- In your homework, you showed that the relativistic Vlasov equation is equivalent to the diffusion-convection equation for CR transport (under the mentioned assumptions:

$$\frac{\partial F}{\partial t} - S_0(Z,p,t) = \frac{\partial}{\partial Z}\left[K_{\parallel}\frac{\partial F}{\partial Z}\right] - V\frac{\partial F}{\partial Z} + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\Gamma\frac{\partial F}{\partial p}\right] + \frac{p}{3}\frac{\partial V}{\partial Z}\frac{\partial F}{\partial p} \qquad (30)$$

Here we use a mixed coordinate system where the space coordinates $\vec{X} = (X,Y,Z)$ measured in the lab system and the momentum coordinates (p,μ) in the rest frame of the streaming plasma,

$$K_{\parallel}(Z,p,t) = \frac{v^2}{8}\int_{-1}^{1} d\mu\, \frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)} \quad \text{... spatial diffusion coefficient}$$

$$\Gamma(Z,p,t) = \frac{1}{2}\int_{-1}^{1} d\mu \left[D_{pp}(\mu) - \frac{D_{\mu p}^2(\mu)}{D_{\mu\mu}(\mu)}\right] \text{... momentum diffusion coefficient} \qquad (31)$$

$$V(Z,p,t) = \underbrace{U}_{\text{plasma bulk speed}} + \frac{1}{4p^2}\frac{\partial}{\partial p}\left(p^2 v A_1\right), \quad A_1 = \int_{-1}^{1} d\mu\,(1-\mu^2)\frac{D_{\mu p}(\mu)}{D_{\mu\mu}(\mu)} \text{... CR bulk speed}$$

- 1) We introduce continuous loss processes by adding the term $-\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2\dot{p}F\right)$, (32)
  with the appropriate negatively counted loss rate $\dot{p}$, and

  2) introduce catastrophic loss processes (i.e. hadronic p-p interactions) by adding the term $-\frac{F}{T_c(p,Z)}$, with the appropriate loss time T_c, and (33)

  3) note that

$$-\frac{\partial}{\partial Z}\left[VF\right] + \frac{1}{p^2}\frac{\partial}{\partial p}\left[\frac{p^3}{3}\frac{\partial V}{\partial Z}F\right] = \frac{p}{3}\frac{\partial V}{\partial Z}\frac{\partial F}{\partial p} - V\frac{\partial F}{\partial Z} \text{ to get the ...}$$

⇒ Generalized convection-diffusion equation = CR transport equation:

$$\frac{\partial F}{\partial t} - S(z,p,t) = \frac{\partial}{\partial z}\left[\kappa_{\parallel}\frac{\partial F}{\partial z} - VM\right] \quad \text{... spatial diffusion and convection}$$

$$+ \underbrace{\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2\,\Gamma\,\frac{\partial F}{\partial p}\right.}_{\substack{\text{momentum diffusion} \\ \text{(Fermi II - acceleration)}}} + \underbrace{\frac{p^3}{3}\frac{\partial V}{\partial z}F}_{\substack{\text{momentum convection} \\ \text{(Fermi I - acceleration)}}} \left. - \,p^2\dot{p}F\right) - \frac{F}{T_c(z,p)} \qquad (34)$$

continuous and catastrophic loss processes

- IC and synchrotron losses: $\dot{p} = -\frac{4}{3}\sigma_T\left(\frac{m_e}{m}\right)^2(\varepsilon_B + \varepsilon_{ph})\,\gamma^2 \qquad (35)$

with $\sigma_T = \frac{8\pi r_e^2}{3}$, $r_e = \frac{e^2}{m_e c^2}$

∼ radiative loss of baryons is suppressed by $\left(\frac{m_e}{m}\right)^2$ such that they can be neglected unless they are ultra-high energy CRs with energies $\gtrsim 10^{18}$ eV!

∼ relativistically charged particle experiences IC scattering with real or virtual photons (that are provided by the magnetic field → synchrotron radiation, photons are emitted in the forward direction into a narrow cone of half-angle γ^{-1} with respect to its momentum ∼ energy loss can be described as friction force in the opposite direction to its momentum

- spatial diffusion: is determined by magnetic irregularities / fluctuations ($\kappa_{\parallel} \propto D_{\mu\mu}^{-1}$), in a turbulent magnetic field with Kolmogoroff-type spectrum on small scales $\kappa \propto p^{-1/3}$ ∼ loss time scale $\propto p^{-1/3}$

  ∼ steepening of observed spectrum within Galactic disk by $p^{-1/3}$ relative to injected spectrum

- First order Fermi process (diffusive shock acceleration):

Need compressible flow with $\operatorname{div} \vec{v} < 0$, non-inertial entrainment due to deceleration of the scattering medium. We have inertial force $F_j = -p_i \left(\frac{\partial v_j}{\partial x_i} \right)$

$\sim$ accelerating power $P_{acc} = - \langle v_i p_j \rangle \frac{\partial v_j}{\partial x_i} = - \frac{vp}{3} \nabla \cdot \vec{v}$.

- Second order Fermi process: electro-magnetic fluctuations cause particles to scatter, because scattering agents are moving targets, particles gain and lo energy to first order at equal rates, net energy gain only second order in $\beta_A = \frac{v_A}{c}$.

- Catastrophic losses: 1) Coulomb and Ionization losses – works effectively at low energies where particles are removed into the thermal pool.

  2) Hadronic, inelastic p-p interactions – production of charged and neutral pions that decay into secondaries (electrons, positrons, neutrinos, γ-rays)

  3) Spallation – destruction of atomic nuclei upon collision with a CR particle

  → production of debris (single nucleon, or light nuclei)

  → responsible for high galactic abundances of Li, Be, B

1.3. Fermi I-process – diffusive shock acceleration

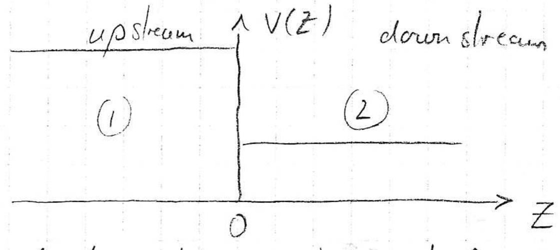
1.3.1 Picture and Assumptions

An energetic particle with a higher individual velocity than the plasma flow in the shock frame may be able to travel against the stream and gets trapped at the shock and experiences the system as a permanently compressing flow. Particles gain energy in going forth and back the shock front, in one cycle (up-, down-, upstream) a relativistic particle gains a fraction $\frac{V_{sh}}{c}$ in momentum. Due to convection of the flow, the CR population loses a fraction $\frac{V_{sh}}{c}$ of members that escape downstream.

We will derive the spatial and momentum distribution from the CR transport equation.

Assumpt.:
- Neglect. Fermi II + continuous / catastrophic losses
- sharp shock transition (shock width $\ll$ CR diffusion length $\frac{\kappa}{v}$)

$$V(z) = V_1 + (V_2 - V_1)\,\theta(z)$$
$$\frac{\partial V}{\partial z} = (V_2 - V_1)\,\delta(z) \qquad (36)$$



velocity profile in shock rest-frame

⇝ Assuming this kinematic structure of the shock implicitly accounts for shock heating through dissipation and acts therefore as a thermodynamical background model.

CR transport equ'n for stationary solution; $n_{CR,inj} \equiv \eta_{CR}\, n_{th,1} = \int G(p)\, 4\pi p^2 dp$

$$V\frac{\partial F}{\partial z} - \frac{p}{3}\frac{\partial V}{\partial z}\frac{\partial F}{\partial p} = \frac{\partial}{\partial z}\left[\kappa(z,p)\frac{\partial F}{\partial z}\right] + V_1\, G(p)\,\delta(z) \qquad (37)$$

⇝ transport of CRs over shock

source term accounts for CR injection @ shock

⇝ thermal particles of M-B distribution are not explicitly followed, implicitly accounted for by $V(z)$-background model.

1.3.2. Spatial structure – CR precursor

Eqn. (37) can be integrated on both sides of the shock front. Convection opposes diffusion ahead of the shock, while it is not possible to balance convection against diffusion in the downstream when neglecting loss processes and accounting for diffusive shock acceleration ~ const. F in downstream

Sol'n.
$$F(z,p) = \begin{cases} F_1(p) + [F_2(p) - F_1(p)] \exp\left[-\int_z^0 \frac{V_1\, dz'}{\kappa(z',p)} \right], & z<0 \\ F_2(p) & z>0 \end{cases} \qquad (38)$$

with $F_1(p) = F(-\infty, p)$ and $F_2(p) = F(\infty, p) = F(0,p)$.

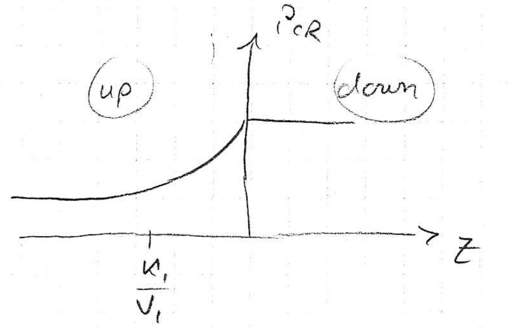
Proof for ①
- $-\frac{1}{3}\frac{\partial U}{\partial z} p \frac{\partial F}{\partial p} = -\frac{p}{3}(V_2 - V_1)\,\delta(z)\frac{\partial F}{\partial p} = 0 \begin{cases} z>0 \\ z<0 \end{cases}$
- $V_1 \frac{\partial F}{\partial z} = \frac{\partial}{\partial z}\left[\kappa \frac{\partial F}{\partial z}\right] \quad / \int dz$.

$$V_1 F = \kappa \frac{\partial F}{\partial z} + C_1 \quad ,$$

$$\underline{F(z,p) = F_1 + (F_2 - F_1)\exp\left[-\int_z^0 \frac{V_1\, dz'}{\kappa(z',p)}\right]} \qquad (39)$$

where $C_1 = -\ln(F_2 - F_1)$ from continuity of F across shock front!

The distribution function exponentially decreases from the shock front into the upstream regime over a diffusion length $\frac{\kappa_1}{V_1}$.



~ The CR pressure exponentially decreases into the upstream and develops a precursor over which the incoming plasma is adiabatically heated

~ "CR modified shock" where the original shock front is being replaced by a subshock!

1.3.3. Momentum power-law distribution

The flux of particles at a given momentum must be continuous. Integrating the transport equation (37) we obtain the continuity condition:

$$\frac{\partial}{\partial z}\left[\kappa \frac{\partial F}{\partial z} + \frac{V}{3}\frac{\partial F}{\partial \ln p}\right] = V\frac{\partial}{\partial z}\frac{\partial (F p^3)}{\partial p^3} + V_1 G(p)\,\delta_D(z) \tag{40}$$

$$\Rightarrow \left[\kappa(z,p)\frac{\partial F}{\partial z} + \frac{V}{3}\frac{\partial F}{\partial \ln p}\right]_{0_-}^{0_+} = V_1\, G(p)$$

Proof:

- $$\frac{\partial}{\partial z}\left[\frac{V}{3}\frac{\partial F}{\partial \ln p}\right] = \frac{p}{3}\frac{\partial V}{\partial z}\frac{\partial F}{\partial p} + V\frac{\partial}{\partial z}\left(\frac{p}{3}\frac{\partial F}{\partial p}\right) = \underbrace{\frac{1}{3}\frac{\partial V}{\partial z}\frac{\partial F}{\partial \ln p}}_{(*)} + V\frac{\partial}{\partial z}\frac{\partial F p^3}{\partial p^3} - V\frac{\partial F}{\partial z} \tag{41}$$

$$\frac{\partial (F p^3)}{\partial p^3} \underset{(dp^3 = 3p^2 dp)}{=} F + \frac{p^3}{3p^2}\frac{\partial F}{\partial p} = F + \frac{p}{3}\frac{\partial F}{\partial p}$$

- transport equation: $$V\frac{\partial F}{\partial z} - \underbrace{\frac{1}{3}\frac{\partial V}{\partial z}\frac{\partial F}{\partial \ln p}}_{(*)\text{ from (41)}} = \frac{\partial}{\partial z}\left(\kappa\frac{\partial F}{\partial z}\right) + V_1 G(p)\,\delta_D(z)$$

$$\cancel{V\frac{\partial F}{\partial z}} - \frac{\partial}{\partial z}\left[\frac{V}{3}\frac{\partial F}{\partial \ln p}\right] - \cancel{V\frac{\partial F}{\partial z}} + V\frac{\partial}{\partial z}\frac{\partial (F p^3)}{\partial p^3} = \frac{\partial}{\partial z}\left(\kappa\frac{\partial F}{\partial z}\right) + V_1 G(p)\,\delta_D(z)$$

$$\Rightarrow \frac{\partial}{\partial z}\left[\kappa(z,p)\frac{\partial F}{\partial z} + \frac{V}{3}\frac{\partial F}{\partial \ln p}\right] = V\frac{\partial}{\partial z}\frac{\partial (F p^3)}{\partial p^3} + V_1 G(p)\,\delta_D(z) \tag{42}$$

- rhs: 1) $$\lim_{\delta\to 0}\int_{-\delta}^{\delta} V\frac{\partial}{\partial z}\frac{\partial (F p^3)}{\partial p^3}\,dz = V\frac{\partial (F p^3)}{\partial p^3}\bigg|_{-\delta}^{+\delta} - \int \frac{\partial (F p^3)}{\partial p^3}\underbrace{\frac{\partial V}{\partial z}}_{=\Delta V\,\delta(z)}\,dz$$

$$= \Delta V\frac{\partial (F p^3)}{\partial p^3} - \Delta V\frac{\partial (F p^3)}{\partial p^3} = 0$$, assuming F is continuous!

2) $$\lim_{\delta\to 0}\int_{-\delta}^{\delta} G(p)\,\delta(z)\,dz = G(p)$$

$$(42) \rightsquigarrow \left[\kappa\frac{\partial F}{\partial z} + \frac{V}{3}\frac{\partial F}{\partial \ln p}\right]_{0_-}^{0_+} = V_1\, G(p) \qquad \text{q.e.d.} \tag{43}$$

Now join the two solutions up- and downstream of the shock while using the continuity equation in order to obtain the differential equation for the transmitted distribution function $F_2(p)$:

$$\lim_{z\to\infty} \left(\kappa \frac{\partial F}{\partial z}\Big|_2 + \frac{V_2}{3}\frac{\partial F}{\partial \ln p} - \kappa \frac{\partial F}{\partial z}\Big|_1 - \frac{V_1}{3}\frac{\partial F}{\partial \ln p} \right) = V_1 G(p)$$

$$0$$

$$\lim_{z\to\infty} \left[\frac{1}{3}\frac{\partial F}{\partial \ln p}(V_2 - V_1) - V_1 \left[F(z,p) - F_1 \right] \right] = V_1 G(p) \qquad \frac{V_1}{V_2} = \frac{\varsigma_2}{\varsigma_1} = r$$

$$\frac{\partial F_2}{\partial \ln p} \cdot \frac{V_2(1-r)}{3} = V_1 (F_2 - F_1) + V_1 G(p)$$

$$\frac{\partial F_2}{\partial \ln p} = \frac{3r}{r-1}(F_1 - F_2) + \frac{3r\,V_1}{(r-1)\,r\,V_2} G(p)$$

$$\underline{\frac{\partial F_2}{\partial \ln p} = \frac{3r}{r-1}\left(F_1 + G - F_2 \right)}$$

sol'n: $$\underline{F_2(p) = \alpha\, p^{-\alpha} \int_0^p dp' \left[F_1(p') + G(p') \right] p'^{\,\alpha-1}} \quad , \quad \alpha \equiv \frac{3r}{r-1}$$

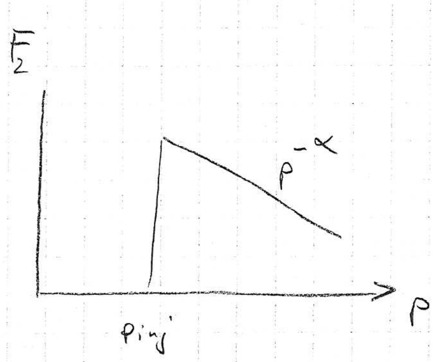
F_2 is independent on κ as long as it is positive! The CR power law is only governed by the kinematic structure of the shock front, i.e. the compression ratio r at the shock and does not depend on the incident kinetic energy flux!

<u>example:</u> $F_1 = 0$, $G(p) = \delta_D(p - p_{inj})$

$$F_2(p) = \alpha\, p^{-\alpha} \int_0^p dp'\, \delta(p' - p_{inj})\, p'^{\,\alpha-1}$$

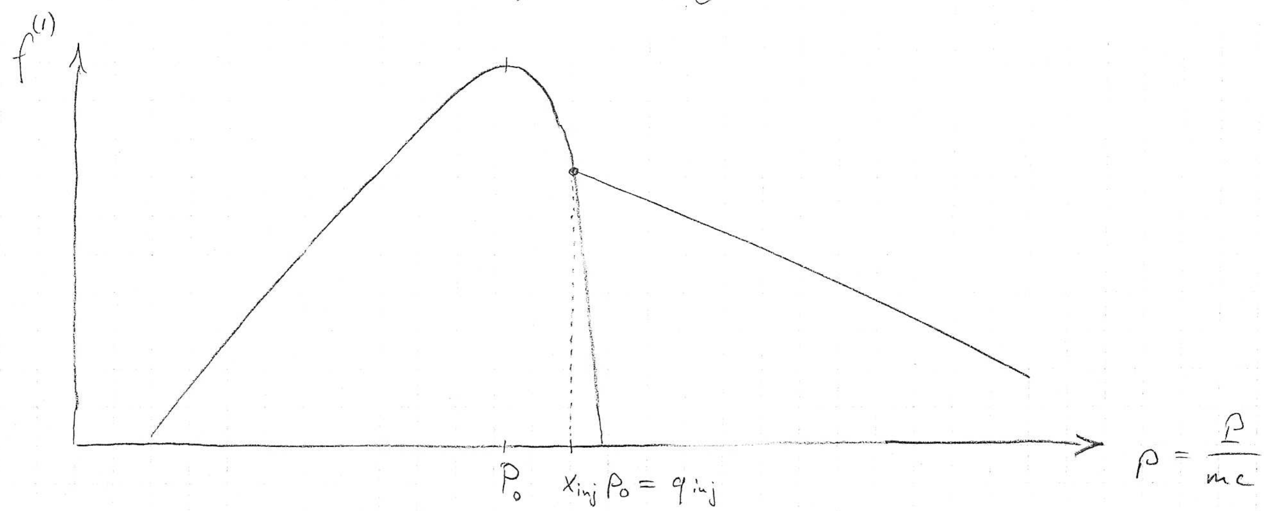
$$= \alpha\, p^{-\alpha}\, p_{inj}^{\alpha-1}\, \Theta(p - p_{inj})$$

$$F_2(p) = \frac{\alpha}{p_{inj}} \left(\frac{p}{p_{inj}} \right)^{-\alpha} \Theta(p - p_{inj})$$



1.3.4 A Recipe for diffusive shock acceleration in hydrodynamical simulations

$f^{(1)} = 4\pi\, p^2 f^{(3)}(p)$, $\quad p \equiv \dfrac{P}{mc}$ is the dimensionless momentum



- $f^{(1)}_{th}(p) = 4\pi n \left(\dfrac{mc^2}{2\pi k T_2}\right)^{3/2} p^2 \exp\left(-\dfrac{mc^2 p^2}{2kT_2}\right)$ denotes the Maxwellian in the post-shock regime with energy kT_2 and number density $n = n_2$.
- Only particles of the high-energy tail are able to return to the upstream shock regime and participate in the Fermi I-process. The momentum threshold of accelerated particles is

$$q_{inj} = x_{inj}\, p_0 = x_{inj}\sqrt{\frac{2kT_2}{mc^2}}\,, \quad \text{where } x_{inj} \simeq 3.5 \text{ has to be determined}$$

by observations or first principle calculations.

$\rightsquigarrow$ $\underline{f_{cr}(p) = f_{th}(q_{inj})\left(\dfrac{p}{q_{inj}}\right)^{-\alpha_{inj}} \Theta(p - q_{inj})}$ employing continuity.

$$f^{(1)} = 4\pi\, p^2 f^{(3)} \propto p^{-\alpha_1} = p^{2-\alpha_3} \longrightarrow \alpha_1 = \alpha_3 - 2 = \frac{3r}{r-1} - \frac{2r-2}{r-1} = \frac{r+2}{r-1}\,; \quad r = \frac{\varrho_2}{\varrho_1}$$

The Rankine-Hugoniot jump conditions yield $r \leq 4$ for $\gamma = \frac{5}{3}$ (assuming negligible cooling or CR modification): strong shock $\to r = 4 \to \alpha_1 = 2$!

In the linear regime, the number density of injected CRs is given by

$$\Delta n_{cr} = \int_0^\infty dp\, f_{cr}(p) = f_{th}(q_{inj})\,\frac{q_{inj}}{\alpha_{inj} - 1}$$

- particle injection efficiency in the linear regime:

$$\eta_{ein} \equiv \frac{\Delta n_{ein}}{n} = \frac{4}{\sqrt{\pi}} \frac{x_{inj}^3}{\alpha_{inj}-1} e^{-x_{inj}^2} \simeq 10^{-3} \ldots 10^{-4} \quad \text{for reasonable params}$$

- injected energy density (lin. regime):

$$\Delta \varepsilon_{ein} = \eta_{ein} \, E_{inj}(\alpha_{inj}, q_{inj}) \, n(T_2)$$, where the average kinetic energy is given by

$$E_{inj} = \frac{\varepsilon_{inj}}{n_{inj}} = n_{inj}^{-1} \int_0^\infty dp \, f_{inj}(p) \, E(p) \quad , \quad E(p) = \left(\sqrt{1+p^2} - 1\right) mc^2$$

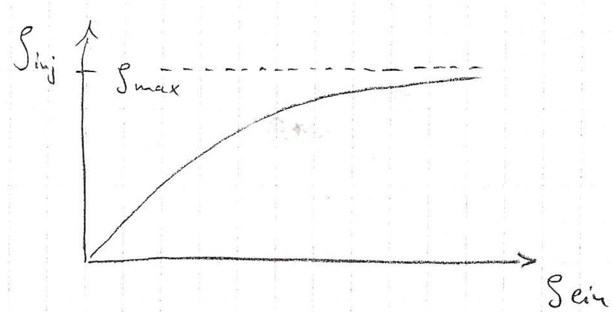
$$= mc^2 \left[\frac{q^{\alpha-1}}{2} \, \mathcal{B}_{\frac{1}{1+q^2}}\left(\frac{\alpha-2}{2}, \frac{3-\alpha}{2}\right) + \sqrt{1+q^2} - 1 \right] \quad , \quad \alpha = \alpha_{inj} \, , \; q = q_{inj}$$

incomplete β-function $(\alpha > 2)$

$\leadsto$ energy injection efficiency $\zeta_{ein} = \frac{\Delta \varepsilon_{ein}}{\Delta \varepsilon_{diss}}$, $\Delta \varepsilon_{diss} = \varepsilon_2 - \varepsilon_1 r^{\gamma}$

shock-dissipated energy in the downstream, corrected for the energy increase due to adiabatic contraction over the shock

- Non-linear saturation for strong shocks:



$$\zeta_{inj} = \left[1 - \exp\left(-\frac{\zeta_{ein}}{\zeta_{max}}\right)\right] \zeta_{max}$$

$$\zeta_{max} \simeq \begin{cases} 0.1-0.3 & \text{for protons} \\ 0.05 & \text{for electrons} \end{cases}$$

- $\Delta \varepsilon_{inj} = \zeta_{inj} \, \Delta \varepsilon_{diss}$, we finally get the injected distribution function:

$$\boxed{\begin{aligned} f_{inj}(p) &= C_{inj} \, p^{-\alpha_{inj}} \, \Theta(p - q_{inj}) \, , \\ C_{inj} &= \left(1 - e^{-\delta}\right) \delta^{-1} f_{th}(q_{inj}) \, q_{inj}^{\alpha_{inj}} \, , \quad \delta = \frac{\Delta \varepsilon_{ein}}{\zeta_{max} \, \Delta \varepsilon_{diss}} \\ f_{th}(q_{inj}) &= \frac{4}{\pi} \, n \, x_{inj}^3 \, q_{inj}^{-1} \, e^{-x_{inj}^2} \end{aligned}}$$