2 Interstellar turbulence

Eddie:
You know, it's funny... you come to someplace new, an'...
and everything looks just the same.
Willie:
No kiddin' Eddie.

— **Stranger than Paradise (1984)**
Jim Jarmusch

Contents

2.1 Fluid mechanics

This section is dedicated to giving the reader a brief introduction to the concept of turbulence emerging from fluid mechanics, by limiting ourselves to incompressible flows[1]. The equations presented here are largely taken from the following book: "turbulence, an introduction for scientist and engineers" of P. A. Davidson (Davidson, 2015). In this short overview, we derive the Navier-Stokes equation and the kinetic energy equation to establish the expression of the internal energy transfer rate which is of primary interest for describing the turbulence cascade acting in the ISM. We finally introduce the concept of turbulence by making a brief description of vorticity dynamics. For complementary discussions of these equations, we refer the reader to the Chapter 2 of Davidson (2015).

2.1.1 The Navier-Stokes equation

The continuity equation First, let us consider a fluid element of volume δV with a density ρ, moving at a local velocity $\mathbf{u}$. Its mass is

$$\delta m = \rho \delta V. \tag{2.1}$$

Assuming that this mass is conserved, its time-rate-of-change is zero and we have

$$\frac{D(\delta m)}{Dt} = 0. \tag{2.2}$$

Combining Eqs. 2.1 and 2.2, we have

$$\frac{D\rho}{Dt} + \rho \left[\frac{1}{\delta V} \frac{D(\delta V)}{Dt} \right] = 0 \tag{2.3}$$

where the term in bracket is the time rate of change of the volume of the fluid element, per unit volume. It can be easily shown (see Sect. 2.4 of Anderson (1995)) that it is the physical meaning of the divergence of the velocity $\boldsymbol{\nabla} \cdot \mathbf{u}$. Equation 2.3 becomes

$$\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{2.4}$$

This is the continuity equation. In the case of an incompressible fluid, Eq. 2.4 reduces to

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{2.5}$$

The momentum equation The Newton's second law applied to the same fluid element can be written in the form

$$(\rho \delta V) \frac{D\mathbf{u}}{Dt} = -\left(\boldsymbol{\nabla} p\right) \delta V + \text{viscous forces} \tag{2.6}$$

where the first term on the right hand is the net pressure force acting on the fluid element and the second term represents the forces arising from the viscous stresses[2]. The viscous stresses are

[1] It is now well known that turbulence in the interstellar medium is far from being considered incompressible. Nevertheless, this framework allows us to first understand the fundamental principles related to turbulence.

[2] Note that we will give a physical meaning of the viscous stresses in the following when introducing Newton's law of viscosity.

composed of shear stresses and normal stresses that, if we consider a cubic fluid element, are noted τ_{xy}, τ_{xz}, ..., τ_{xx}, τ_{yy} and τ_{zz}. Any imbalance in stress arising between faces of a fluid element implies a net viscous force. As a result, the fluid element experiences a deformation that leads to a change in its trajectory. The viscous force in each direction i of the cube is then proportional to the sum of the variation of stresses (dependent of i) between the top and the bottom of each faces. Following the summation over repeated index i and j (convention), the viscous forces are

$$f_i = \frac{\partial \tau_{ji}}{\partial x_j} \, \delta V. \tag{2.7}$$

Combining Eqs. 2.6 and 2.7, we obtain

$$\rho \, \frac{D\mathbf{u}}{Dt} = -\, \boldsymbol{\nabla} p \, + \, \frac{\partial \tau_{ji}}{\partial x_j}. \tag{2.8}$$

We shall now use Newton's law of viscosity (shear stress is directly proportional to velocity gradient) to link τ_{ij} to the rate of deformation of the fluid element:

$$\tau_{ij} = \rho \, \nu \left\{ \frac{\partial u_i}{\partial x_j} \, + \, \frac{\partial u_j}{\partial x_i} \right\}. \tag{2.9}$$

Note that the coefficient of proportionality is simply the absolute viscosity μ=$\rho\nu$, where ν is the kinematic viscosity of the fluid. Physically, we can get an idea of what a viscous stress represents by considering a shear flow $\mathbf{u}$=($u_x(y)$,0,0). Each layer of fluid slides over each other, causing a deformation of fluid elements. In this case, the viscous stress τ_{yx} is proportional to a rate of sliding (or distortion rate) which is simply the velocity gradient $\frac{\partial u_x}{\partial y}$ and the coefficient of proportionality is the absolute viscosity μ of the fluid. Introducing the *strain-rate tensor* S_{ij}

$$S_{ij} = \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} \, + \, \frac{\partial u_j}{\partial x_i} \right] \tag{2.10}$$

we rewrite τ_{ij} in its well-know compact form

$$\tau_{ij} = 2 \rho \nu S_{ij} \tag{2.11}$$

Substituting Eq. 2.11 in Eq. 2.8 with ρ constant, we obtain the so-called Navier-Stokes equation

$$\frac{D\mathbf{u}}{Dt} = -\, \boldsymbol{\nabla} \left(\frac{p}{\rho} \right) \, + \, \nu \, \boldsymbol{\nabla}^2 \mathbf{u} \tag{2.12}$$

Note that Eq. 2.12 is expressed using the convective derivative D( )/Dt used in Eq. 2.3, that takes into account both temporal and spatial variations of each component of the vector field $\mathbf{u}$. The convective derivative can be developed using the chain rule (see Sect. 2.1.2 of Davidson (2015)) to obtain an explicit formula of the acceleration of the fluid element

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} \, + \, (\mathbf{u} \, . \, \boldsymbol{\nabla}) \, \mathbf{u} \tag{2.13}$$

The Navier-Stokes equation presented in Eq. 2.12 then becomes

$$\frac{\partial \mathbf{u}}{\partial t} \, + \, (\mathbf{u} \, . \, \boldsymbol{\nabla}) \, \mathbf{u} = -\, \boldsymbol{\nabla} \left(\frac{p}{\rho} \right) \, + \, \nu \, \boldsymbol{\nabla}^2 \mathbf{u} \tag{2.14}$$

We can already note that terms $(\mathbf{u}\,.\,\boldsymbol{\nabla})\,\mathbf{u}$ and $\nu\,\boldsymbol{\nabla}^2\mathbf{u}$ of Eq. 2.14 can be, respectively, seen as an advection term and a diffusion term. We will see in the following Sect. 2.2 that the development from a laminar to a turbulent regime depends on the ratio of these two terms, defined as the Reynolds number Re. It is important to note here that the advection term is quadratically non-linear in $\mathbf{u}$. It is via this non-linearity that instabilities can grow and eventually, if Re is sufficiently high, cause turbulence.

The energy equation The third and last fundamental equation of fluid mechanics results from the first law of thermodynamics: the total energy of the moving fluid element is conserved. However, its derivation is not of primary interest for the present chapter and we shall come back to it in the following Chapt.3 to understand the concept of thermal instability in astrophysical plasma. Instead, we focus here on the kinetic energy equation which allows us to introduce the concept of dissipation of mechanical energy per unit of mass.

2.1.2 Kinetic energy equation

Let us now derive the kinetic energy equation by multiplying Eq. 2.12 by $\mathbf{u}$

$$\mathbf{u}\,.\,\frac{D\mathbf{u}}{Dt} = \frac{D}{Dt}\left(\frac{u^2}{2}\right) = -\,\boldsymbol{\nabla}\,.\,\left[\frac{p}{\rho}\,\mathbf{u}\right] + \nu\,\mathbf{u}\,.\,\left(\boldsymbol{\nabla}^2\mathbf{u}\right) \tag{2.15}$$

The second term of the right hand side of Eq. 2.15 can be decomposed as

$$\nu\,\mathbf{u}\,.\,\left(\boldsymbol{\nabla}^2\mathbf{u}\right) = u_i\,\frac{\partial}{\partial x_j}\,[\tau_{ij}\,/\,\rho] = \frac{\partial}{\partial x_j}\,[u_i\,\tau_{ij}\,/\,\rho] - 2\nu S_{ij}S_{ij}. \tag{2.16}$$

Equation 2.15 then becomes

$$\frac{\partial\left(u^2\,/\,2\right)}{\partial t} = -\,\boldsymbol{\nabla}\,.\,\left[\left(u^2\,/\,2 + p/\rho\right)\mathbf{u}\right] - \frac{\partial}{\partial x_j}\,[u_i\,\tau_{ij}\,/\,\rho] - 2\nu S_{ij}S_{ij}. \tag{2.17}$$

Integrating Eq. 2.17 over an arbitrary volume V gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \left(u^2\,/\,2\right)\,\mathrm{d}V = \underbrace{-\int_V \boldsymbol{\nabla}\,.\,\left[\left(u^2\,/\,2\right)\mathbf{u}\right]\,\mathrm{d}V}_{\text{- (rate at which kinetic energy is transported across the boundary)}} \tag{2.18}$$

$$\underbrace{-\int_V \boldsymbol{\nabla}\,.\,\left[(p/\rho)\,\mathbf{u}\right]\,\mathrm{d}V}_{\text{+ (rate at which the pressure forces do work on the boundary)}} \tag{2.19}$$

$$\underbrace{-\int_V \frac{\partial}{\partial x_j}\,[u_i\,\tau_{ij}\,/\,\rho]\,\mathrm{d}V}_{\text{+ (rate at which the viscous forces do work on the boundary)}} \tag{2.20}$$

$$\underbrace{-\int_V 2\nu S_{ij}S_{ij}\mathrm{d}V}_{\text{+ (rate of loss of mechanical energy to heat)}} \tag{2.21}$$

where the meaning of the last term of the right hand side is directly inferred from the conservation of energy. When applied to a small volume δV, this term is the dissipation of mechanical energy per unit of mass and is noted

$$\epsilon = 2\nu S_{ij} S_{ij}. \tag{2.22}$$

This quantity is of primary interest since, as we will see in the following Sect. 2.3, it allows one to characterize the turbulence cascade acting in the fluid. It is also possible, after some algebra, to write the total rate of dissipation of mechanical energy as function of the curl of the velocity field $\mathbf{u}$

$$\int \epsilon \mathrm{d}V = \nu \int (\boldsymbol{\nabla} \times \mathbf{u})^2 \, \mathrm{d}V = \nu \int \boldsymbol{\omega}^2 \mathrm{d}V \tag{2.23}$$

where ω=$\boldsymbol{\nabla} \times \mathbf{u}$ is the vorticity field. We can anticipate via this result that the vorticity field ω plays an important role in describing the properties of a turbulent flow since it is directly related to the rate of dissipation of mechanical energy. Note that ω can be seen as a measure of the local rotation of a fluid element. Therefore, a fluid element can be distorted/strained at a rate S_{ij} **and** rotated at a rate ω. These two quantities are naturally linked by the gradient of the velocity field at any point in the fluid

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = S_{ij} - \frac{1}{2}\epsilon_{ijk}\boldsymbol{\omega} \tag{2.24}$$

where ϵ_{ijk} is the Levi-Civita symbol.

2.1.3 Vorticity dynamics

The idea that the vorticity field plays an important role in describing turbulence can be exploited by looking at its dynamics. In particular, we shall compare in this section the behaviour of governing equations of velocity and vorticity fields. This comparison will allow us to develop a first definition of turbulence.

Let us write its governing equation[3]

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times [\mathbf{u} \times \boldsymbol{\omega}] + \nu \boldsymbol{\nabla}^2 \boldsymbol{\omega} \tag{2.25}$$

that can also, using the identity

$$\boldsymbol{\nabla} \times (\mathbf{u} \times \boldsymbol{\omega}) = (\boldsymbol{\omega} \,.\, \boldsymbol{\nabla})\, \mathbf{u} - (\mathbf{u} \,.\, \boldsymbol{\nabla})\, \boldsymbol{\omega} \tag{2.26}$$

be rewritten as

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \,.\, \boldsymbol{\nabla})\, \mathbf{u} + \nu \boldsymbol{\nabla}^2 \boldsymbol{\omega}. \tag{2.27}$$

It is important to note that unlike the governing equation of the velocity field $\mathbf{u}$ expressed in Eq. 2.14, Eq. 2.27 is only composed of a diffusion term and a term of creation of vorticity fuelled by variations of the velocity field along the vorticity tubes, and that these two mechanisms are *local*. Therefore, unlike the velocity field, the vorticity can only be locally spread.

[3] The vorticity equation can be easily obtained by rewriting Eq. 2.14 using the identity $\boldsymbol{\nabla}\left(\mathbf{u}^2 \,/\, 2\right) = (\mathbf{u} \,.\, \boldsymbol{\nabla})\, \mathbf{u} + \mathbf{u} \times \omega$ and by taking its curl.

To understand why the evolution of velocity field is non-local, it is useful to compare Eqs. 2.14 and 2.27 which respectively are the governing equations of the velocity field $\mathbf{u}$ and the vorticity field $\boldsymbol{\omega}$. The main difference between Eqs. 2.14 and 2.27 is that the governing equation of the velocity field contains the term $-\boldsymbol{\nabla}\,(p/\rho)$. To better understand its nature and impact on the fluid, we can take the divergence of Eq. 2.12. In this way, we can relate directly the pressure field to the velocity field. It follows

$$\boldsymbol{\nabla}^2\,(p\,/\,\rho) = -\,\boldsymbol{\nabla}\,.\,(\mathbf{u}\,.\,\boldsymbol{\nabla}\mathbf{u}) \tag{2.28}$$

that can be reversed using the Biot-Savart law, giving

$$p(\mathbf{x}) = \frac{\rho}{4\pi}\int \frac{[\boldsymbol{\nabla}\,.\,(\mathbf{u}\,.\,\boldsymbol{\nabla}\mathbf{u})]'}{|\mathbf{x}\,-\,\mathbf{x}'|}\,\mathrm{d}x'. \tag{2.29}$$

The pressure field can be calculated at every point of space using the velocity field $\mathbf{u}$. An important property of incompressible fluid is that pressure waves travel at infinite speed. The information contained in $p(\mathbf{x})$ is therefore instantaneously transferred to the entire fluid. It makes the pressure field a non-local one, unlike the vorticity field $\boldsymbol{\omega}$. In other words, any modification of $\mathbf{u}$ in space is instantaneously felt by the fluid whose response is controlled, again at any point in space, by the term $-\boldsymbol{\nabla}\,(p/\rho)$ in Eq. 2.14. This instantaneous propagation of the information makes a representation of $\mathbf{u}$ impossible in terms of sub-regions of the fluid. On the other hand, it is natural to think about the vorticity field as a group of local regions called vortices evolving together in space. In general, these vortices are called "eddies". Following Davidson (2015), we shall now be able to define turbulence, as suggested by Stanley Corrsin in 1961, by the following

> *Incompressible hydrodynamic turbulence is a spatially complex distribution of vorticity which advects itself in a chaotic manner in accordance with Eq. 2.27. The vorticity field is random in both space and time, and exhibits a wide and continuous distribution of length and time scales.*

2.1.4 The closure problem of turbulence

We have seen previously that the pressure p can be determined instantaneously using the velocity field u using Eq. 2.29. It therefore appears that for incompressible flows, the Navier-Stokes equation presented in Eq. 2.14 is not anymore a function of p and $\mathbf{u}$ but can be formally written as

$$\frac{\partial \mathbf{u}}{\partial t} = F(\mathbf{u}) \tag{2.30}$$

where $F(\mathbf{u})$ contains the inertial, pressure and viscous forces. It turns out that Eq. 2.30 is deterministic and can be solved over time to obtain $\mathbf{u}(\mathbf{x}, t)$. So why is it well known that "turbulence is the most unsolved problem of classical physics" (Richard Feynman) ? The answer to that question is called *the closure problem*. Because of the chaotic behaviour of $\mathbf{u}(\mathbf{x}, t)$ in the turbulent regime, most of the turbulence theories developed so far are based on statistical modelling, involving the *Reynolds decomposition*

$$\mathbf{u}(\mathbf{x}, t) = \overline{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(\mathbf{x}, t) \tag{2.31}$$

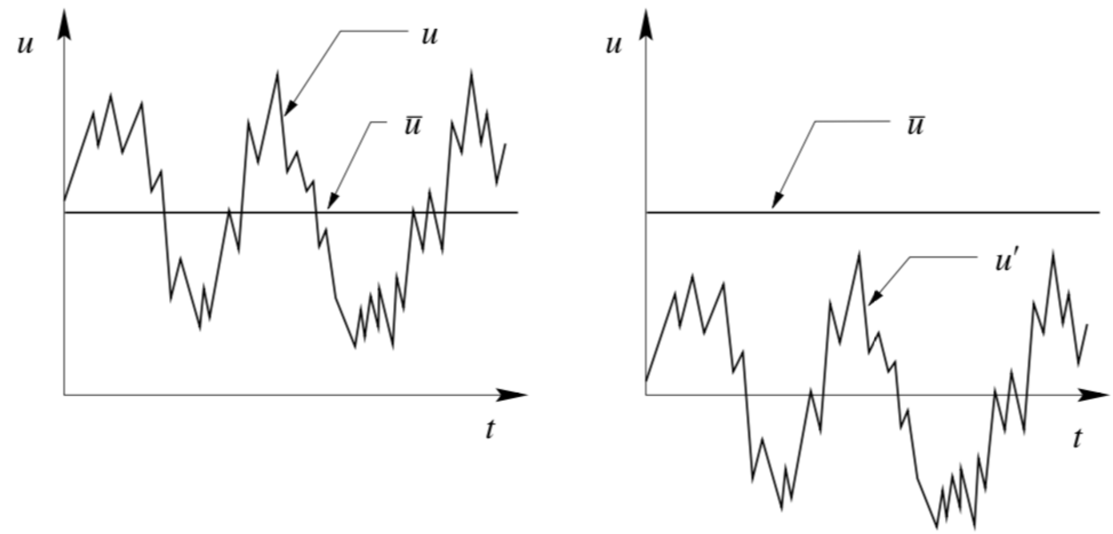

Fig. 2.1.: Plots of parts of Reynolds decomposition. *From Introductory Lectures on Turbulence: Physics, Mathematics and Modeling (James M. McDonough)*

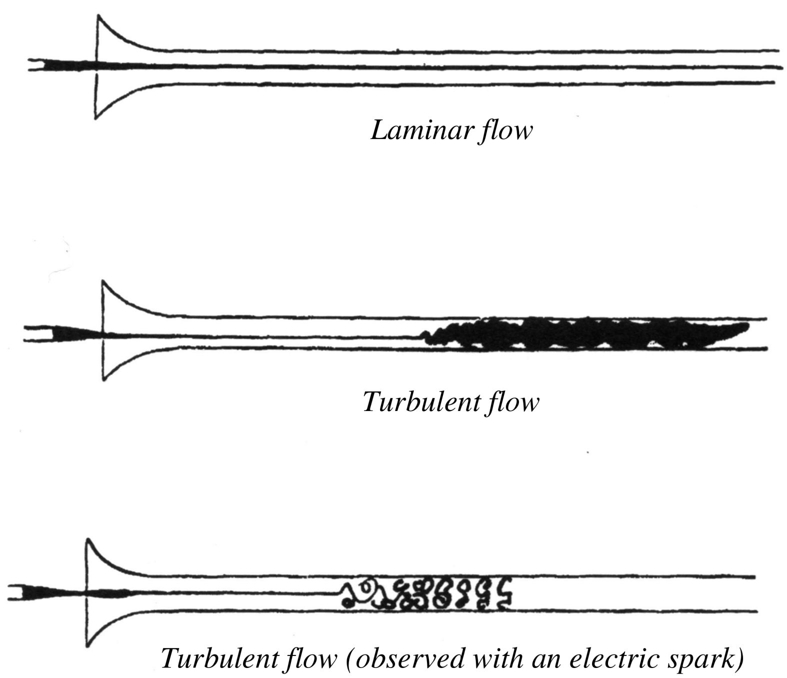

Fig. 2.2.: Sketch of the transition from laminar (top) to turbulent flow (middle) from Reynolds (1883). The bottom panel shows the same turbulent flow when observed with the light of an electric spark to reveal the "eddies" (now viewed as vorticies) of the flow.

where $\overline{\mathbf{u}}$ is the mean flow and $\mathbf{u}'$ is a random fluctuation also called the "*fluctuating part*". Figure 2.1 shows a sketch of the temporal evolution of $\overline{\mathbf{u}}(\mathbf{x})$ and $\mathbf{u}'(\mathbf{x}, t)$. The traditional way of solving Eqs. 2.14

Fig. 2.3.: Drawing made by Leonardo da Vinci (*circa* 1510) about the movement of water.

statistically is by using a decomposition of the form of 2.31 involving in particular the two-point and three-point correlation function[4]. The formulation of an equation of the three-point correlation function involves the four-point correlation function, and so on. The set of equations inferred from this method is therefore not a closed system. It turns out that it is a direct consequence of the non-linearity of the term $(\mathbf{u}\,.\,\boldsymbol{\nabla})\,\mathbf{u}$, which is also mostly responsible for the development of turbulence. Therefore, no exact solution can be found for the velocity field $\mathbf{u}(\mathbf{x},\mathbf{t})$ in space and time, even when considering its statistical properties. We will limit ourselves here to mentioning that a wide variety of approximations have been developed to close this system since the first try by Joseph Valentin Boussinesq at the end of the nineteenth century.

2.2 Reynolds's experiment

Now that we have an idea of the equations that govern incompressible fluids, we can ask ourselves under what precise conditions turbulence develops to form this collection of "eddies" called vortices ? Or in other words, under what conditions do we switch from a laminar regime to a turbulent regime ? Osborne Reynolds, in 1883, was the first to address the question by studying a simple flow passing through a pipe. The sketch of his experiment is shown in Fig. 2.2 where the top and middle panels represent the laminar and turbulence cases. He argued that the transition of turbulence represented here depends on the viscosity ν of the fluid, the velocity u of the flow and the diameter d of the pipe

[4]The notion of correlation function will be introduced in following Sect. 2.4.1

which corresponds to the characteristic length l of the flow. The dimensionless combination of these quantities is now called the Reynolds number Re

$$Re = \frac{ul}{\nu}. \tag{2.32}$$

Reynolds found that turbulence appears at Re ~2000 if initial perturbations applied to the flow are not too small. On the other hand, when minimizing these perturbations, the flow can remains in a laminar regime up to $Re \sim$ 13 000. This wide range over which turbulence may eventually appear highlights the crucial role played by the amplitude of initial disturbances.

A modern interpretation of the Reynolds number can be obtained with a simple dimensional analysis of the ratio of the advection term and the diffusion term of Eq. 2.14. Using the characteristic length l of the system, the ratio is approximately

$$\frac{(\mathbf{u}\,.\,\boldsymbol{\nabla})\,\mathbf{u}}{\nu\,\boldsymbol{\nabla}^2\mathbf{u}} \sim \frac{u^2/l}{\nu u/l^2} = \frac{ul}{\nu} = Re \tag{2.33}$$

As shown in Fig. 2.2 (bottom), when observed with the light of an electric spark, the turbulent regime described by Reynolds results in a collection of "eddies" (vorticies). It is interesting to note, however, that the first visual representation of a turbulent flow should be attributed to Leonardo da Vinci (~1510) whose infinite detail drawing shown in Fig. 2.3 exhibits a complex collection of "eddies" emerging from a laminar flow of water. It is also astonishing to read the annotation associated to the drawing,

> *Observe the motion of the surface of the water which resembles that of hair, and has two motions, of which one goes on with the flow of the surface, the other forms the lines of the eddies; thus the water forms eddying whirlpools one part of which are due to the impetus of the principal current and the other to the incidental motion and return flow.*

which is reminiscent of Eq. 2.31.

2.3 The phenomenology of Richardson and Kolmogorov

One striking thing that appears when you look at da Vinci's drawing is the variety of size of the eddies. It is now well known, as we will see in the following, that the size of the largest eddies is comparable to the characteristic length l of the flow and the size of the smallest eddies depends on the Reynolds number Re. It was by attempting to describe the properties of this variety of "eddies" that Richardson introduced the concept of energy cascade at high Re. His commentary on the behaviour of clouds in the atmosphere (1922) is probably the most quoted since the emergence of turbulence theories.

> *One gets a similar impression when making a drawing of a rising cumulus from a fixed point; the details change before the sketch can be completed. We realize that big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity.*

The idea has evolved since 1922, thanks in particular to Geoffrey Ingram Taylor and Andrey Nikolaevich Kolmogorov, whose contributions allow us to express the modern vision of this energy cascade.

When instabilities are generated on a large scale in the flow, this creates eddies (a collection of vortices). These coherent structures in space can then themselves be subject to new instabilities, which cause the transfer of their respective energies (ideally without creation or destruction because the viscous stresses acting on large scales are negligible) to smaller scales. In other words, the eddies break into smaller ones. The same process of energy accumulation subjected to new instabilities generates a new transfer on a lower scale and so on until the viscous forces become dominant (i.e. $Re \sim$1) and dissipate the energy.

This vision of the energy cascade led Kolmogorov to introduce in 1941 the two following hypothesis:

1. At sufficiently high Reynolds numbers, there is a region of high wave numbers, where the turbulence is statistically in equilibrium and uniquely determined by the dissipation of mechanical energy per unit of mass ϵ and the viscosity of the fluid ν. This state of equilibrium is universal and at this state, the turbulence is statistically homogeneous and locally isotropic.
2. At sufficiently high Reynolds numbers, the statistics of the motions of scale l in the range $l_0 \ll l \ll \eta$ have a universal form that is uniquely determined by the dissipation of mechanical energy per unit of mass ϵ, independent of the viscosity of the fluid ν.

These are now called *Kolmogorov's hypothesis*. The range $l_0 \ll l \ll \eta$ is called the *inertial range* where l_0 is the *integral length scale* and η is the *Kolmogorov length scale*. In other words, it corresponds to the range of scales where the statistical properties of the flow are self-similar. Having this in mind, it is now possible with a simple dimensional analysis to determine the dissipation scale η of the flow. Let us consider the turn-over time τ_{l_0}=l_0/u_0 of the largest eddies which correspond approximately to their timescale or lifespan. The rate of energy per unit mass transferred to the next lower scale is then

$$\Pi \sim u_0^2/(l_0/u_0) = u_0^3/l_0. \tag{2.34}$$

This was introduced by Taylor in 1935, whose idea was that a large eddy loses a significant fraction of its kinetic energy within one eddy turn-over time. Considering now the velocity of the smallest eddies v, the rate of dissipation associated is given by Eq. 2.22 and can be approximated using $S_{ij} \sim v/\eta$ as

$$\epsilon \sim \nu S_{ij} S_{ij} \sim \nu(v^2/\eta^2). \tag{2.35}$$

The energy cascade as expressed above (no accumulation of energy at any intermediate scale, i.e. statistically steady turbulence) implies that the energy transfer rate is constant over scales and so Π=ϵ,

$$u^3/l \sim \nu(v^2/\eta^2). \tag{2.36}$$

Knowing that the Reynolds number on the dissipation scale is

$$Re_\eta = v\eta/\nu \sim 1, \tag{2.37}$$

We obtain the following relations

$$\eta \sim l_0 Re^{-3/4} \sim (\nu^3/\epsilon)^{1/4} \tag{2.38}$$

and

$$v \sim u_0 Re^{-1/4} \sim (\nu\epsilon)^{1/4} \tag{2.39}$$

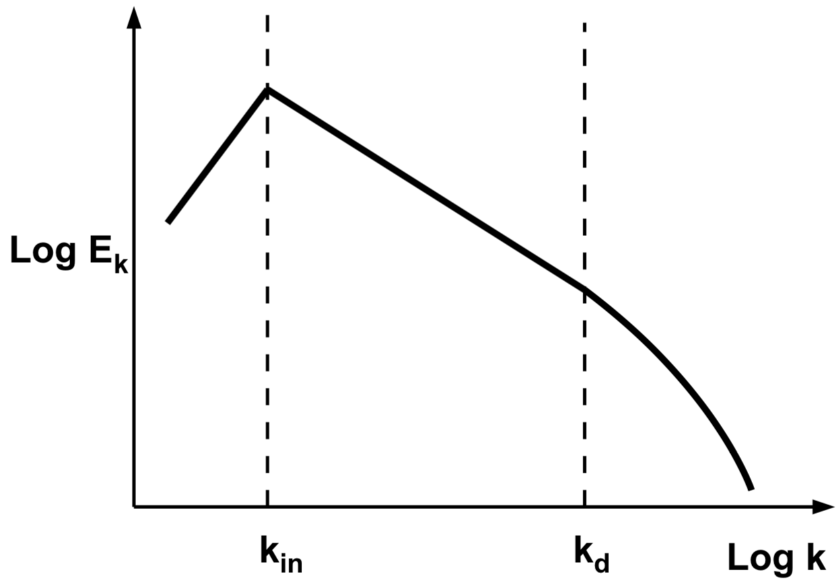

Fig. 2.4.: From Saury (2012): Sketch of the energy spectrum of a fully developed turbulence where k_{in} ~$1/l_0$ and $k_d \sim 1/\eta$ are the limit of the inertial range.

where Re=$u_0 l_0/\nu$ is taken at large scales. It is useful to note that, since $\epsilon \sim u_0^3/l_0$ and ϵ is considered constant over the cascade, we have the proportionality relationship

$$u_l \propto l^{1/3} \tag{2.40}$$

where u_l is the characteristic velocity at scale l. Using Eq. 2.40 it follows that the scaling law for the dispersion velocity of u_l with a zero-mean distribution is

$$\sigma_{u_l} = \left\langle u_l^2 \right\rangle^{1/2} \propto l^{\beta}. \tag{2.41}$$

where β=1/3. This relation is well-know in astrophysics as the "σ_{u_l}-l relation". In particular, this was observed for the first time in HI and molecular clouds by Richard Larson (Larson, 1979; Larson, 1981). Therefore, it is often referred to as Larson's law in molecular clouds. It is interesting to note that the exponent β can vary for different type of turbulence such as compressible turbulence (see Boldyrev (2002) and references within).

With Eq. 2.40 in hand, it becomes straightforward to infer the scaling law of the kinetic energy $E(k)$, where k ~$1/l$ is the wavenumber associated to the length l. Since $E(k)k \sim u_l^2$, we have

$$\epsilon \sim u_l^3/l \sim k\,(E(k)k)^{3/2} \sim E(k)^{3/2}k^{5/2} \tag{2.42}$$

The kinetic energy spectrum in the spectral range is then

$$E(k) \sim \epsilon^{2/3}k^{-5/3}. \tag{2.43}$$

Equation 2.43 is called the Kolmogorov's "5/3" law. A sketch of this energy spectrum in the inertial range is presented in Fig. 2.4.

2.4 Statistical properties of turbulence

We present in this section a brief overview of the Fourier analysis of homogeneous turbulence based on the Chapter 5 of Lesieur (1987). The formalism presented here allows us to introduce two fundamental tools used for studying the statistical properties of turbulence, the *spectral* tensor and the *second-order structure function*. We refer the reader to Lesieur (1987) for complementary discussions.

2.4.1 Spectral tensor

In this section, we introduce the spectral tensor $U_{ii}(\mathbf{k})$ of homogeneous turbulence which is an observable quantity in interstellar turbulence. In particular, we shall derive here the relation between $U_{ii}(\mathbf{k})$ and the kinetic energy spectrum $E(k)$ derived in Eq. 2.43. Note that $\hat{U}_{ii}(\mathbf{k})$ is usually called *power spectrum* in interstellar turbulence and we shall refer to it using the notation $P(k)$ in the following chapters. It is a quantity of primary interest since it is measurable using astronomical observation.

We assume here a purely solenoidal velocity field $u(\mathbf{r})$, meaning that we are still working under the incompressibility condition. Furthermore, we assume that the field is isotropic and therefore homogeneous (because a translation can be decomposed as the product of two rotations, see after the definition of homogeneity and isotropy). Considering these assumptions, the spectral analysis of this particular case of turbulence can be easily described in the Fourier space by the following development. Let us consider $\langle u_i(\mathbf{x_1})\, u_j(\mathbf{x_2})\rangle$, the velocity correlation tensor at point $\mathbf{x_1}$ and $\mathbf{x_2}$. The assumption of homogeneity implies that all quantities built with $(\mathbf{x_1}, \mathbf{x_2})$ are invariant by translation of the set $(\mathbf{x_1}, \mathbf{x_2})$. Therefore, the velocity correlation tensor is

$$\langle u_i(\mathbf{x_1})\, u_j(\mathbf{x_2})\rangle = \langle u_i(\mathbf{x_1}+\mathbf{r})\, u_j(\mathbf{x_2}+\mathbf{r})\rangle\,, \tag{2.44}$$

and the second order velocity correlation tensor is

$$U_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x_1})\, u_j(\mathbf{x_1}+\mathbf{r})\rangle\,. \tag{2.45}$$

The spectral tensor of homogeneous turbulence is given by the Fourier transform of the second order velocity correlation tensor

$$\hat{U}_{ij}(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^3 \int e^{-i\,\mathbf{k}\,.\,\mathbf{r}}\, U_{ij}(\mathbf{r})\, d^3\mathbf{r}. \tag{2.46}$$

Equation 2.46 can be simplified considering isotropy (all quantities build with $(\mathbf{x_1}, \mathbf{x_2})$ are invariant by rotation of the set $(\mathbf{x_1}, \mathbf{x_2})$) and incompressibility. Incompressibility condition implies $k_i\, \hat{U}_{ij}(\mathbf{k}) = k_j\, \hat{U}_{ij}(\mathbf{k}) = 0$. For isotropic turbulence, $\hat{U}_{ij}(\mathbf{k})$ must be an isotropic tensor and after some algebra described in Lesieur (1987), the spectral tensor, neglecting the helicity[5] term, can be written as a function of $E(k)$ the kinetic energy spectrum which is the density of kinetic energy at wavenumber k

$$\hat{U}_{ij}(\mathbf{k}) = \frac{1}{2}\hat{U}(k)\, Q_{ij}(\mathbf{k}) = \frac{E(k)}{4\,\pi\, k^2}\, Q_{ij}(\mathbf{k}) \tag{2.47}$$

where $\hat{U}(k)$ is the trace of the spectral tensor $\hat{U}_{ij}(\mathbf{k})$ and $Q_{ij}(\mathbf{k})$ is a projected tensor introduced thanks to the fact that for incompressible turbulence, the Fourier transform $\hat{\underline{u}}$ of the velocity is in the plane $\mathbf{\Pi}$ perpendicular to the vector $\mathbf{k}$, i.e $\mathbf{k}\,.\,\hat{\underline{u}}(\mathbf{k}) = 0$, and can be written as

$$Q_{ij}(\mathbf{k}) = \delta_{ij} \,-\, \frac{k_i\, k_j}{k^2} \tag{2.48}$$

with $\delta_{ij} = \delta_i^j = [i \,=\, j]$, using the Iverson bracket, is the Kronecker tensor. We note that for the case δ_i^i, it implies a summation over indice i and therefore $\delta_i^i = 3$. It follows

$$Q_{ii}(\mathbf{k}) = 3 \,-\, \frac{k_x^2 \,+\, k_y^2 \,+\, k_z^2}{k^2} = 2. \tag{2.49}$$

Therefore, according to this condition, the spectral tensor is equal to its trace and equation 2.47 becomes

$$\hat{U}_{ii}(\mathbf{k}) = \hat{U}(k) = \frac{E(k)}{2\,\pi\, k^2}. \tag{2.50}$$

We can now rewrite the Kolmogorov "5/3" law in term of $\hat{U}(k)$. Equation 2.43 becomes

$$\hat{U}(k) \sim \frac{\epsilon^{2/3} k^{-5/3}}{2\pi k^2} \sim \frac{1}{2\pi}\epsilon^{2/3} k^{-11/3}. \tag{2.51}$$

2.4.2 Second order velocity structure function

In this section, we derive the second order velocity structure function $S_2(r)$ which is, as the spectral tensor, measurable using astronomical observation. We shall derive here the link between $S_2(r)$ and the kinetic energy spectrum $E(k)$.

The *second-order structure function* is defined as

$$S_2(r) = \left\langle \left[u(\mathbf{x}+\mathbf{r}) - u(\mathbf{x})\right]^2 \right\rangle \tag{2.52}$$

where $\mathbf{r}$ is a vector pointing between two nearby locations of "measurement" of u and $\langle . \rangle$ denotes a spatial ensemble average. For the velocity field $\mathbf{u}$ and the vorticity field $\boldsymbol{\omega}$, the *transverse* and *longitudinal* structure functions can be defined as

$$S_{2\perp}(r) = \left\langle \left[u_\perp(\mathbf{x}+r\hat{\mathbf{e}}_\perp) - u_\perp(\mathbf{x})\right]^2 \right\rangle \tag{2.53}$$

[5] Note that we deliberately omit to introduce here the concept of helicity since it is not of primary interest for the understanding of this section.

and

$$S_{2\parallel}(r) = \left\langle \left[u_{\parallel}(\mathbf{x} + \mathbf{r}\hat{\mathbf{e}}_{\parallel}) - u_{\parallel}(\mathbf{x}) \right]^2 \right\rangle \tag{2.54}$$

where $\hat{\mathbf{e}}_{\parallel}$ and $\hat{\mathbf{e}}_{\perp}$ are unit vectors in the transverse and perpendicular direction with respect to $u_{\perp}$ and $u_{\parallel}$. It is interesting to note that in astronomy, only the projection of the velocity field along the line of sight (i.e. perpendicular to the plane of the sky) is generally available. This is particularly the case when hyper-spectral imaging is used to observe atomic and molecular lines in the interstellar medium. Any structure function calculated from hyper-spectral data is therefore a transverse structure function.

Generally, $S_2(r)$ can be approximately seen as the energy per unit of mass contained in eddies of size r or less. A simple way of getting a hint of the link between $S_2(r)$ and an energy is by considering the Taylor expansion of the velocity field u at small r,

$$u(\mathbf{x} + \mathbf{r}) - u(\mathbf{x}) \simeq \frac{\partial u}{\partial x} r. \tag{2.55}$$

The right hand side can be seen as a small-scale fluctuation corresponding to a change in u over a small distance r. The square of Eq. 2.55 therefore represents the energy of this small fluctuation. The formal relationship between the spectral tensor and the second-order structure function can be derived using the spectral tensor $\hat{U}_{ij}(\mathbf{k})|_{i=j}$. Equation 2.52 becomes

$$S_2(r) = 2 \int \hat{U}_{ii}(\mathbf{k}) \, \left(1 - e^{i\,\mathbf{k}\,.\,\mathbf{r}}\right) \, d^3\mathbf{k}. \tag{2.56}$$

To simplify this equation, let us now express it using spherical coordinates, $d^3k = k^2\, dk\, d(cos\theta)\, d\phi$. Combining Eq. 2.50 and 2.56, it follows

$$S_2(r) = 2 \int_{-1}^{+1} d(cos\theta) \int dk\, E(k) \, \left(1 - e^{i\,\mathbf{k}\,.\,\mathbf{r}}\right). \tag{2.57}$$

If $\mathbf{k}$ is aligned with (O_z), then we can decompose the exponential $e^{i\,\mathbf{k}\,.\,\mathbf{r}}$ as

$$e^{i\,\mathbf{k}\,.\,\mathbf{r}} = \sum_l (2\,l + 1)\, i^l\, j_l(kr)\, P_l(cos\theta) \tag{2.58}$$

where P_l is the Legendre Polynomial. Furthermore, since

$$\int_{-1}^{+1} d(cos\theta)\, P_l(cos\theta) = \begin{cases} 2, & \text{for } l = 0 \\ 0, & \text{for } l \geq 1 \end{cases} \tag{2.59}$$

the second order structure function can be written as

$$S_2(r) = 4 \int_0^{+\infty} dk\, E(k) \left(1 - \frac{sin\, kr}{kr}\right). \tag{2.60}$$

2.5 Turbulence in the ISM

2.5.1 Beyond the incompressible turbulence

It is important to keep in mind that all the formalism used previously results from the Navier-Stokes equation with very specific conditions, namely incompressibility and isotropy. Things get considerably more complicated when it comes to interstellar turbulence because assumptions used previously break down. In particular, the interstellar medium is compressible, magnetized and self-gravitating. Gravity, in general, will be introduced in Eq. 2.14 by adding a force $\mathbf{F}$ on the right hand side. The magnetic field, on the other hand, requires a rewriting of the Navier-stokes equations using magnetohydrodynamics (MHD). The fields associated with these components are distorted by the velocity field, which then provides feedback on it (Elmegreen and Scalo, 2004). We will limit ourselves here to saying that generally, all these considerations have the effect of changing the scaling laws that we have previously deduced in the incompressible case. Moreover, it is even possible to question the fluid approximation used in all cases mentioned so far when the *Kolmogorov scale* is very close to the mean free path of atoms and molecules constituting the fluid (Lequeux, 2012).

2.5.2 Energy injections

The complexity of turbulence that we have just mentioned remains, however, a field of research that has been studied extensively for decades from both a theoretical and experimental point of view by fluid dynamicists. The most important difficulty when it comes to interstellar turbulence is undoubtedly the wide variety of physical processes by which kinetic energy can be converted into turbulence. To illustrate this, we use here the list of the main sources compiled by Elmegreen and Scalo (2004):

1. Protostellar winds, expanding HII regions, O star and Wolf-Rayet winds, supernovae, and combinations of these producing superbubbles.
2. Shocks of spiral arms or bars due to Galactic rotation, the Balbus-Hawley (1991) instability, and the gravitational scattering of cloud complexes at different epicyclic phases.
3. Gaseous self-gravity through swing-amplified instabilities and cloud collapse.
4. Kelvin-Helmholtz and other fluid instabilities.
5. Galactic gravity during disk-halo circulation, the Parker instability, and galaxy interactions.
6. Sonic reflections of shock waves hitting clouds.
7. Cosmic ray streaming.
8. Field star motions.

It is striking to note that the energy injections via all these processes are carried out at very variable scales ranging from kiloparsec to tenths of a parsec. It is even more surprising to note that all the scaling laws observed so far in various environments of the interstellar medium are generally[6] single power laws. Note that most of these studies used density fields instead of velocity fields. Whatever the

[6] Note that Elmegreen et al. (2001) observed a broken power law in the Large Magellanic Cloud (LMC) but this behavior is more considered as a signature of the transition from 2D to 3D turbulence rather then a real energy injection or dissipation.

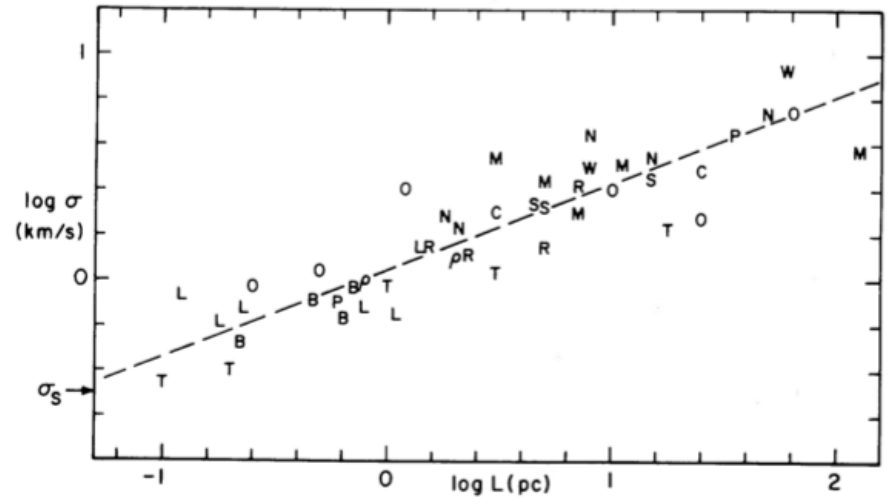

Fig. 2.5.: From Larson (1981): The three-dimensional internal velocity dispersion σ plotted versus the maximum linear dimension L of molecular clouds and condensations, based on data from Table 1; the symbols are identified in Table 1. The dashed line represents equation (1), and σ_s is the thermal velocity dispersion.

nature of the field studied, the signature of these processes in the power spectrum statistics remains non-trivial and not well understood. We will discuss this further in the following Sect. 2.5.3.

2.5.3 Observations of interstellar turbulence

Early works The first optical observations suggesting that the interstellar medium is turbulent date back to the middle of the 20th century. Ten years after Kolmogorov described the scaling laws of incompressible turbulence, astronomical observations of emission-line velocities in the Orion Nebula revealed their self-similar nature (Hoerner, 1951), suggesting that interstellar fluid follows the same statistics. Note that the second order velocity structure function introduced in Eq. 2.52 was used to perform the statistical analysis by measuring the scaling law[7] $S_{2\perp} \propto r^{\beta}$. In line with this idea, Hoerner (1951) introduced the idea of a hierarchical cloud structure in the interstellar medium. Several years after, still using the second order velocity structure function, Wilson et al. (1959) inferred from a sample of $\sim$ 10 000 lines-of-sight a higher exponent $\beta \sim$ 0.66, suggesting that it results from a compressible turbulence. Then, Miville-Deschenes et al. (1995) found $\beta \sim$ 0.8 in an HII region, suggesting again a compressible turbulence with a possible energy input from a stellar wind. In parallel with these observations, the theory of interstellar turbulence began. Based on Hoerner (1951) suggestions, Weizsäcker (1951) suggested that this hierarchy of structures is formed in interacting shock waves by large-scale supersonic turbulent flows and then dissipated at small-scales by atomic viscosity (Elmegreen and Scalo, 2004). As noted by Elmegreen and Scalo (2004), the vision proposed by Weizsäcker (1951) is similar to what is believed today. In this model, the density ρ_ν and the cloud size l_ν are connected by the following relation

$$\frac{\rho_\nu}{\rho_{\nu+1}} = \left(\frac{l_\nu}{l_{\nu+1}}\right)^{-3k_\nu} \tag{2.61}$$

[7] Note that following the same reasoning as for velocity dispersion, we find that β=1/3 for incompressible turbulence.

where k_ν is the degree of compression at the step ν, ρ_ν is the average density at the step ν being an integer increasing with the cloud size l_ν. This model was the first attempt to describe a turbulent astrophysical fluid by its density fluctuations instead of velocity fluctuations. Since this work, several models have aimed at understanding the scaling laws of the density field in a turbulent fluid. Notably, Von Weizsacker's model was the basis of the Flech model whose goal was to incorporate density fluctuations into predictions for power spectra of incompressible turbulence (Vogel, 2011).

Larson's law The study of interstellar turbulence underwent a considerable change when the first observations in the millimetre range appeared. In particular, the CO line revealed considerably large widths that were incompatible with the temperature of the gas ($T_k \sim$ 10 K) measured using ^{12}CO, ^{13}CO and OH (Lequeux, 2012). It turned out that this incompatibility was due to a turbulent broadening of the line which is quadratically added to its thermal broadening. It was in 1979 and 1981 (Larson, 1979; Larson, 1981) that Richard Larson published the first relationship between the line widths of atomic and molecular lines and the respective maximum linear dimension of the clouds, thus revealing the scaling laws of the turbulent cascade presented in Eq. 2.41 in different environments of the interstellar medium. Notably, he measured a scaling exponent $\beta \sim$ 0.38 close to an incompressible turbulence. Figure 2.5 from Larson (1981) shows this relationship for a set of molecular clouds in the Milky-Way. In contrast, Solomon et al. (1987) found $\beta \sim$ 0.5, suggesting a compressible turbulence. More recently, Miville-Deschênes et al. (2017) found $\beta \sim$ 0.63 for 8107 molecular clouds in the Milky-Way disk, closer to the result of Solomon et al. (1987). It is important to note here that this relationship is likely to be influenced also by the gravity of clouds, which can modify the value of the exponent. This is still a matter of debate.

Power spectrum of the integrated density The notion of a power spectrum of the density introduced here must be absolutely dissociated from the power spectrum introduced previously in Eq. 2.50 which was directly related to the energy cascade acting in the fluid. The fundamental reason for this is that the scaling law described above are derived for incompressible fluids. Nevertheless, as discussed in Sect. 2.5.1, interstellar fluids are often compressible and since density fields are observable quantities in astronomy, their statistical properties have generated some interest in characterizing interstellar turbulence. Using numerical simulation of isothermal, compressible and subsonic turbulence, Kim and Ryu (2005) showed that the power spectral index of the density field follows that of the velocity field, i.e. $\propto k^{-11/3}$. This work also showed that when simulating a supersonic regime, it causes the creation of small scale density fluctuations (due to shocks), which has the effect of flattening the power spectrum of the density field. It is interesting to note that the same behaviour was observed in numerical simulations of subsonic bi-stable turbulence of atomic gas (Saury et al., 2014; Gazol and Kim, 2010) which are highly non-isothermal cases. High density contrasts in these simulations result from the condensation of diffuse warm gas into cold dense clouds caused by the thermal instability. Even if it occupies a very small fraction of the volume, these dense structures are likely to have exactly the same impact, flattening the power spectrum of the density field.

Power spectrum of the centroid velocity The statistical properties of the 3D velocity field can be inferred from its 2D projection, also called the centroid velocity. However, the statistical properties of these fields can differ drastically since the centroid velocity results from a weighting by the density field (Ossenkopf et al., 2006). Several studies have shown that the statistics of the centroid velocity field reflects those of the velocity field only when the fluctuations of the density field are small, i.e. $\sigma(\rho)/\rho \lesssim$

1 (Miville-Deschênes et al., 2003b; Esquivel and Lazarian, 2005; Ossenkopf et al., 2006). It must be noted, for the same reason as discussed above for the density power spectrum, that this condition is very rarely satisfied for interstellar turbulence. For that reason, the statistical properties of the velocity field of interstellar turbulence is still not well characterized.