High-Energy Astrophysics : Lecture 1

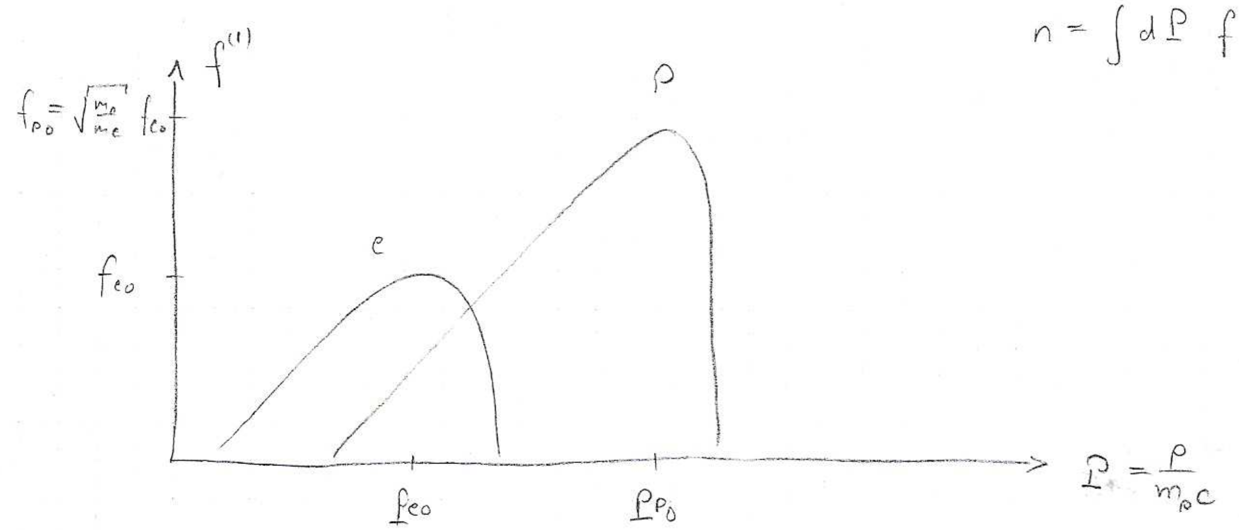
1. Phase space density

1.1 General considerations

- take a hydrogen gas, heat it up to 0.1 keV that it ionises and comes into equilibrium → Maxwell-Boltzmann distribution for e + p :

$$f_{th}^{(1)} = 4\pi n \left(\frac{m_p c^2}{2\pi kT}\right)^{3/2} \mathrm{P}^2 \exp\left(-\frac{m_p c^2 \mathrm{P}}{2kT}\right), \qquad \mathrm{P} = \frac{p}{m_p c}$$

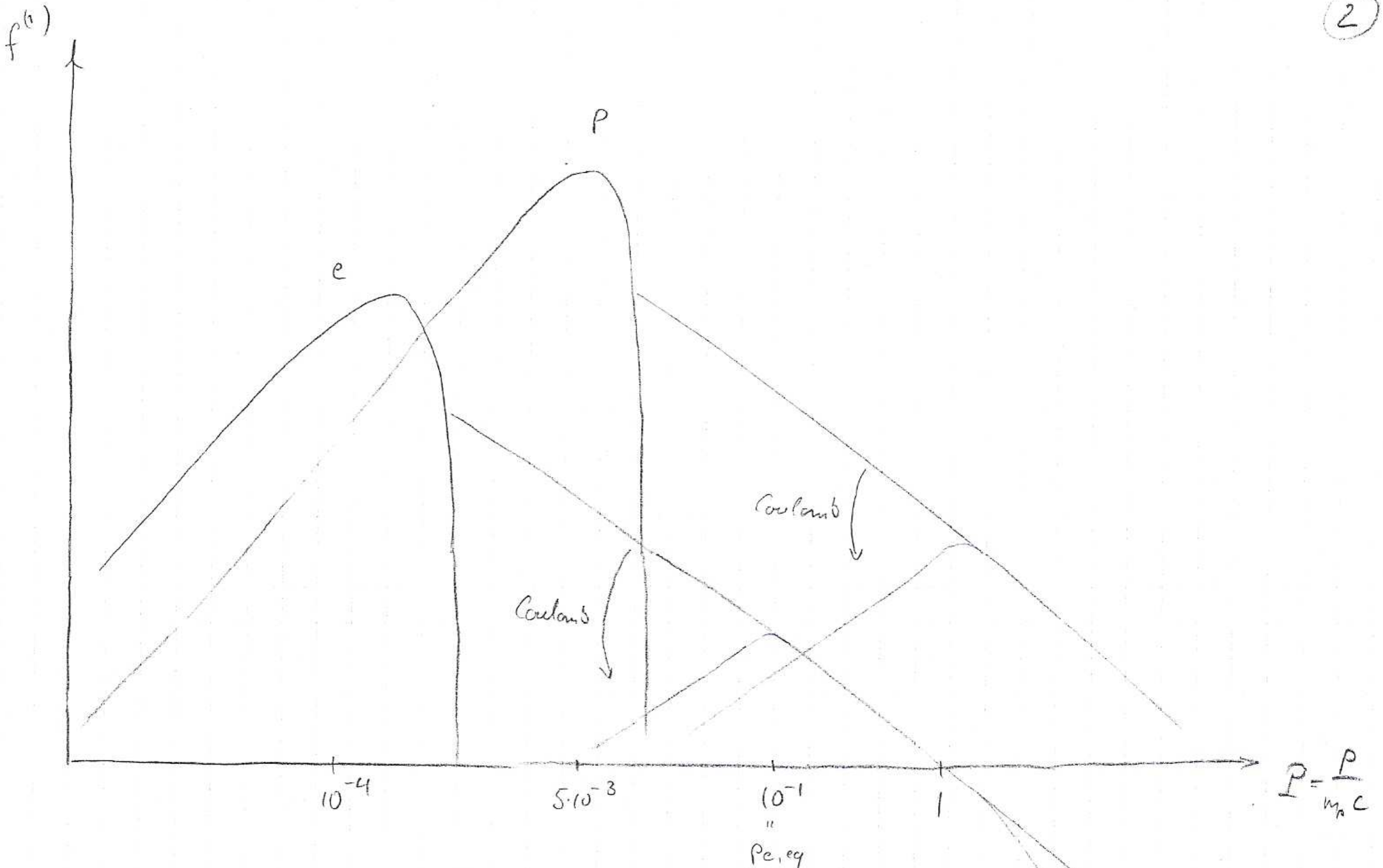
$$n = \int d\mathrm{P}\, f^{(1)}$$



$$\mathrm{P}_{po} = \sqrt{\frac{2kT}{m_p c^2}} = 5 \cdot 10^{-4} \quad \xRightarrow{\text{shock}} \quad 5 \cdot 10^{-3}$$

$$\mathrm{P}_{eo} = \sqrt{\frac{m_e}{m_p}}\, \mathrm{P}_{po} = 10^{-5} \qquad\qquad 10^{-4}$$

- Suppose a shock wave energizes the plasma and heats it up to $kT = 10$ keV
  - ⇝ dissipation of heat + diffusive shock acceleration
  - ⇝ injection of a power-law CR population (→ lecture 2)



• <u>non-radiative / radiative losses</u>:

Coulomb losses: $-\dot{E}_{Coul} = \dfrac{3\sigma_T m_e c^3 n_e}{2\beta} \ln \Lambda_{p,e}$, $\sigma_T = \dfrac{8\pi r_e^2}{3}$

$r_e = \dfrac{e^2}{m_e c^2}$

IC/synchr. losses: $-\dot{E}_{IC,synchr} = \frac{4}{3}\sigma_T c \frac{m_e^2}{m^2} (\varepsilon_B + \varepsilon_{ph}) \gamma^2 \beta^2 \propto E^2$

for protons suppressed by $4\cdot10^6$

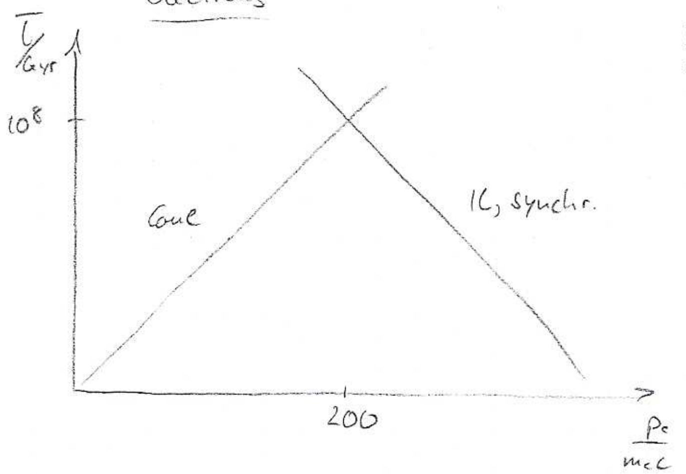
hadronic pp interactions:

$$-\dot{E}_{pp} = E c \sigma_{pp} n$$

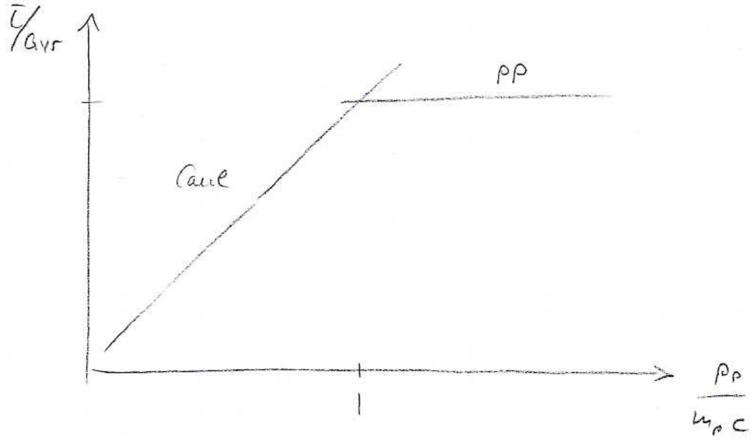
$\rightsquigarrow$ which effects dominate which momentum regime?

- timescales: $\tau = \frac{E}{\dot{E}}$

electrons:



protons:



$n = 10^{-3}\,\mathrm{cm}^{-3}$

$$\frac{p_{e,\mathrm{equ}}}{m_p c} = \frac{m_e c \beta \gamma}{m_p c} = \frac{m_e}{m_p}\, 200\, \frac{\beta_1}{1}\, \frac{\gamma_{200}}{200} = 0.1$$

- In addition there are even more processes that modify the distribution function:
  - adiabatic gains + losses
  - interactions with MHD turbulence: pitch angle scattering + acceleration
  - diffusion
  - ...

$\leadsto$ we need a systematic approach to deal with all these processes:

Since we deal with a collisionless plasma in various astrophysical problems, take Vlasov equation and study evolution of phase space density to derive a CR transport equation!

High-Energy Astrophysics

1.2 Cosmic ray transport equations – quasilinear theory

Starting point:

relativistic Vlasov equ.'n for a particle species described by its distribution function $f(t, \vec{x}, \vec{p})$:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \dot{\vec{p}} \cdot \nabla_p f = S(\vec{x}, \vec{p}, t) \qquad (1)$$

EoM:

$$\dot{\vec{p}} = q \left[\vec{E}(\vec{x}, t) + \frac{\vec{v} \times \vec{B}(\vec{x}, t)}{c} \right] \qquad (2)$$

$$\dot{\vec{x}} = \vec{v} = \frac{\vec{p}}{\gamma m}$$

S denotes sources / sinks of particles

- high conductivity → large scale $\vec{E}$-field can be neglected; take $\vec{B}_0 = B_0 \vec{e}_z$

$$\vec{B} = \vec{B}_0 + \delta \vec{B}(\vec{x}, t), \quad \vec{E} = \delta \vec{E}(\vec{x}, t) \qquad (3)$$

- in uniform magn. field ~ particles perform gyrorotation with guiding center

$$\vec{R} = (X, Y, Z) = \vec{x} + \frac{\vec{v} \times \vec{e}_z}{\varepsilon \Omega} \qquad (4)$$

gyrofrequencies $\Omega = \dfrac{eB}{\gamma mc} = \begin{cases} 1.78 \cdot 10^7 \, \frac{B}{G} \, Hz & (\text{electrons}, \gamma = 1) \\ 9.3 \cdot 10^3 \, \frac{B}{G} \, Hz & (\text{protons}, \gamma = 1) \end{cases}$

$\varepsilon = \dfrac{q}{|q|}$ charge sign

~ spherical coordinates in momentum space $(p, \mu = \cos\vartheta, \varphi)$:

$$p_x = p \cos\varphi \sqrt{1-\mu^2}, \quad p_y = p \sin\varphi \sqrt{1-\mu^2}, \quad p_z = p\mu \qquad (5)$$

$$\sim \quad X = x + \frac{v\sqrt{1-\mu^2}}{\varepsilon \Omega} \sin\varphi \qquad (6)$$

$$Y = y - \frac{v\sqrt{1-\mu^2}}{\varepsilon \Omega} \cos\varphi$$

$$Z = z$$

Excurs:

transforming into general curvilinear coordinate system:

X_i ... six orthogonal coordinates

- $N = \int \prod_{i=1}^{6} (h_i \, dX_i) \, f(X_i, t)$, h_i are square roots of the metric tensor

$h_i \equiv 1$ in Cartesian system

- $\nabla_k S = (\text{grad} S)_k = \frac{1}{h_k} \frac{\partial S}{\partial X_k}$ [summation over h_k not counted, otherwise Einstein summation convention]

$\nabla_k V_k = \text{div} \vec{U} = \frac{1}{h} \frac{\partial}{\partial X_k} \left(\frac{h V_k}{h_k} \right)$, $h = \prod_{i=1}^{6} h_i = p^2$ for spherical coordinates

, $\frac{\partial f}{\partial t} + \nabla \cdot \left[\frac{d\vec{X}}{dt} f \right] = \frac{\partial f}{\partial t} + \frac{1}{h} \frac{\partial}{\partial X_k} \left[h \frac{dX_k}{dt} f \right] = 0$ $[d^6X = d^3x \, dp \, p^2 d\mu \, d\varphi]$

(Vlasov equ.) note cancellation

with EoM: $\frac{dX_k}{dt} = G_k(X_i(t), t) + g_k(X_i(t), t)$

- transforming to $x_\sigma = (p, \mu, \varphi, X, Y, Z)$; Vlasov equation reads: (7)

$$\frac{\partial f}{\partial t} + \underbrace{v\mu \frac{\partial f}{\partial Z}}_{\text{drift term}} - \underbrace{\varepsilon \Omega \frac{\partial f}{\partial \varphi}}_{\text{gyromotion}} + \frac{1}{p^2} \frac{\partial}{\partial x_\sigma} \left(p^2 g_{x_\sigma} f \right) = S(\vec{x}, \vec{p}, t) \quad (8)$$

where generalized force term g_{x_σ} includes the effect of the randomly fluctuating magnetic fields: (9)

$$g_p = \dot{p} = \frac{m c \gamma \varepsilon \Omega}{p B_0} \vec{p} \cdot \delta\vec{E} = \frac{\varepsilon \Omega p c}{v B_0} \left[\mu \, \delta E_\parallel + \sqrt{\frac{1-\mu^2}{2}} \left(\delta E_L e^{-i\varphi} + \delta E_R e^{i\varphi} \right) \right]$$

$$g_\mu = \dot{\mu} = \frac{\varepsilon \Omega \sqrt{1-\mu^2}}{B_0} \left[\frac{c}{v} \sqrt{1-\mu^2} \, \delta E_\parallel + \frac{i}{\sqrt{2}} \left[e^{i\varphi} \left(\delta B_R + i\mu \frac{c}{v} \delta E_R \right) - e^{-i\varphi} \left(\delta B_L - i\mu \frac{c}{v} \delta E_L \right) \right] \right]$$

$g_Z = 0$

with $\delta B_{L,R} \equiv \frac{1}{\sqrt{2}} \left(\delta B_x \pm i \, \delta B_y \right)$, $\delta B_\parallel = \delta B_z$, (10)

$\delta E_{L,R} \equiv \frac{1}{\sqrt{2}} \left(\delta E_x \pm i \, \delta E_y \right)$, $\delta E_\parallel = \delta E_z$.

(right-/lefthanded pol. states

Largest contribution to g_p, g_μ:

1) Faraday law: $\mathrm{curl}\,\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B} \rightarrow -i\vec{k}\times\tilde{\vec{E}} = -i\frac{\omega}{c}\tilde{\vec{B}} \sim \vec{k}\times\tilde{\vec{E}} = \frac{\omega}{c}\tilde{\vec{B}}$

$$\vec{E} = \tilde{\vec{E}}\, e^{i(\omega t - \vec{k}\vec{x})}$$

$$\frac{|B|}{|E|} = \frac{ck}{\omega} = \frac{c}{c_A} \gg 1$$ ~ magnetic fluctuations much larger than electric ones!

$$\omega = k v_{ph} = k c_A$$

2) $$\vec{k}\times(\vec{k}\times\tilde{\vec{E}}) = \frac{\omega}{c}\vec{k}\times\tilde{\vec{B}}$$

$$\underbrace{\vec{k}\cdot(\vec{k}\cdot\tilde{\vec{E}})}_{0} - \tilde{\vec{E}}(\vec{k}\cdot\vec{k}) = \frac{\omega}{c}\vec{k}\times\tilde{\vec{B}} \Rightarrow \delta\tilde{\vec{E}} = -\frac{\omega}{ck^2}\vec{k}\times\vec{B}_0$$

transverse EM waves $\Rightarrow \delta E_{\parallel} = 0$

$$\Rightarrow \quad g_p \propto \sqrt{1-\mu^2}\left(\delta E_L e^{-i\varphi} + \delta E_R e^{i\varphi}\right)$$
$$g_\mu \propto \sqrt{1-\mu^2}\left(\delta B_L e^{-i\varphi} + \delta B_R e^{i\varphi}\right)$$

electric fluctuations change the energy of CRs,
magnetic fluctuations change its pitch angle !

- f develops in an irregular way under the influence of g_σ, but detailed fluctuations are not of interest -> expectation value for f must be found in terms of the statistical properties of g_σ:
consider an ensemble of functions f, all beginning with identical values at some time $t = t_0$. Each of these function evolves according to a different realization of g_σ that is drawn from an ensemble of g_σ with the same statistical properties. Seek equation for

$$\underbrace{\langle f(\vec{x},\vec{p},t)\rangle}_{\text{ensemble average}} \equiv F(\vec{x},\vec{p},t) \qquad (11)$$

use

$$< \delta \vec{B}(\vec{x},t) > = < \delta \vec{E}(\vec{x},t) > = \vec{0} \quad (12)$$

$$< \vec{B}(\vec{x},t) > = \vec{B}_0 \quad , \quad < \vec{E}(\vec{x},t) > = \vec{0}$$

taking average of (8):

$$< p^2 g_{x_\sigma} (F + \delta f) > = \underbrace{< p^2 g_{x_\sigma} >}_{0 \ (12)} F + < p^2 g_{x_\sigma} \delta f >$$

$$\leadsto \frac{\partial F}{\partial t} + v\mu \frac{\partial F}{\partial z} - \varepsilon \Omega \frac{\partial F}{\partial \varphi} = S(\vec{x},\vec{p},t) - \frac{1}{p^2} \frac{\partial}{\partial x_\sigma} \left(< p^2 g_{x_\sigma} \delta f > \right) \quad (13)$$

where $\delta f(\vec{x},\vec{p},t) = f(\vec{x},\vec{p},t) - F(\vec{x},\vec{p},t)$ (14)

subtracting (13) from (8): $\left[\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} - \varepsilon \Omega \frac{\partial f}{\partial z} + \frac{1}{p^2} \frac{\partial}{\partial x_\sigma} (p^2 g_{x_\sigma} f) = S(\vec{x},\vec{p},t) \ (8) \right]$

$$\leadsto \frac{\partial \delta f}{\partial t} + v\mu \frac{\partial \delta f}{\partial z} - \varepsilon \Omega \frac{\partial \delta f}{\partial \varphi} = - g_{x_\sigma} \frac{\partial F}{\partial x_\sigma} - g_{x_\sigma} \frac{\partial \delta f}{\partial x_\sigma} + < g_{x_\sigma} \frac{\partial \delta f}{\partial x_\sigma} > \quad (15)$$

using $- \frac{1}{p^2} \frac{\partial}{\partial x_\sigma} \underbrace{\cancel{(p^2 g_{x_\sigma})}}_{0} \cdot f - g_{x_\sigma} \frac{\partial f}{\partial x_\sigma} + < \frac{1}{p^2} \frac{\partial}{\partial x_\sigma} \underbrace{\cancel{(p^2 g_{x_\sigma})}}_{0} \delta f > + < g_{x_\sigma} \frac{\partial \delta f}{\partial x_\sigma} >$

$$= - g_{x_\sigma} \frac{\partial f}{\partial x_\sigma} + < g_{x_\sigma} \frac{\partial \delta f}{\partial x_\sigma} >$$

since $\frac{1}{p^2} \frac{\partial}{\partial x_\sigma} (p^2 g_{x_\sigma}) = 0 \iff \frac{\partial}{\partial \vec{p}} \vec{F}_L = \frac{\partial}{\partial \vec{p}} \frac{d\vec{p}}{dt} = \frac{d}{dt} \frac{\partial \vec{p}}{\partial \vec{p}} = \frac{d}{dt} 1 = 0$ (16)

Now: perturbation method – quasilinear approximation:

EM fluctuations are assumed to have small amplitude, g_σ significantly small so that there exists a time scale T:

$$t_c \ll T \ll t_F \quad , \quad (17)$$

t_F is the time scale on which g_σ affects the evolution of the distribution function. EoM $\leadsto$

$$t_F = \frac{F}{g_{x_\sigma} \frac{\partial F}{\partial x_\sigma}} \quad (18)$$

(15) $\rightsquigarrow$ $\delta f \overset{!}{<} F$ for variation δf generated by F within T

$$\frac{\partial \delta f}{\partial t} + v\mu \frac{\partial \delta f}{\partial Z} - \varepsilon \Omega \frac{\partial \delta f}{\partial \varphi} \simeq - g_{x_\sigma} \frac{\partial F}{\partial x_\sigma} \qquad (19)$$

- This equation can be solved by the method of characteristics, where we take t as a parameter and express Z and φ in terms of t:

$$\frac{\partial Z}{\partial t} = \mu v \, , \quad \frac{\partial \varphi}{\partial t} = - \varepsilon \Omega$$

$$\rightsquigarrow \quad \frac{d \delta f}{dt} \simeq - g_{x_\sigma} \frac{\partial F}{\partial x_\sigma}$$

$$\delta f(t) = \delta f(t_0) - \int_{t_0}^{t} ds \left[g_{x_\sigma}(x_\nu, s) \frac{\partial F(x_\nu, s)}{\partial x_\sigma} \right]' \qquad (20)$$

prime indicates, that the bracketed quantities are to be evaluated along the characteristics, i.e. unperturbed particle orbit in a uniform B-field given by

$$\bar{X} = x_0 \, , \quad \bar{Y} = y_0 \, , \quad \bar{Z} = z_0 + v\mu(s-t)$$

$$\bar{x} = x_0 - \frac{v\sqrt{1-\mu^2}}{\varepsilon \Omega} \sin(\bar{\varphi}) \qquad \bar{p} = p$$

$$\bar{y} = y_0 + \frac{v\sqrt{1-\mu^2}}{\varepsilon \Omega} \cos(\bar{\varphi}) \qquad \bar{\mu} = \mu$$

$$\bar{z} = z_0 + v\mu(s-t) \qquad \bar{\varphi} = \varphi_0 - \varepsilon \Omega (s-t) \qquad (21)$$

@ t_0: $\quad \langle \delta f \, \underbrace{g_{x_\sigma}}_{\text{turbulent field}} \rangle = 0$

- (20) in (13):

$$\frac{\partial F}{\partial t} + v\mu \frac{\partial F}{\partial z} - \varepsilon \Omega \frac{\partial F}{\partial \varphi} = S(\vec{x}, \vec{p}, t) + \frac{1}{p^2} \frac{\partial}{\partial x_\sigma} \left\langle\!\left\langle p^2 g_{x_\sigma} \int_{t_0}^{t} ds \left[g_{x_\nu}(x_\nu, s) \frac{\partial F(x_\nu, s)}{\partial x_\nu} \right]' \right\rangle\!\right\rangle$$

where $t - t_0 = T$

- Rearranging the last term:

$$M_1 \equiv \frac{1}{p^2}\frac{\partial}{\partial x_\sigma}\left(\left\langle p^2 g_{x_\sigma} \int_{t_0}^{t} ds \left[g_{x_\nu}(x_\nu, s) \frac{\partial F(x_\nu,s)}{\partial x_\nu}\right]'\right\rangle\right)$$

$$= \frac{1}{p^2}\frac{\partial}{\partial x_\sigma}\left(p^2 \int_{t_0}^{t} ds \left\langle g_{x_\sigma} g_{x_\nu}'(x_\nu,s)\right\rangle \left[\frac{\partial F(x_\nu,s)}{\partial x_\nu}\right]'\right) \tag{24}$$

- More assumptions:

1) $\exists$ correlation time t_c, such that $\langle g_{x_\sigma} g_{x_\nu}\rangle < \varepsilon$ (small) for $s < t - t_c$

   $\sim$ contribution to integral comes from $t - t_c$ to t

2) variation of $\left[\frac{\partial F}{\partial x_\sigma}\right]'$ over $(t - t_c, t)$ small $\sim$ value equal to that at $s = t$

$$M_1 \simeq \frac{1}{p^2}\frac{\partial}{\partial x_\sigma}\left(p^2 \left[\int_{t-t_c}^{t} ds \left\langle g_{x_\sigma} g_{x_\nu}(x_\nu,s)\right\rangle\right]' \frac{\partial F(x_\nu,t)}{\partial x_\nu}\right)$$

$$= \frac{1}{p^2}\frac{\partial}{\partial x_\sigma}\left(p^2 \left[\int_0^{t} ds \left\langle g_{x_\sigma} g_{x_\nu}(x_\nu,s)\right\rangle\right]' \frac{\partial F(x_\nu,t)}{\partial x_\nu}\right) \tag{25}$$

3) requirement (17), $t - t_0 \gg t_c$ makes (25) a function of t alone: $t - t_c \to 0$

- (23) is now a diffusion equation w/ 2nd-order correlation functions of g_{x_σ} integrated along unperturbed orbit; the Fokker-Planck equation:

$$\frac{\partial F}{\partial t} + v\mu \frac{\partial F}{\partial z} - \varepsilon \Omega \frac{\partial F}{\partial \varphi} = S(\vec{x}, \vec{p}, t) + \frac{1}{p^2}\frac{\partial}{\partial x_\sigma}\left(p^2 D_{x_\sigma x_\nu} \frac{\partial F}{\partial x_\nu}\right) \tag{26}$$

with FP coefficients $D_{x_\sigma x_\nu}(x_\eta, t) = \int_0^{t} ds \langle \bar{g}_{x_\sigma}(t)\, \bar{g}_{x_\nu}(s)\rangle$ (27)

- $g_{x_\sigma} \in \mathbb{R}$: $g^*_{x_\sigma} = g_{x_\sigma}$, using $\xi = t - s$

$$D_{x_\sigma x_\nu}(x_\eta, t) = \mathcal{R}\int_0^{t} ds \langle \bar{g}_{x_\sigma}(t)\, \bar{g}^*_{x_\nu}(s)\rangle = -\mathcal{R}\int_t^{0} d\xi \langle \bar{g}_{x_\sigma}(t)\, \bar{g}^*_{x_\nu}(t-\xi)\rangle$$

$$= \mathcal{R}\int_0^{t} d\xi \langle \bar{g}_{x_\sigma}(t)\, \bar{g}^*_{x_\nu}(t-\xi)\rangle \tag{28}$$

, depends on difference $|\xi| \gg t_c$

$$\sim D_{x_\sigma x_\nu} = \mathcal{R}\int_0^{\infty} d\xi \langle \bar{g}_{x_\sigma}(t)\, \bar{g}^*_{x_\nu}(t+\xi)\rangle \tag{29}$$

, bar denotes that g_{x_σ} have to be calculated along unperturbed orbits

Recap of Lecture 1:

We derived the Fokker Planck equation for the evolution of the ensemble-averaged distribution function:

$$\frac{\partial F}{\partial t} + v\mu \frac{\partial F}{\partial z} - \varepsilon \Omega \frac{\partial F}{\partial \varphi} = S(\vec{x}, \vec{p}, t) + \frac{1}{p^2} \frac{\partial}{\partial x_\sigma} \left(p^2 D_{x_\sigma x_\nu} \frac{\partial F}{\partial x_\nu} \right)$$

with the Fokker - Planck coefficients

$$D_{x_\sigma x_\nu} = \Re \int_{t_0 \to 0}^{t \to \infty} d\xi \langle \bar{g}_{x_\sigma}(t) \, \bar{g}_{x_\nu}(t+\xi) \rangle$$

, the bar denotes that g_{x_σ} have to be calculated along unperturbed orbits.

Assumptions:

1) $\exists$ correlation time t_c, $\langle g_{x_\sigma} g_{x_\nu} \rangle < \varepsilon$ for $s < t - t_0$

→ integral is dominated by times $t - t_c$ to t

2) variation of $\left[\frac{\partial F}{\partial x_\sigma}\right]$ over $(t - t_c, t)$ small → value equal to that at $s = t$

3) $t - t_0 \gg t_c$ ∼ lower integration limit → 0

∼ upper — " — → ∞

General motivation:

- Where do CRs come from ?
- What causes non-thermal radiation — radio, X-ray, γ-ray ?
- What does this tell us about underlying astrophysical objects ?

→ Sun, ISM, radio gal's, AGN, clusters, ...