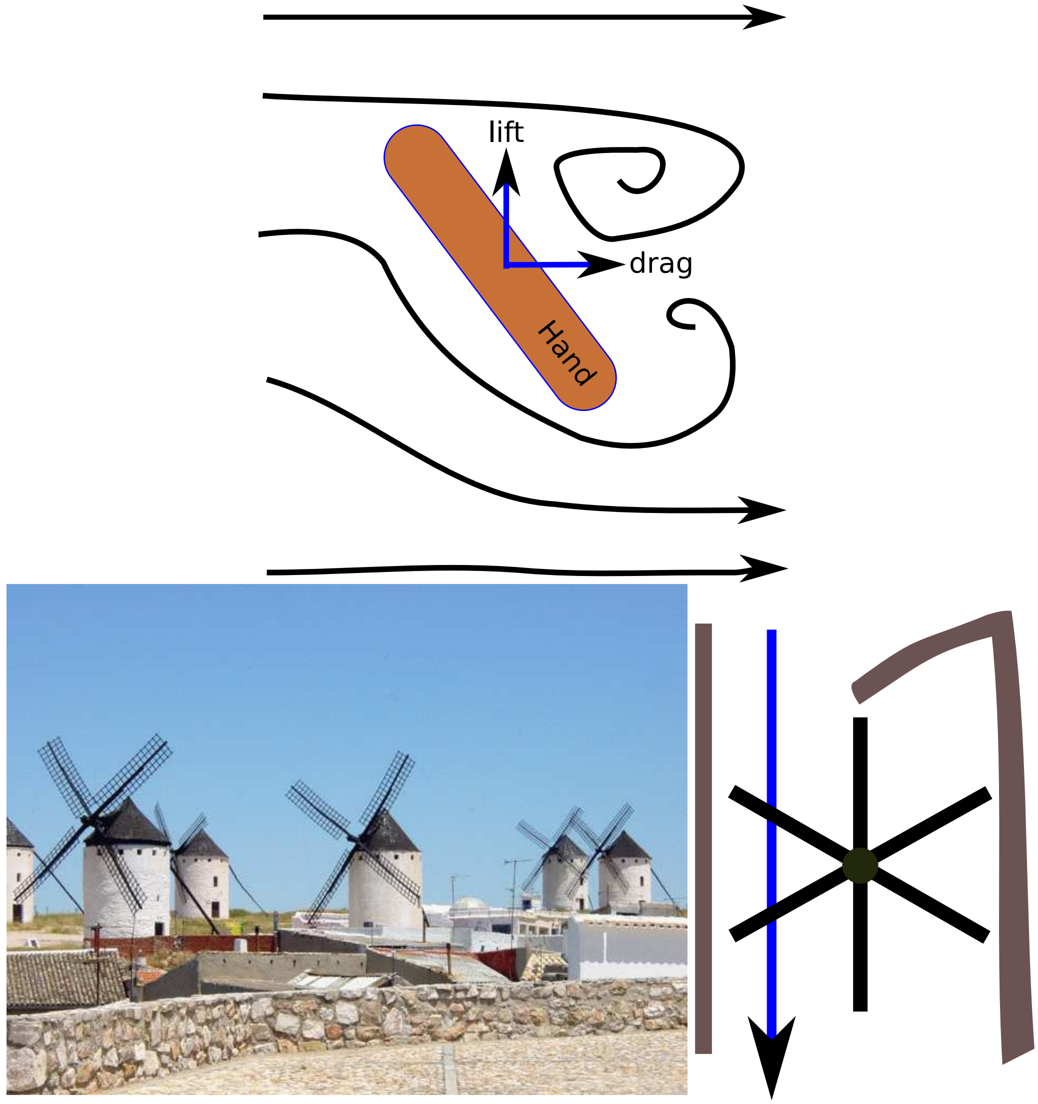

Figure 2: **Top:** when holding your hand out a car window, you feel both a lift and a drag due to the wind. In equations, the drag force is $F_d = (1/2)\rho v^2 C_d A$, where ρ is the air density, v is the velocity, A is the area facing the wind, and C_d is the dimensionless drag coefficient. The lift is given similarly by $F_l = (1/2)\rho v^2 C_l A$, but here with a lift coefficient C_l. If you tip your hand, you know that the ratio of drag to lift changes. At some attack angles you can feel a considerable lift. Turbine designs can exploit either the force of the lift (modern designs) or drag (the first wind devices). **Bottom left:** The lift-type mills of La Mancha (image: wikipedia). The La Mancha mills appear to have a “flat hand” to the wind. Fig. 3 describes the reason for their geometry – in the moving frame of the blade, the resultant wind is both from the incoming wind and the blade rotation. **Bottom right:** A 9th century Persian drag-type mill.

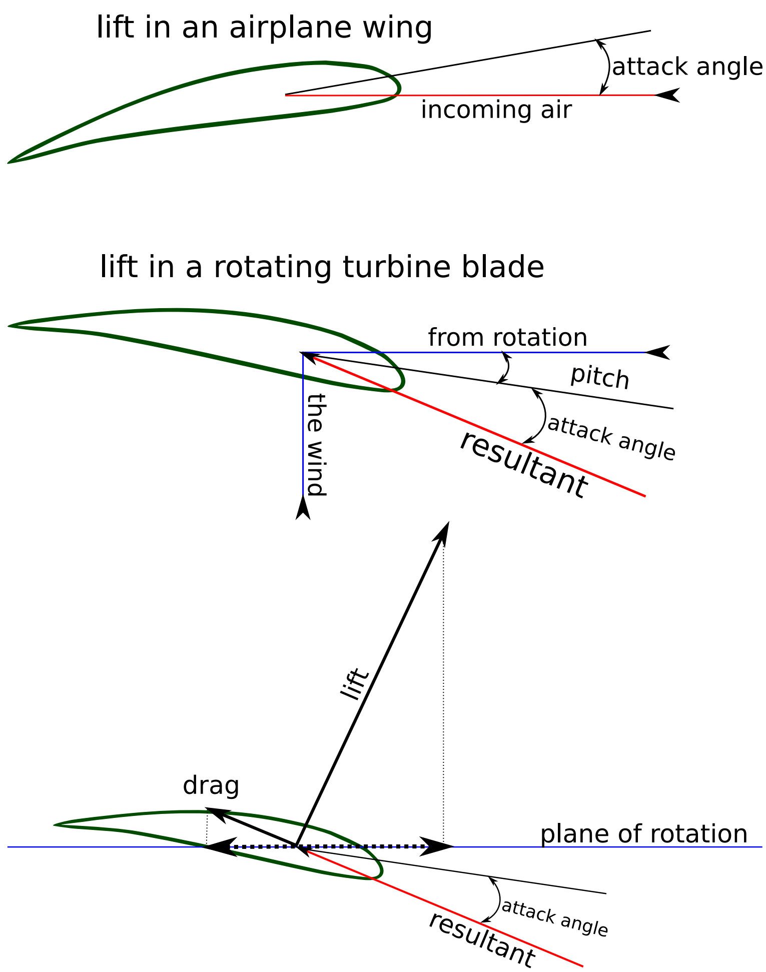

Figure 3: Forces on airfoils in a lift-type turbine. **Top:** In an airfoil such as in aircraft's, the ratio of lift to drag is a function of the attack angle between the chord of the wing and the incoming air. **Middle:** looking at an airfoil cross section from the center of the rotor. In a rotating mill, the incoming air will be both the wind and from rotation (in the frame of the airfoil). This direction is shown by the red, resultant, line. In a lift-type turbine, the attack angle is between the airfoil's chord line and the *resultant* wind velocity. The angle between the plane of rotation and the chord line is called the blade pitch, and is adjustable in most large turbines. **Bottom:** the forces on the airfoil. The airfoil along a segment of the rotor produces drag and lift (like the hand out the window). Only the components of these forces in the plane of rotation yield a torque which turns the generator and the rotor. Because the lift greatly exceeds drag, there is a net torque which drives the rotation.

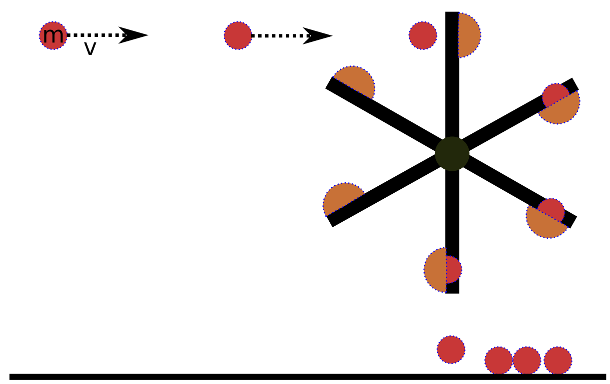

Figure 4: A catching device to illustrate the power in some flow of mass. Recall that the kinetic energy of a mass moving with velocity v is $E = (1/2)mv^2$. Each time the device catches a ball, a ratchet device extracts the energy $(1/2)mv^2$. The power is just the rate that energy is received. In this case, suppose that there is a one pound ball every second. Call the mass flow rate $\dot{m}$, so that there is then one pound per second flowing to the catcher. The power can then be understood as the mass flowing to the catcher, or $P = (1/2)\dot{m}v^2$. This describes the exchange of energy, but from catching we also have some intuition that there is a *force*. When the catching device grabs the ball, a force pushes it backwards, causing it to rotate. This is due to the exchange of momentum – *force is just the rate at which momentum is exchanged*. If the catcher completely stops the horizontal motion of the ball, then each ball imparts some momentum $\Delta p = mv$. The rate that the catcher receives momentum is the force that it experiences, $F = \dot{m}v$. Unlike this stream of balls, the wind is a much more continuous phenomenon. Recall from Fig. 1 that the mass flow rate $\dot{m}$ is $\dot{m} = A\rho v$. Thus, the power embodied in the wind is $P = (1/2)\rho A v^3$, proportional to the area A swept out and the *cube* of the wind's velocity v. The force due to a wind is $F \propto \dot{m}v = \rho A v^2$. Because of aerodynamic effects, there is some dimensionless drag coefficient C_d that multiplies $\rho A v^2$ to give the drag force $F_d = (1/2)\rho v^2 C_d A$. This is just the force I wrote down for a hand out the car window, Fig. 2. For a sphere, $C_d \sim 0.1$, while for a biker, $C_d \sim 0.9$.

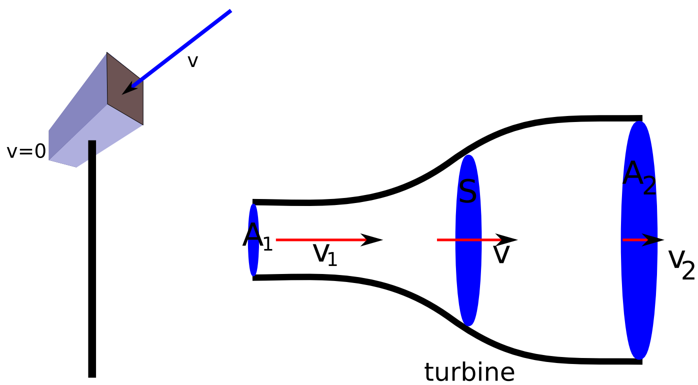

Figure 5: **Left:** an optimistic wind generator design. Here, air enters at some velocity, but exits with zero velocity, so all of the energy of the incoming wind is extracted. Yet, if the output velocity is zero, how does any air flow into such a hypothetical generator? Wind must exit the generator with some velocity, and hence not all of its power can be used. **Right:** a schematic of the flow around a large turbine. The turbine extracts energy from the wind, so the air slows down. We also know that if 10 lb/sec flows into the turbine, 10 lb/sec must flow out because the turbine does not remove molecules from the air! If the density of the air is constant and the flow slows down, the flow must also broaden. In terms of equations (see reasoning in Fig. 1), the mass flow rate into the turbine and out of the turbine must be the same $\dot{m}_{\rm in} = \dot{m}_{\rm out}$, or, using the fact that $\dot{m} = \rho v A$ (the mass flow rate of air with density ρ moving at some velocity v through an area A), $\rho v_{\rm in} A_{\rm in} = \rho v_{\rm out} A_{\rm out}$. If the wind slows down, the area must increase (if the density is constant), producing a wake.