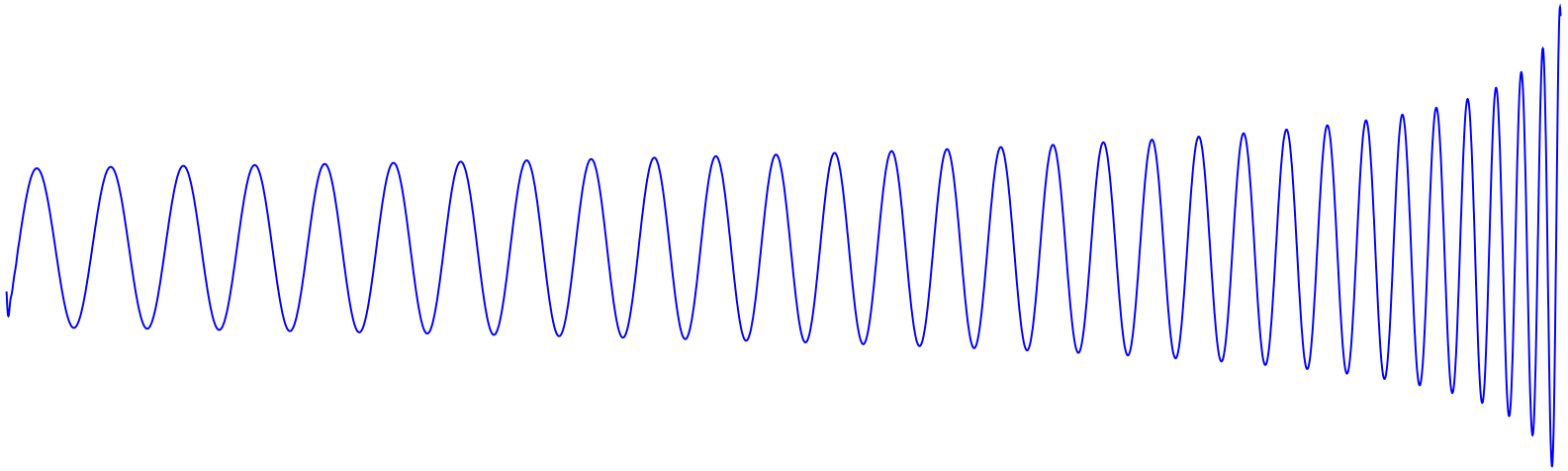
Binary black hole simulations and the validity of post-Newtonian theory

Harald Pfeiffer

California Institute of Technology

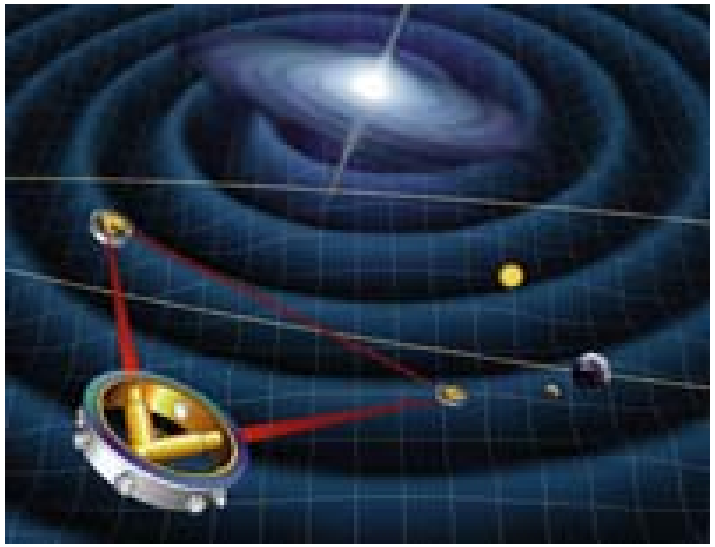
TAPIR Seminar, Caltech, Mar 17, 2007



Gravitational wave detectors

LIGO (Hanford)

LISA (201x)



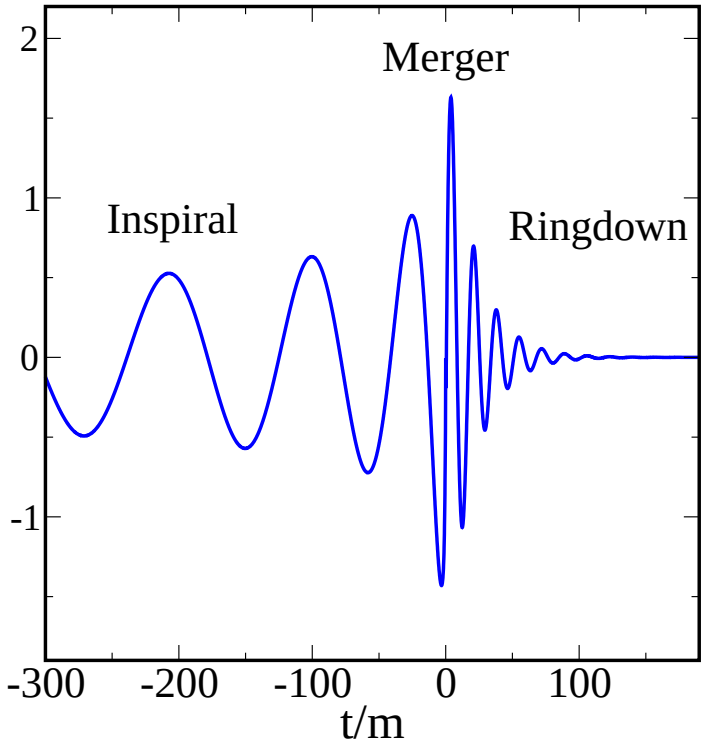
GEO 600

VIRGO

- Prime target: Binary black hole systems

Stages of binary inspiral

- Inspiral
  – post-Newtonian (PN) expansions
- "Late" inspiral
  – PN $\longrightarrow$ Numerical relativity
- Merger
  – Numerical relativity
- Ringdown
  – Perturbation theory
  – Numerical relativity

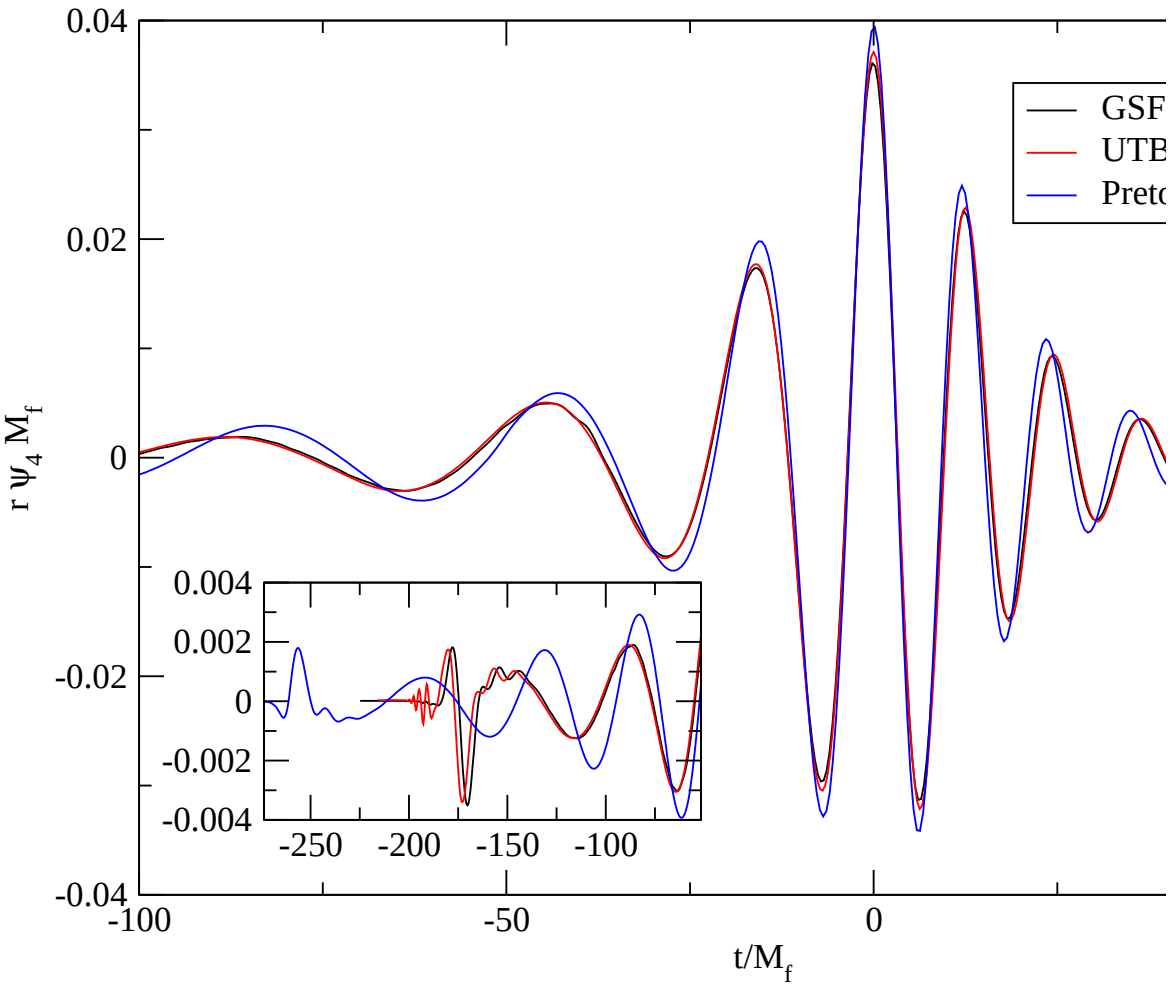


Role of numerical relativity

- Support for GW detectors
  - Waveforms for templates and vetos
  - Parameter estimation
  - Test general relativity
- Explore properties of general relativity
  - Toroidal black holes (Shaprio, Teukolsky)
  - Critical behavior in BH formation (Choptuik)
  - Verify analytic approximations
- Solve the two-body problem
- Binary black hole simulations began in earnest two years ago

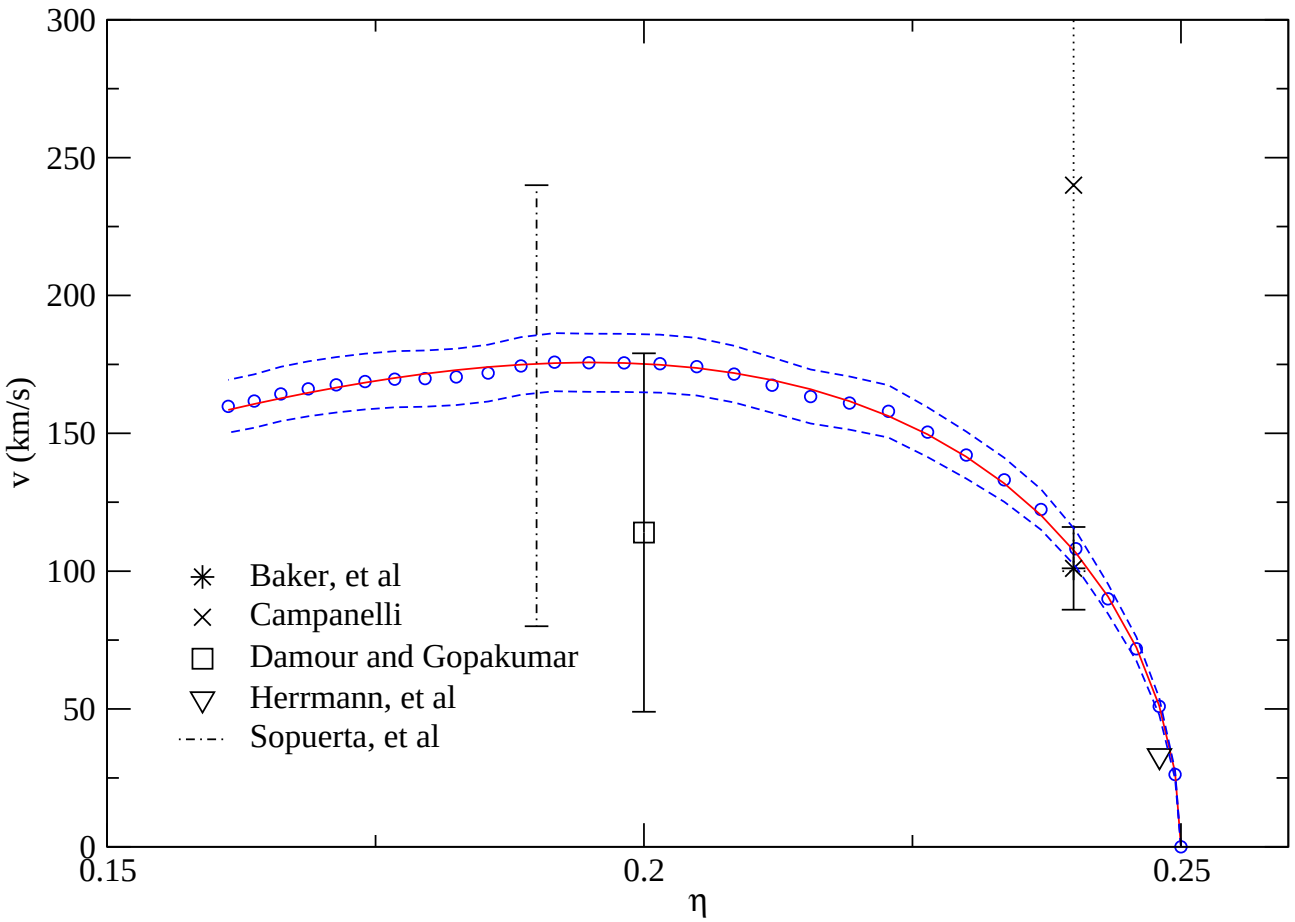
Examples of Merger simulations

Equal-mass, low-spin merger waveform: Baker, Campanelli, Pretorius, Zlochower, 2007



Examples of Merger simulations

Black hole kicks: Gonzalez *et. al.*, 2006



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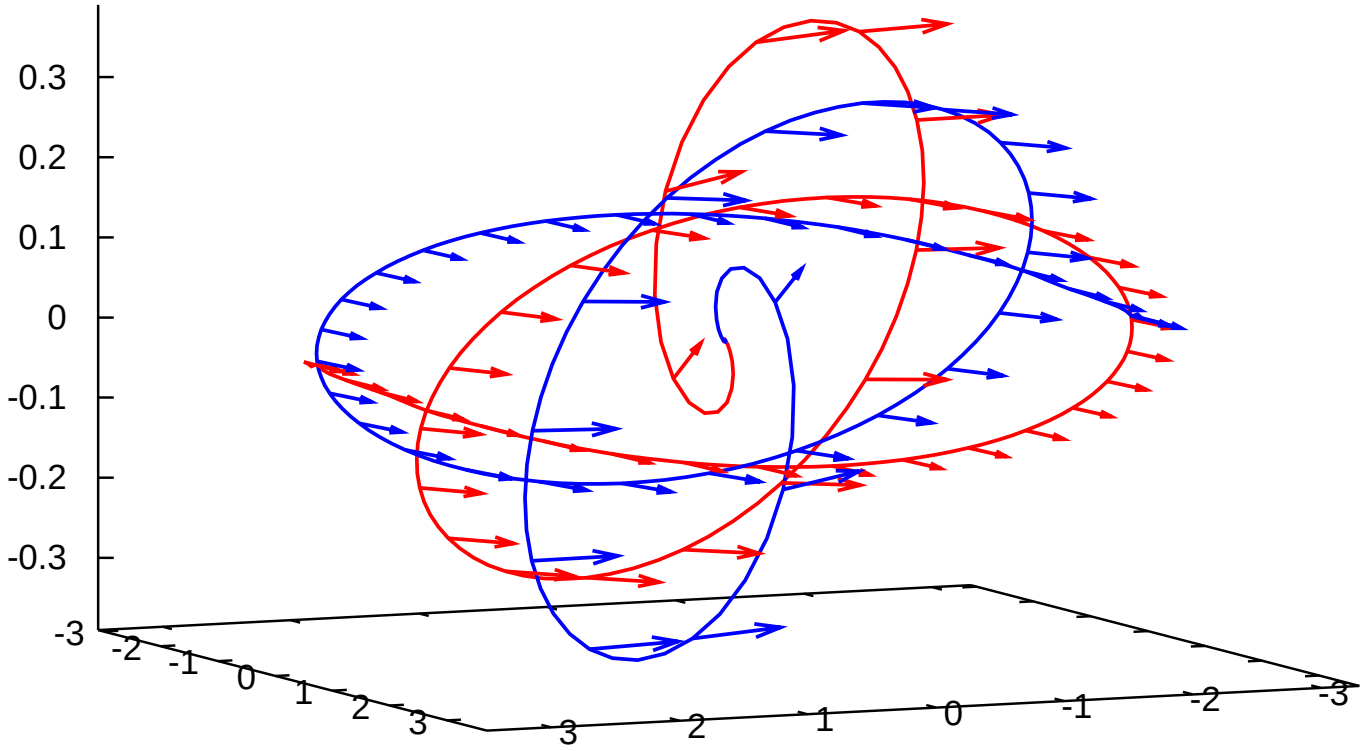
Spinning black holes: Campanelli *et. al.*, 2006



Mergers vs. Inspirals

- Mergers
  - highly dynamic
  - short duration: several $100M$
  - large energy flux: $\Psi_4 rM \sim 0.01$
- Inspirals
  - mildly dynamic
  - long duration: several $1000M$
  - small energy flux: $\Psi_4 rM \sim 0.0002$
  - absolute phase-error relevant for template missmatch

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  - Matching to post-Newtonian waveforms
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- Kip's challenge:

  Simulate last 15 orbits with phase error 0.01rad.

Who are we?

Simulation of eXtreme Spacetimes

- Collaboration between Cornell & Caltech
- TAPIR

  Mike Boyle, Mike Cohen, Tony Chu,
  Lee Lindblom, Geoffrey Lovelace, Rob Owen,
  Harald Pfeiffer, Oliver Rinne, Mark Scheel.
- Cornell

  Matt Duez, Francois Foucart, Larry Kidder,
  Francois Limousin, Abdul Mroue, Nick Taylor,
  Saul Teukolsky.
- Further collaborators: Duncan Brown, Luisa Buchman,
  Gregory Cook, Jan Hesthaven, Stephen Lau

How to simulate black holes: Essential ingredients.

Generalized Harmonic evolution system

- Task: Find space-time metric g_{ab} such that $R_{ab}[g_{ab}] = 0$

$$0 = R_{ab} = -\frac{1}{2}\Box g_{ab} + \nabla_{(a}\Gamma_{b)} + \text{lower order terms} \qquad \Gamma_a = -g_{ab}\Box x^b$$

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$$0 = -\frac{1}{2}\Box g_{ab} + \nabla_{(a}C_{b)} + \gamma\left[t_{(a}C_{b)} - \frac{1}{2}g_{ab}t^c C_c\right] + \text{l. o.}$$

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- Rewrite in first order form (Lindblom, et al 2005). More constraints and more constraint damping (R. Owen)

Spectral Methods

PDE $\mathcal{L}(u) = 0$

- Approximate solution by truncated series

$$u(x,t) \approx u_N(x,t) \equiv \sum_{k=1}^{N} \tilde{u}_k(t)\, \Phi_k(x).$$

- Analytic differentiation

$$\frac{du_N(x,t)}{dx} = \sum_{k=1}^{N} \tilde{u}_k(t)\, \frac{d\Phi_k(x)}{dx}.$$

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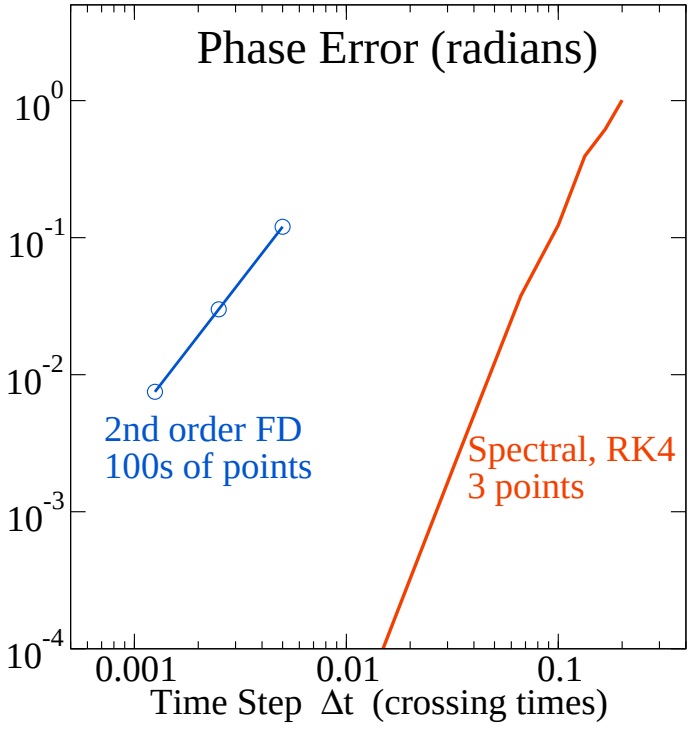
- Recast hyperbolic PDE as grid-based "method of lines":

$$\partial_t u(x_i) = \mathrm{RHS}_i[u].$$

Spectral Methods

Accuracy

- Accuracy determined by how well $u(x,t)$ can be approximated by $u_N(x,t)$.
- Sinusoidal linearized gravitational wave
  - "Mexico City tests" (Mike Boyle et al. 2006)
  - 3 grid points!



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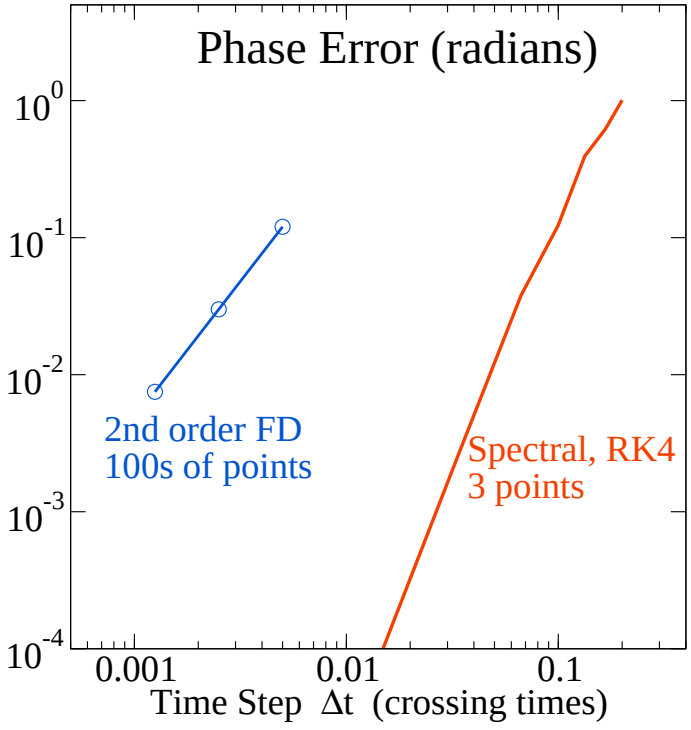

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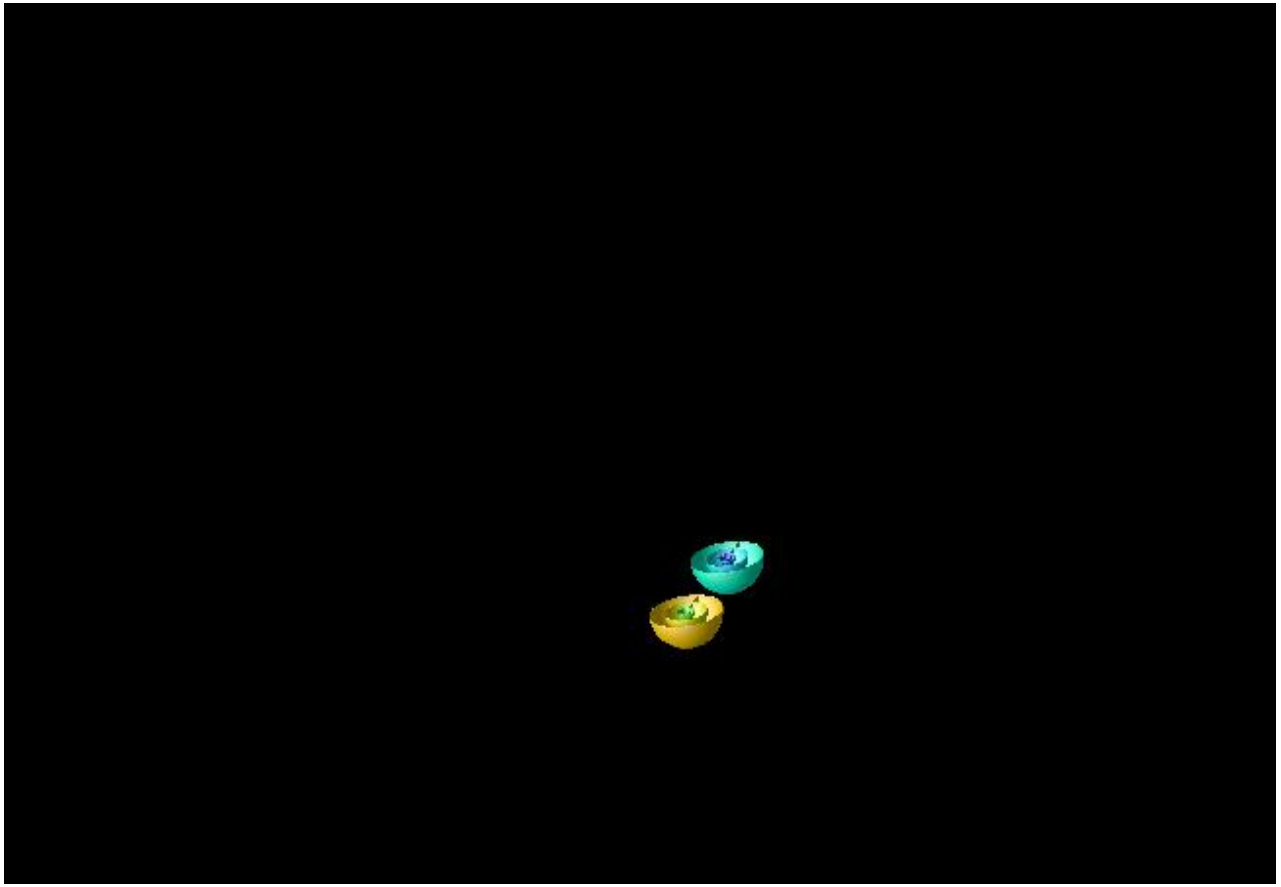
- Sinusoidal linearized gravitational wave
  - "Mexico City tests" (Mike Boyle et al. 2006)
  - 3 grid points!
- General features
  - exponential convergence for smooth solutions $\mathrm{err} \propto e^{-\lambda N}$
  - Low phase-errors

Spectral methods

Domain-decomposition for binary black hole

- Expansions easiest in simple topologies: Blocks, shells, cylinders,...
- Cover complicated domain by multiple subdomains.



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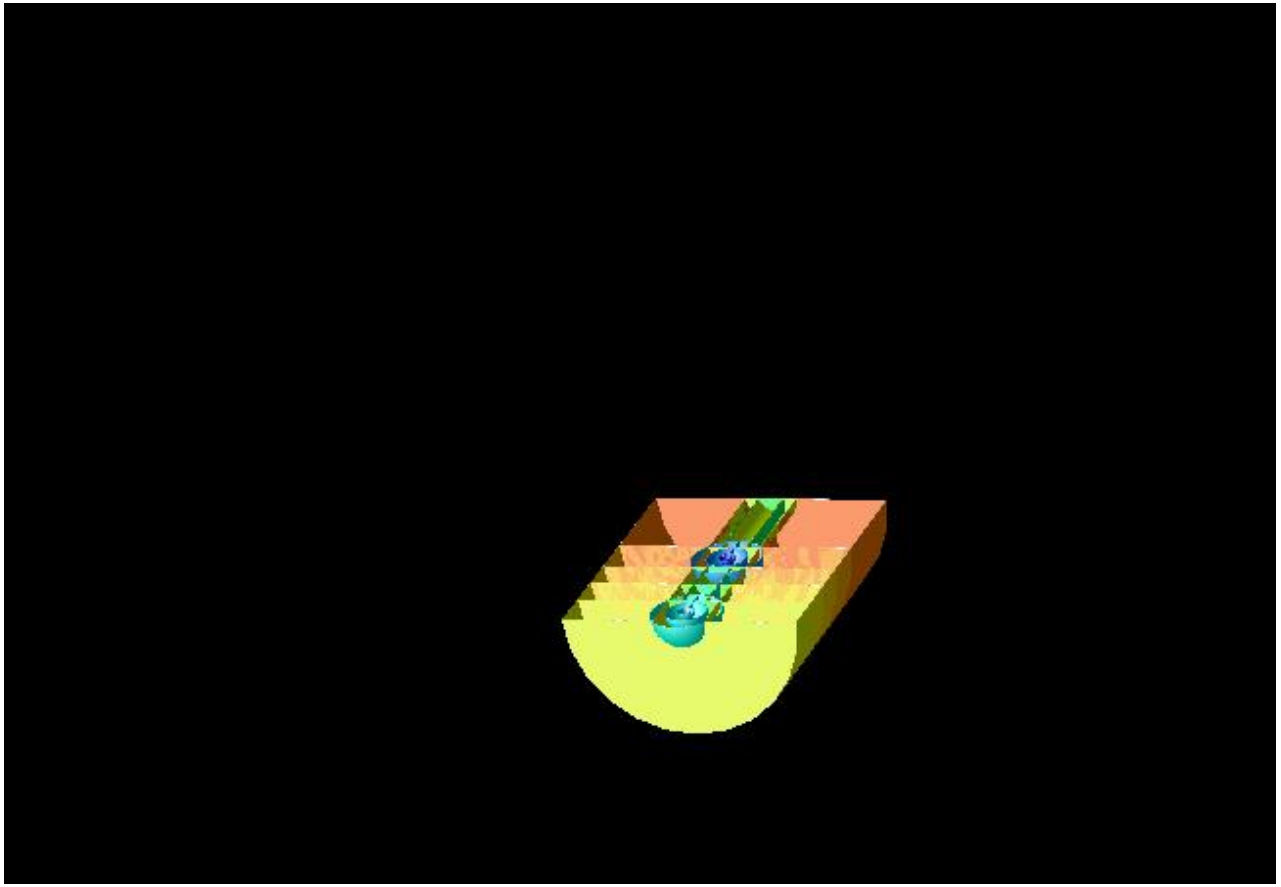

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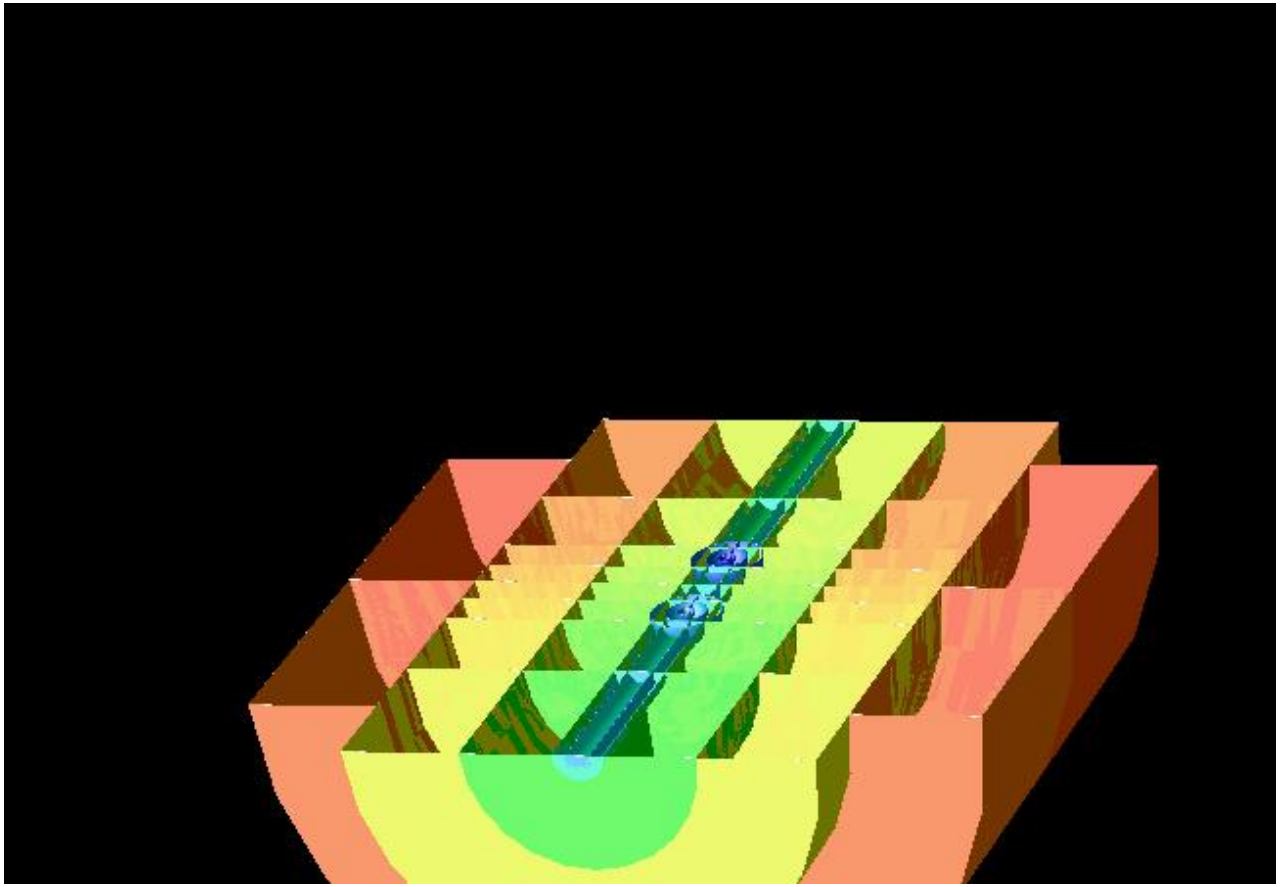

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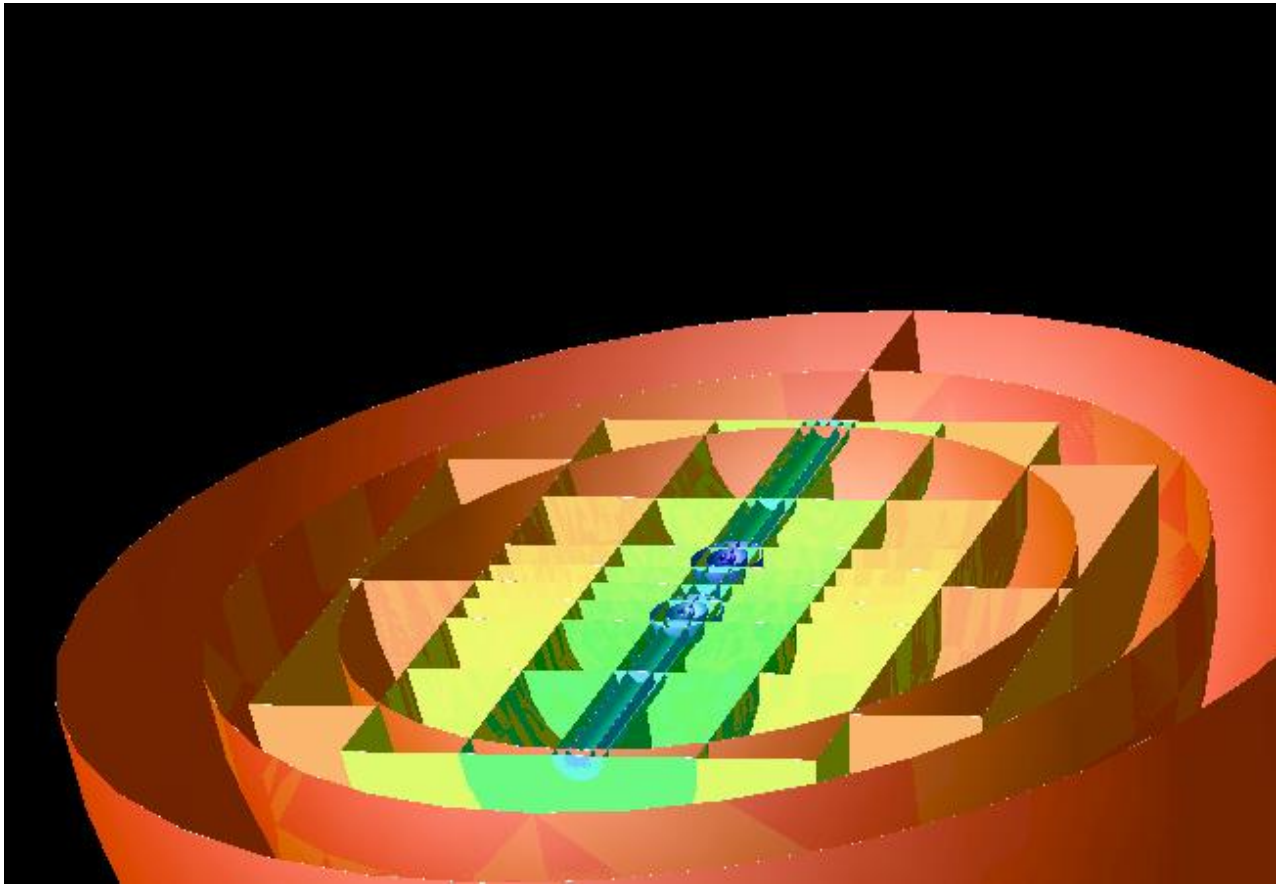

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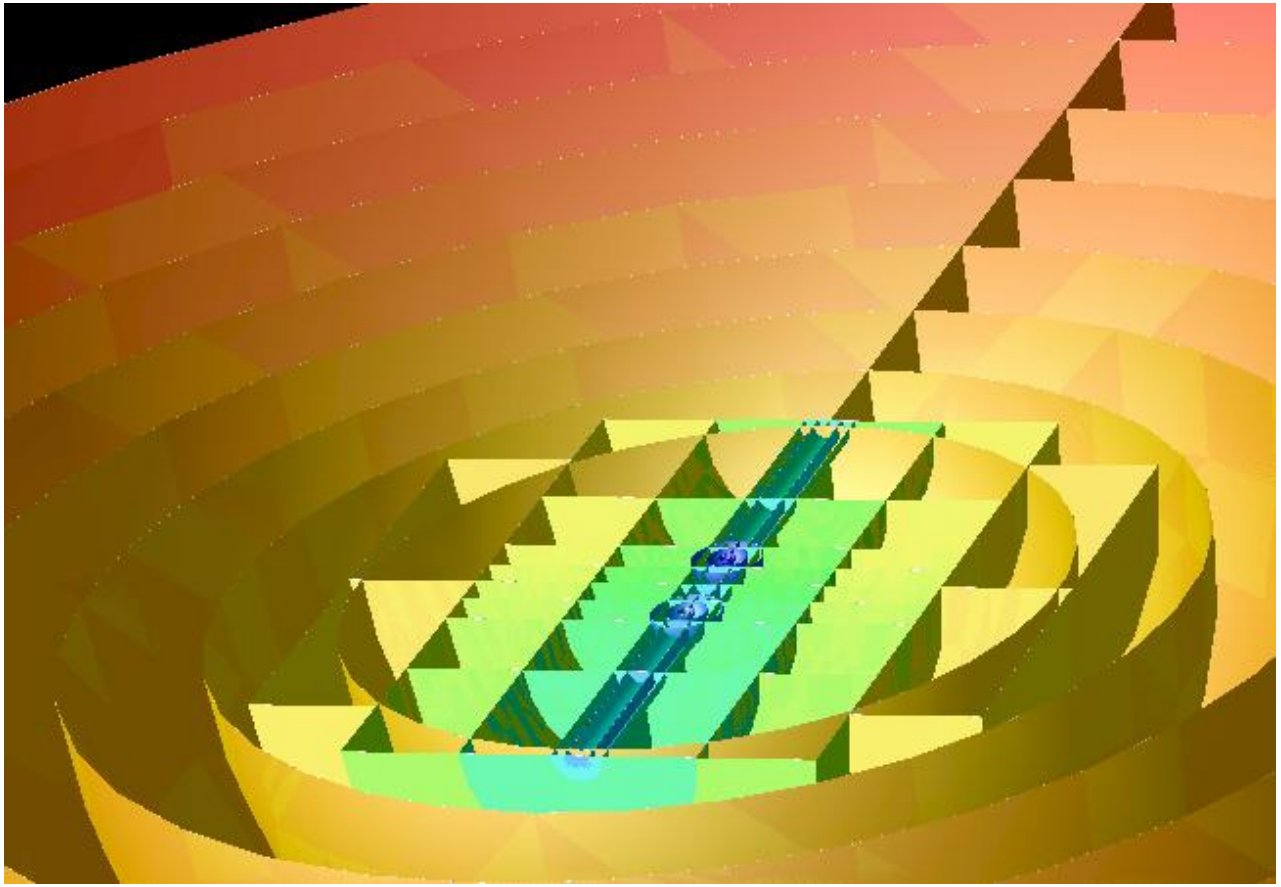


- **Sp**ectral **E**instein **C**ode: mostly C++, 2,000 files, 250,000 lines, principal developers L. Kidder, M. Scheel, H.P.

Spectral methods

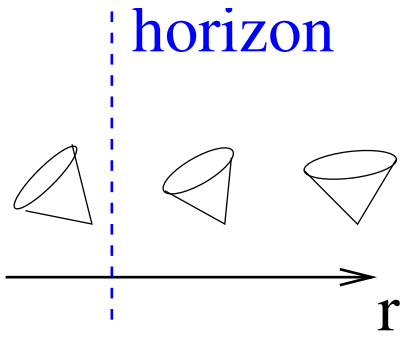
Black magic – the art of making codes fast

- Cost of high-resolution BBH simulations
  - Nov ’05: 4 days for 1/4 orbit (100 CPU-h/M)
  - Nov ’06: 4 weeks for 5 orbits (30 CPU-h/M)
  - Feb ’07: 3 weeks for 15 orbits (4 CPU-h/M)
- Improvements:
  - Cubes -> Cylinders
  - balancing of resolution
  - understanding of constraint damping parameters
  - Code optimization
  - Outer boundary conditions

Boundary conditions & BH excision

- Generalized harmonic evolution system is symmetric hyperbolic

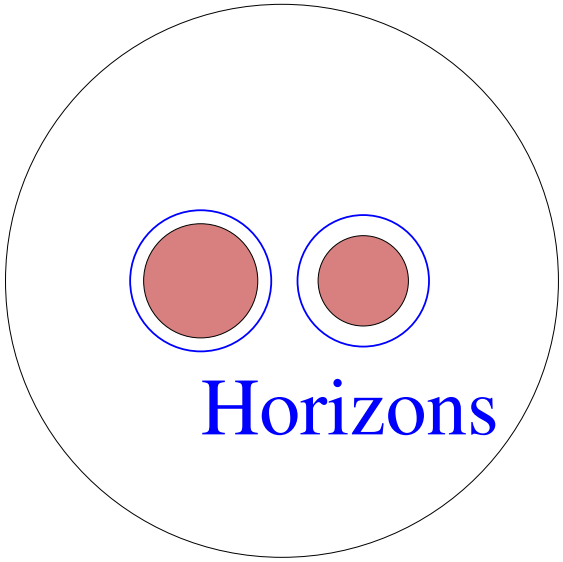
$$u^\alpha + A^{k\alpha}{}_\beta \partial_k u^\beta = F^\beta$$



- Boundary conditions
  - Decompose into characteristic fields

$$e^{\hat\alpha}{}_\alpha\, n_k A^{k\alpha}{}_\beta = v_{(\hat\alpha)} e^{\hat\alpha}{}_\beta$$

  - Impose BCs on incoming fields



- All modes propagate inside light cone
  $\Rightarrow$ Excision boundaries inside horizon do not require any BC

Dual coordinate method

- Changing domain-decomposition is difficult
  – *localize* horizons in coordinate space
  (Scheel *et al.*, 2006):

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  1. Evolve **inertial frame** components of tensors
  2. Represent solution at grid-points which **move** relative to inertial coordinates:

$$x^{\bar{\imath}} = a(t, |x|) R(t)^{\bar{\imath}}{}_j x^j$$

$R(t)^{\bar{\imath}}{}_j$ rotation matrix, $a(t, |x|)$ radial scaling

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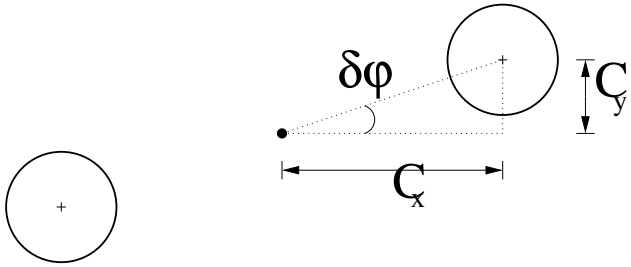

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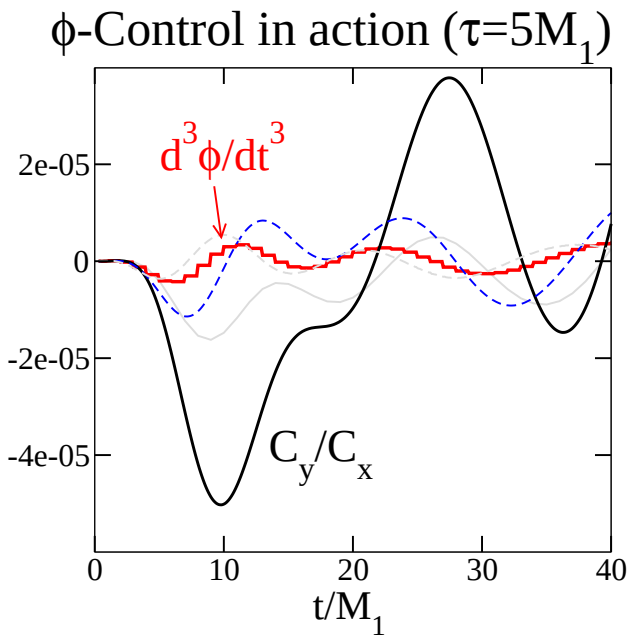
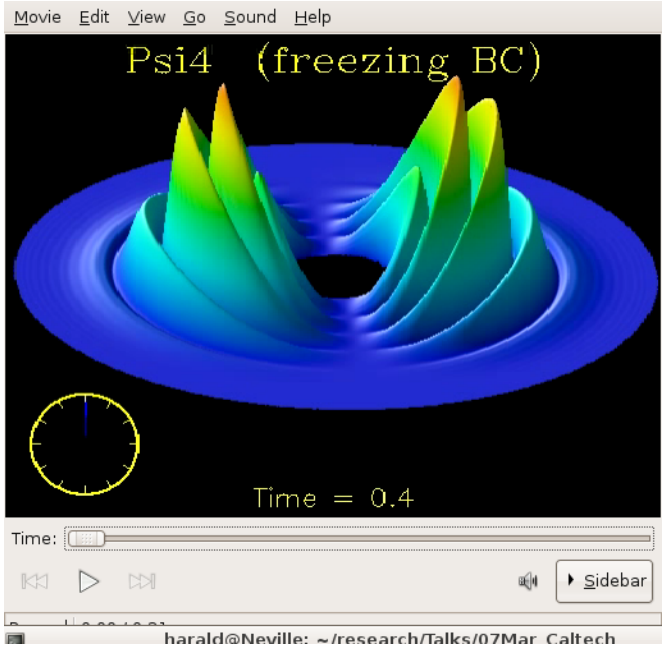
3. R(t) and a(t) determined by dynamic control based on current AH location

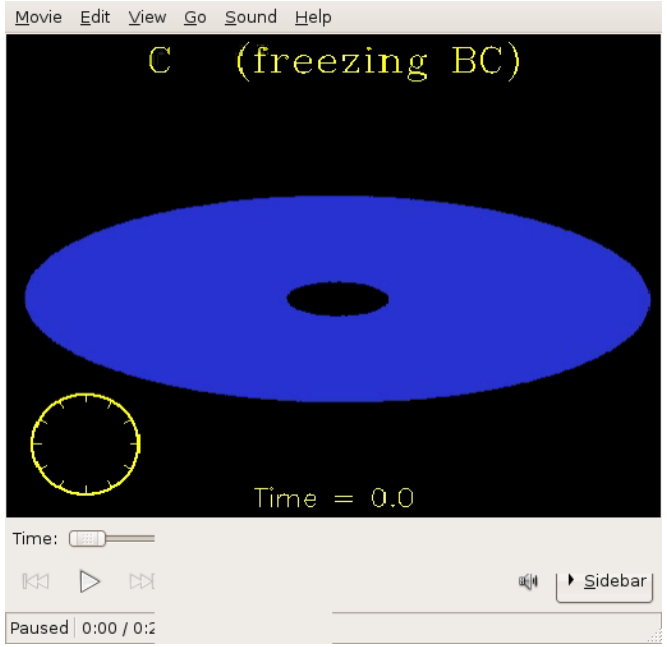


Outer boundary conditions

All physics lies in the boundary conditions!

- Must prevent influx of constraint violations





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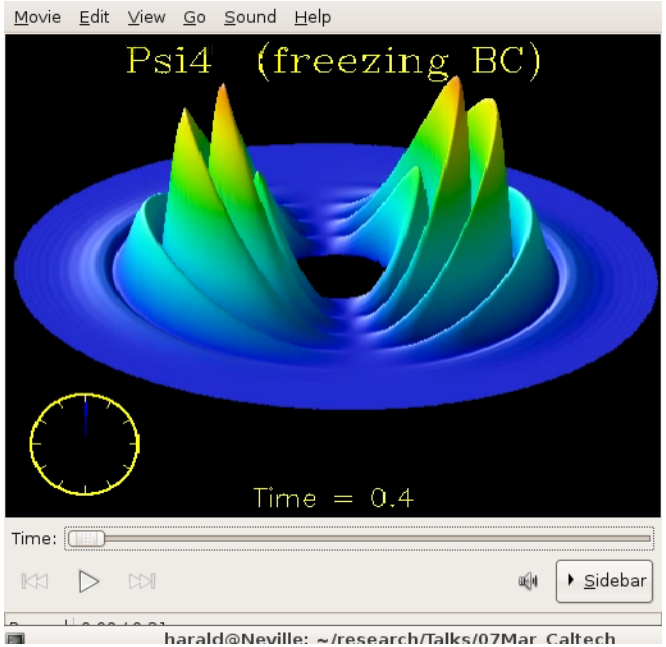

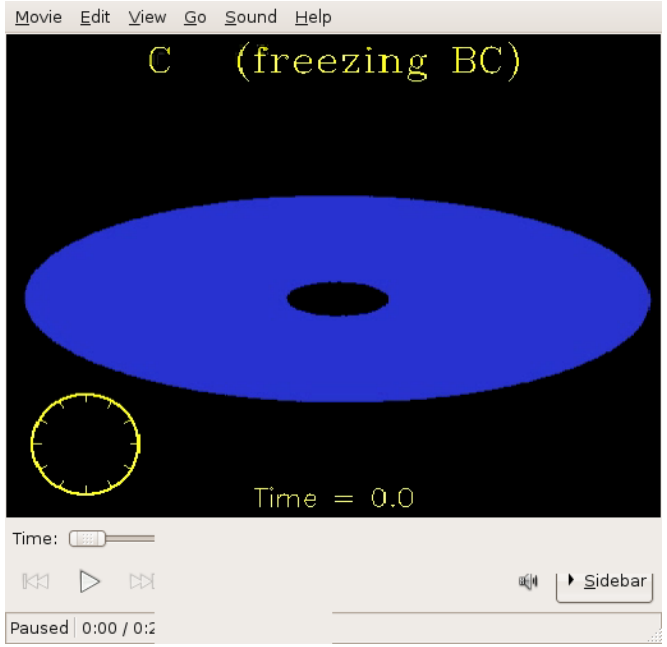

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  - Consider Newman-Penrose scalars
    - Ψ_4 is represented by outgoing characteristic fields (good!)
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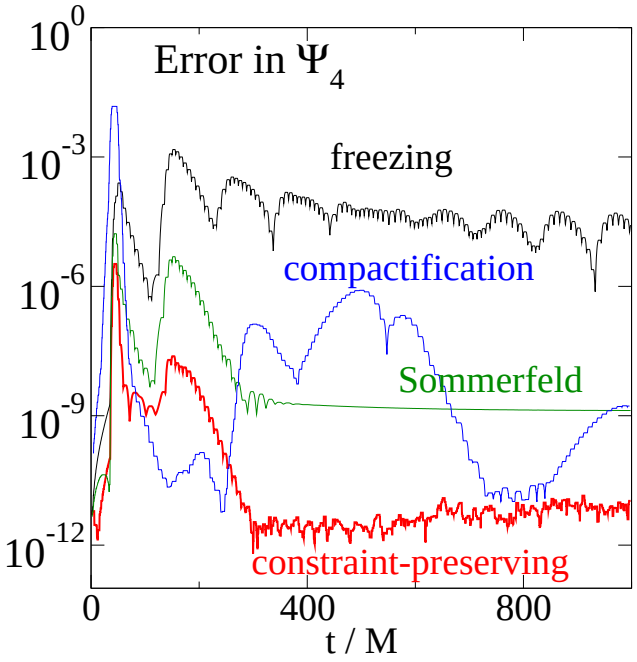

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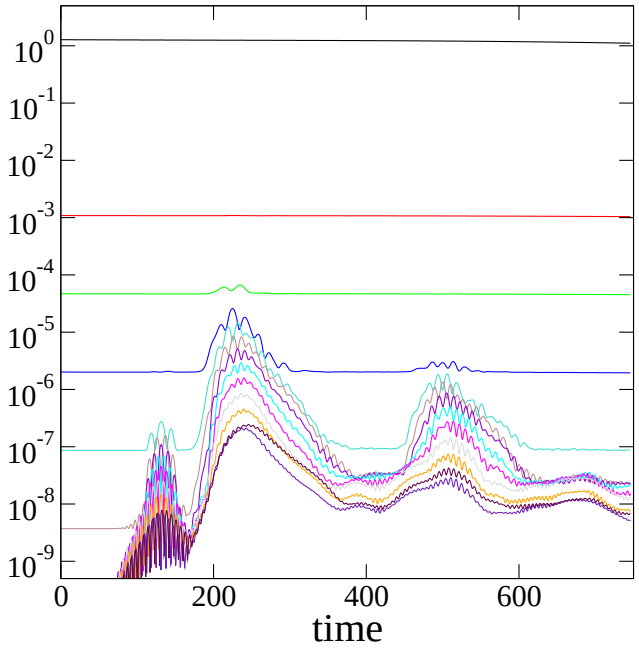

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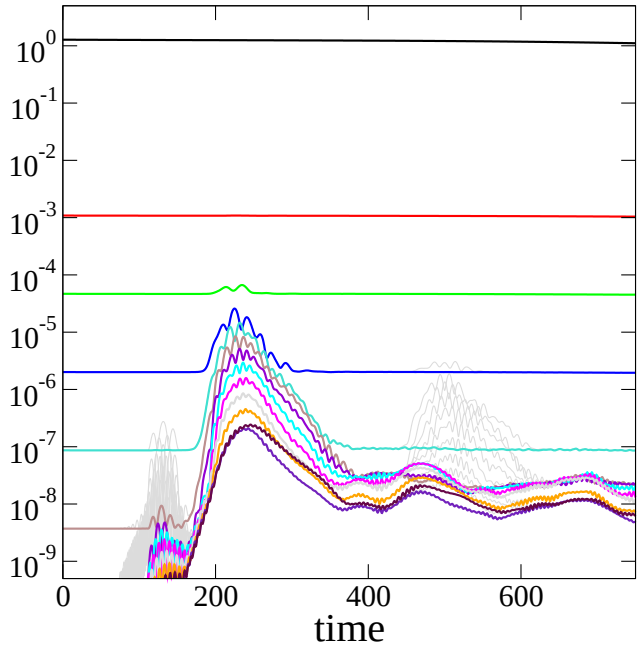
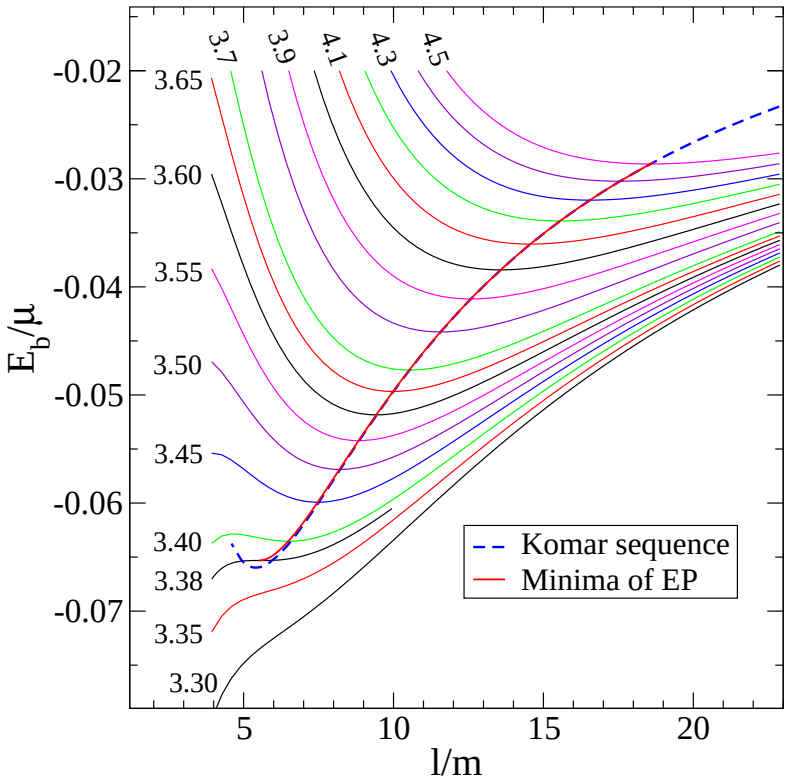
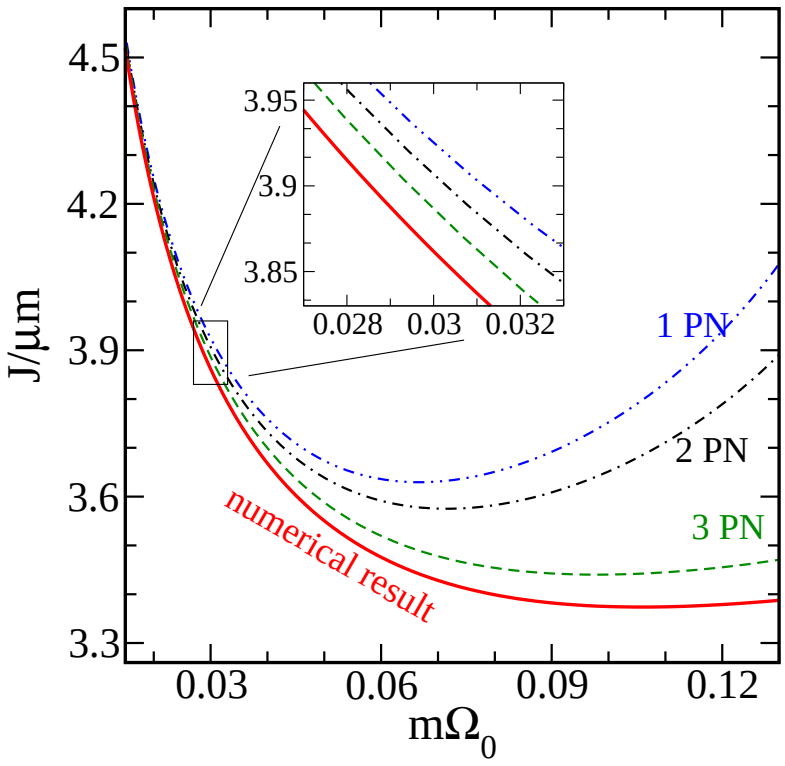


Initial data

Zero radial velocity

- Quasiequillibrium data (Cook, HP, et al, 02-06)
- Equal mass, no spin
- Black holes in *circular* orbits
- Sequences of initial data sets at different separation





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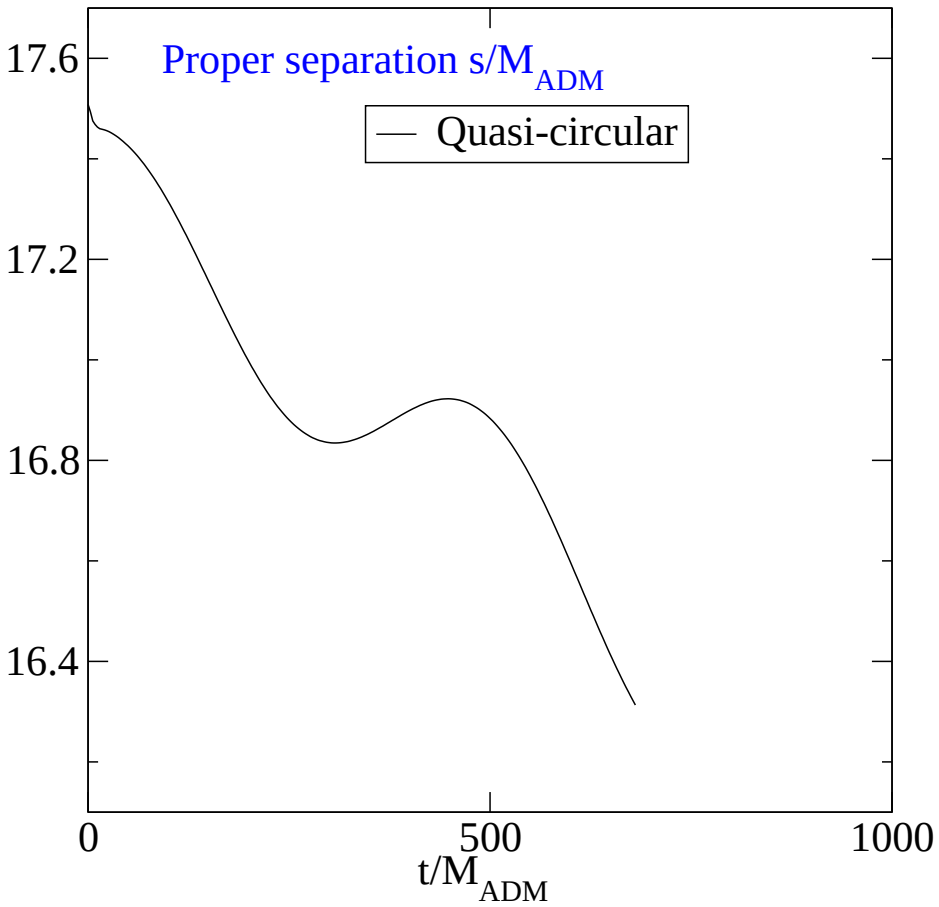

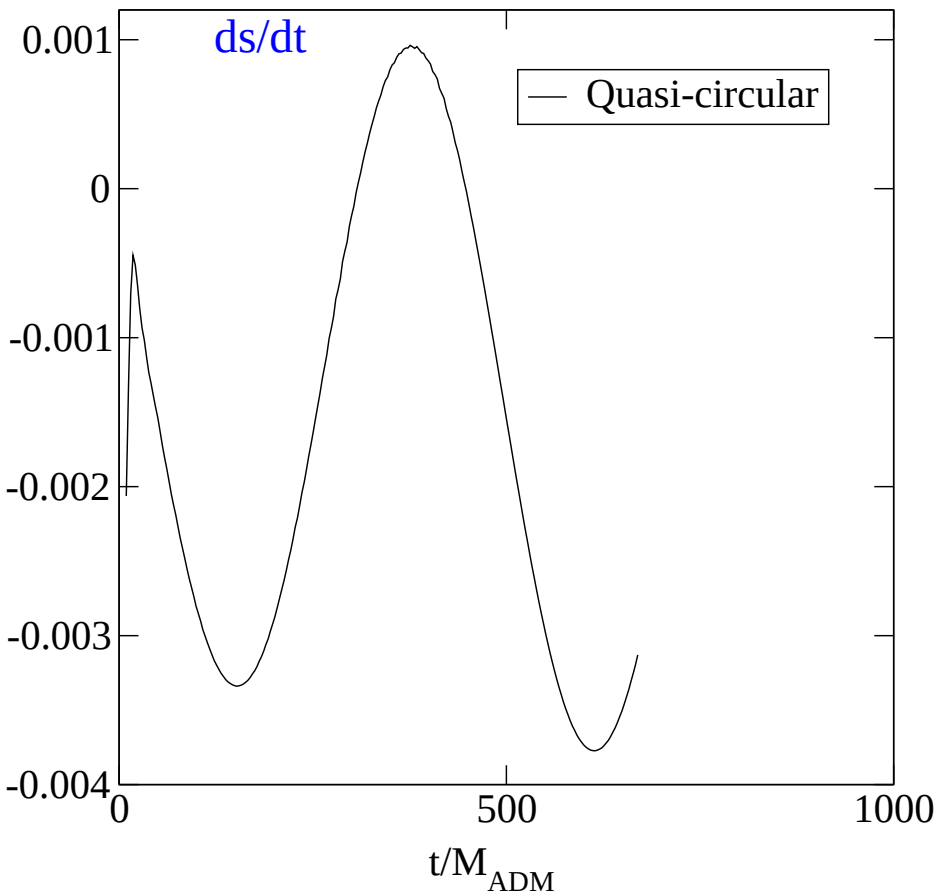

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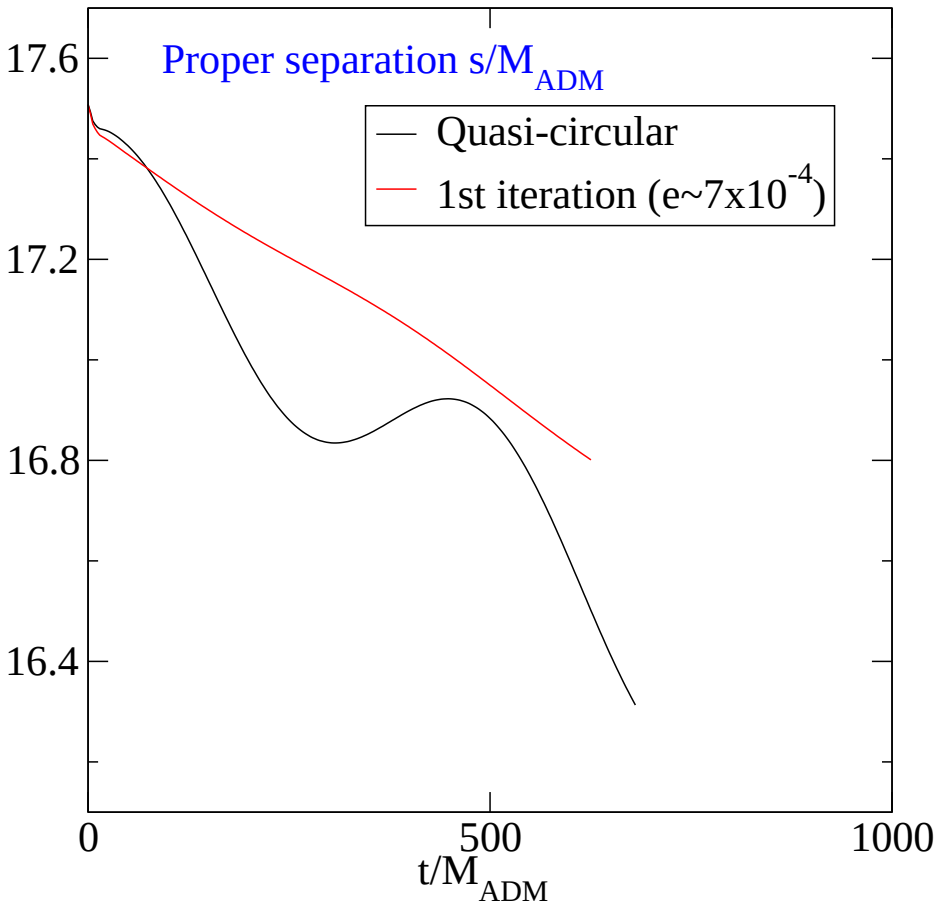

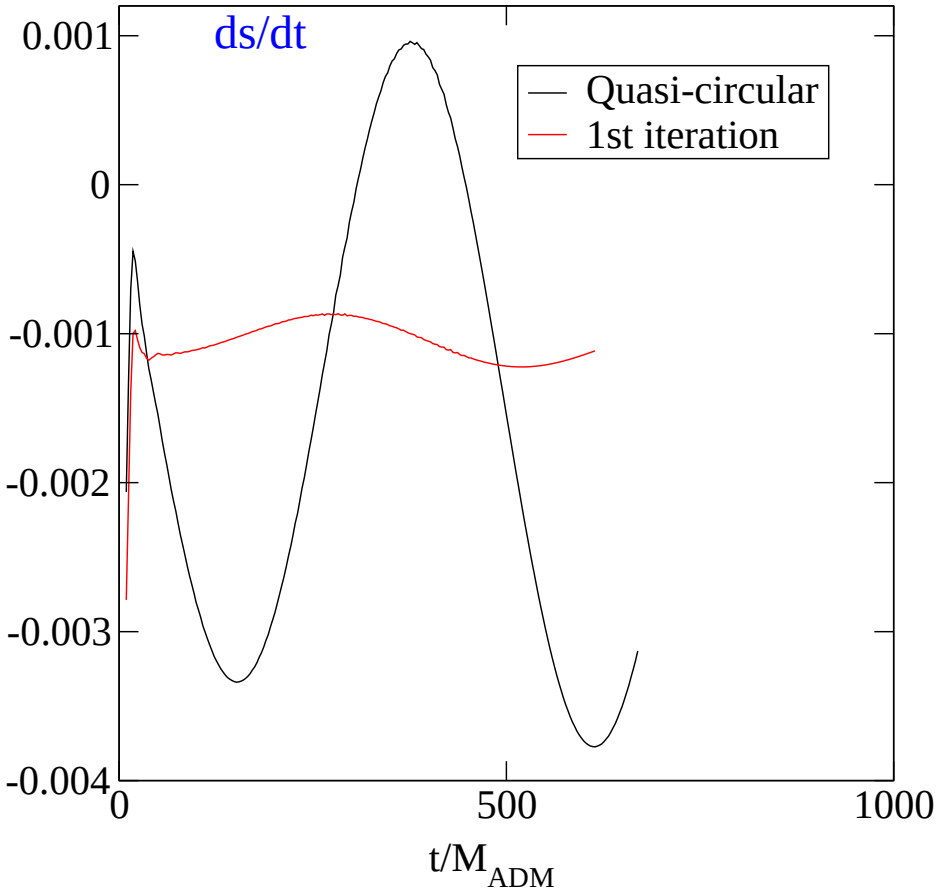

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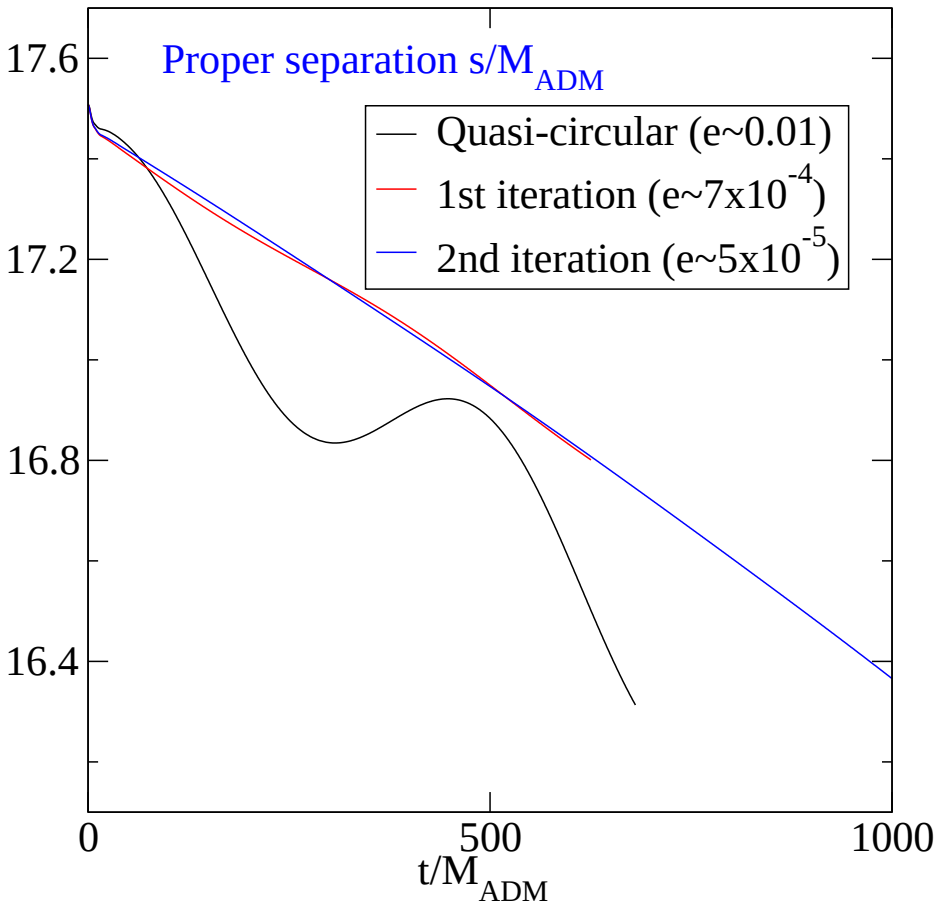
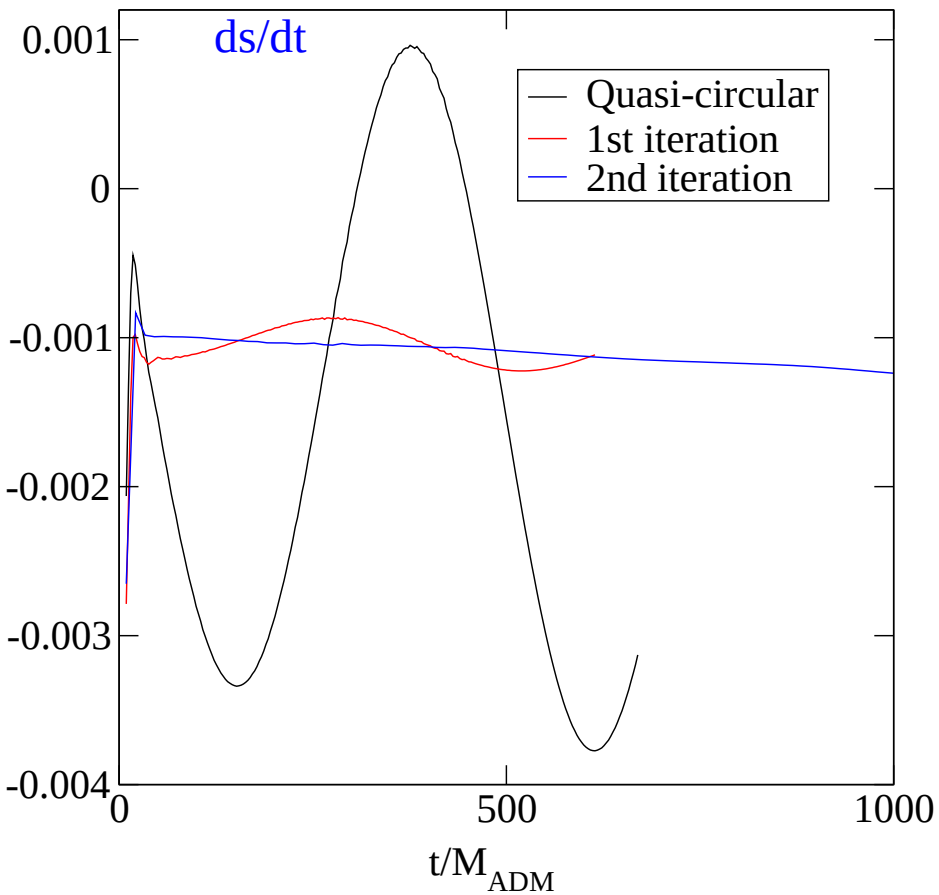
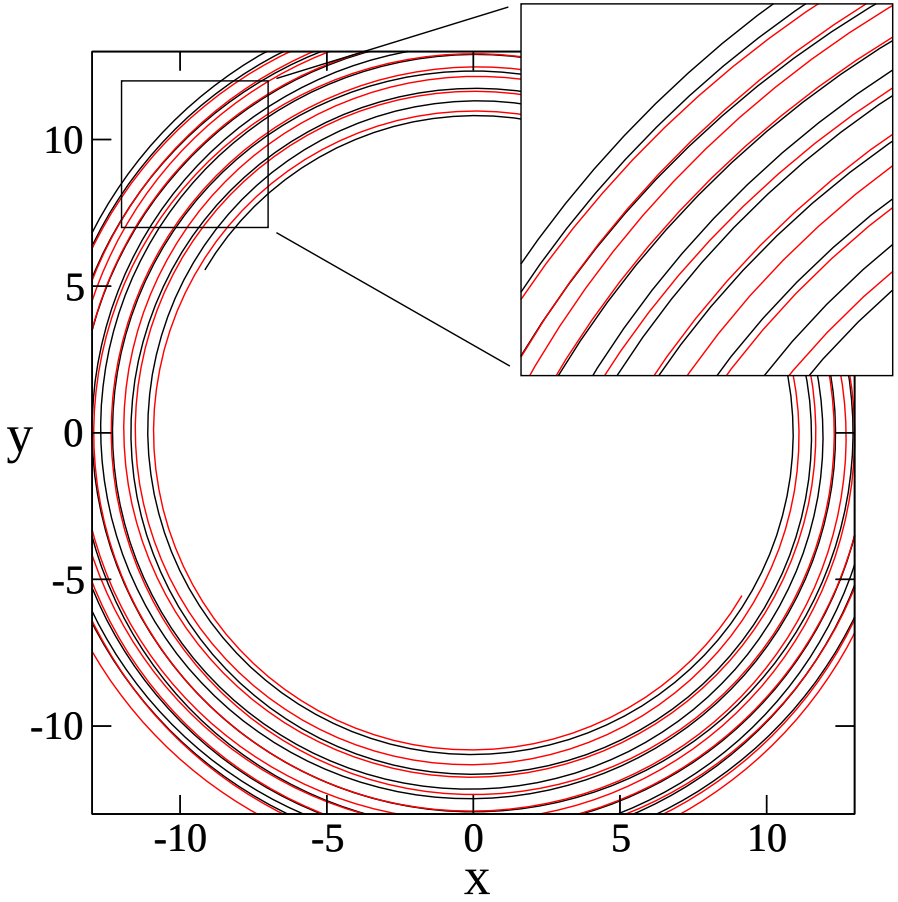




Orbital trajectory

Quasi-circular



Orbital trajectory

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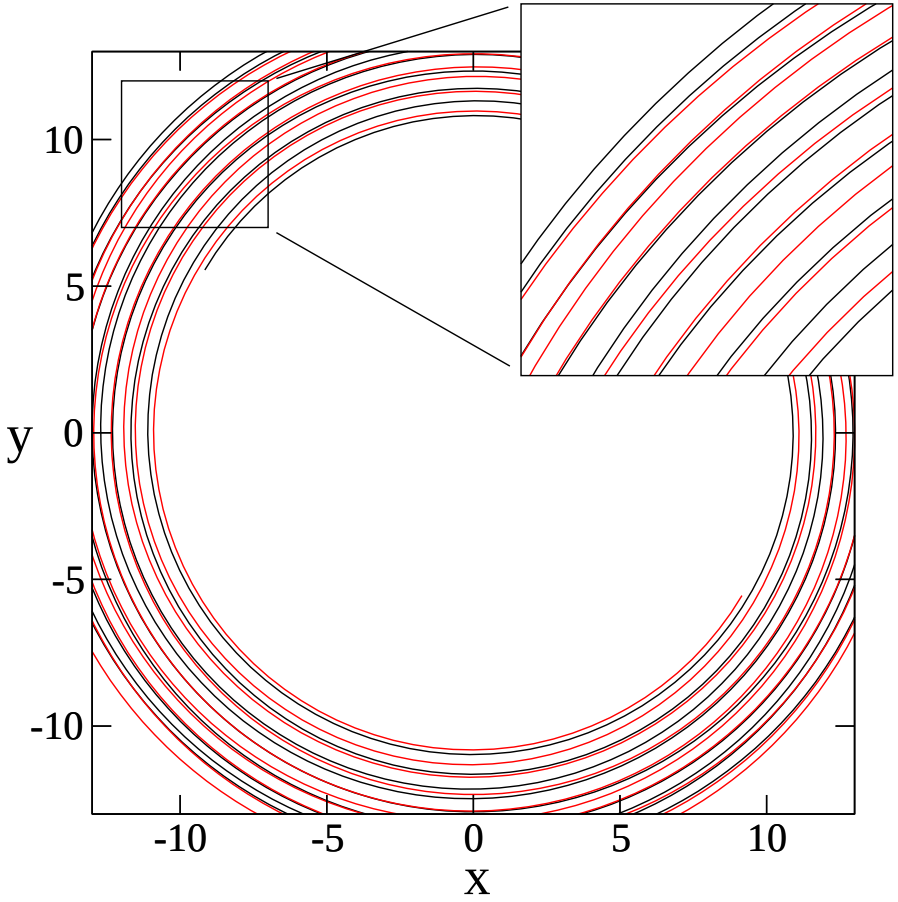
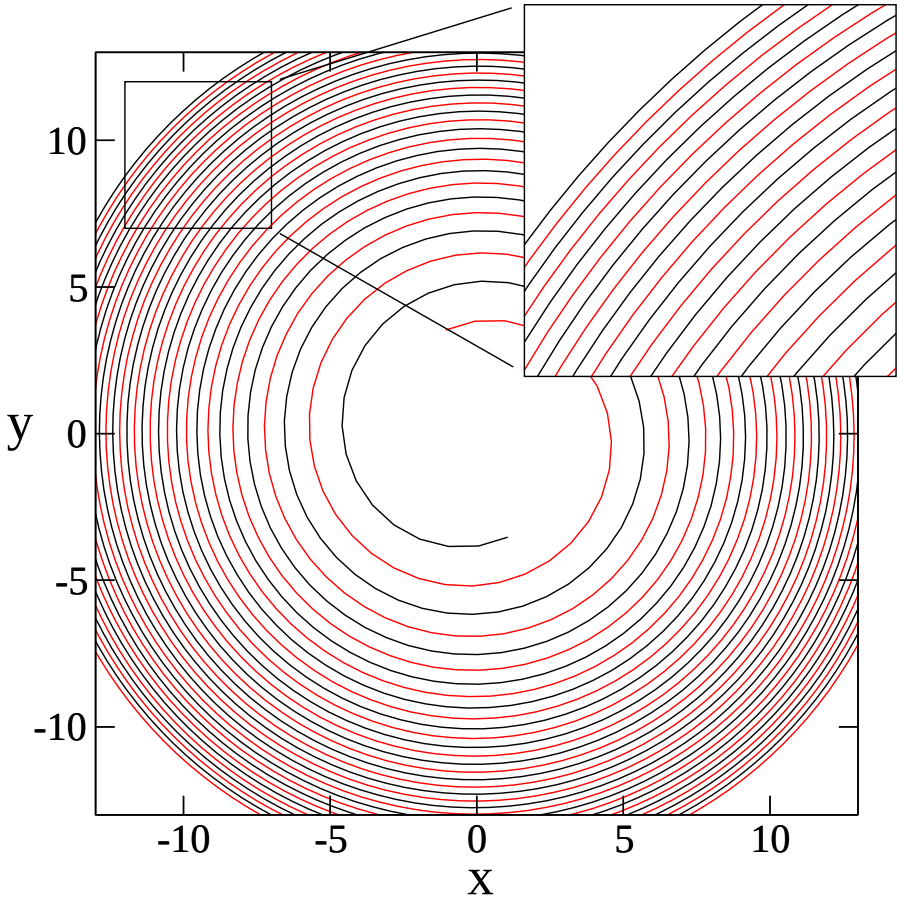
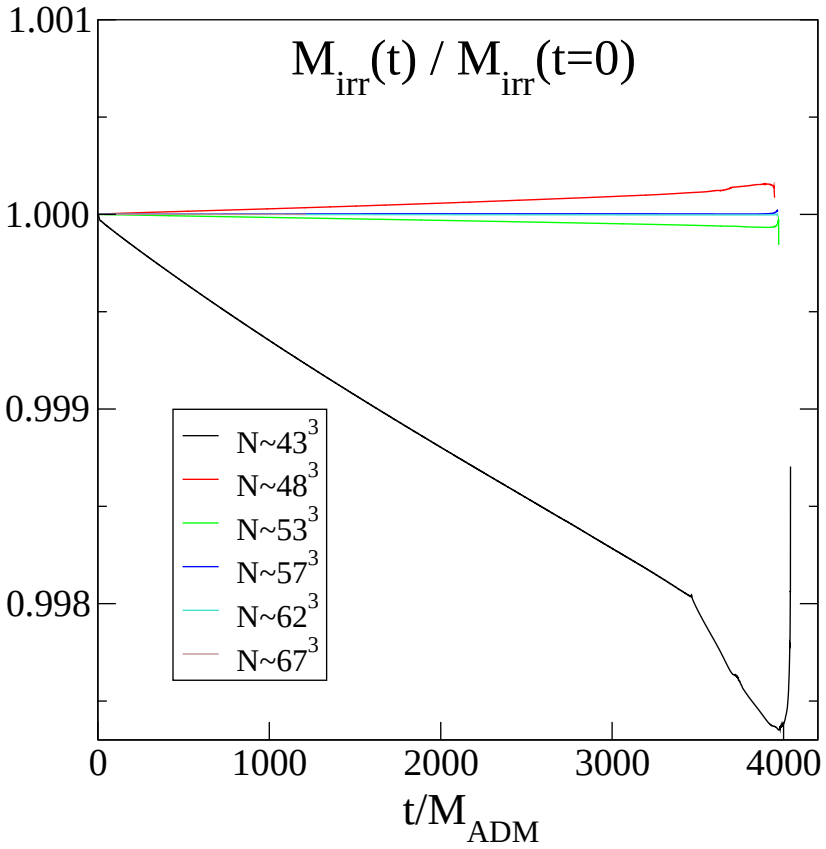


Eccentricity reduced

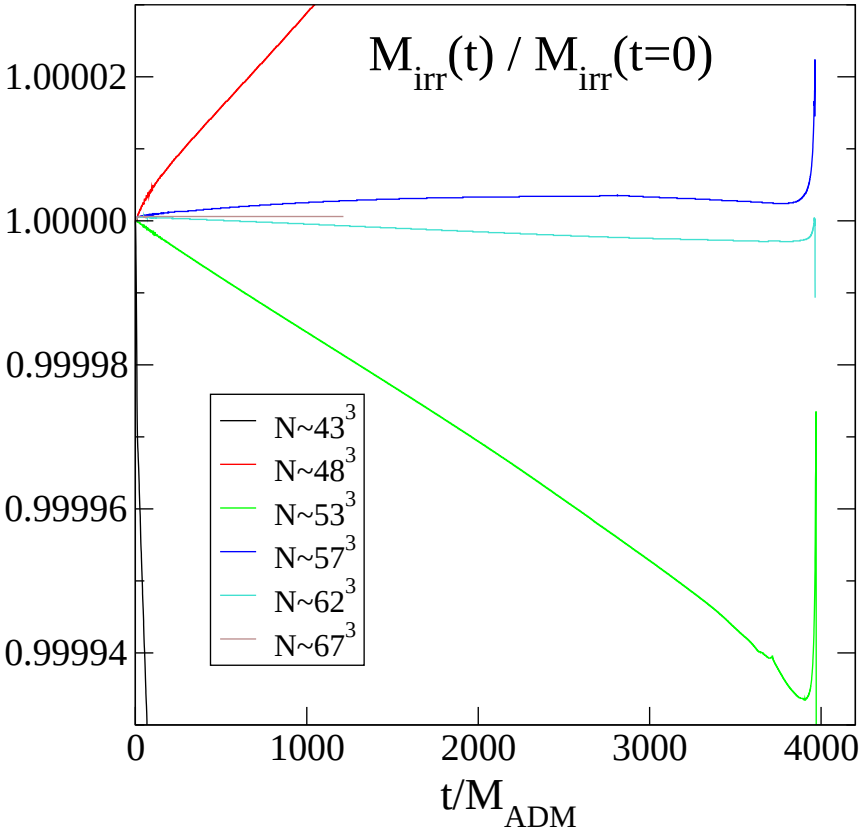


Results

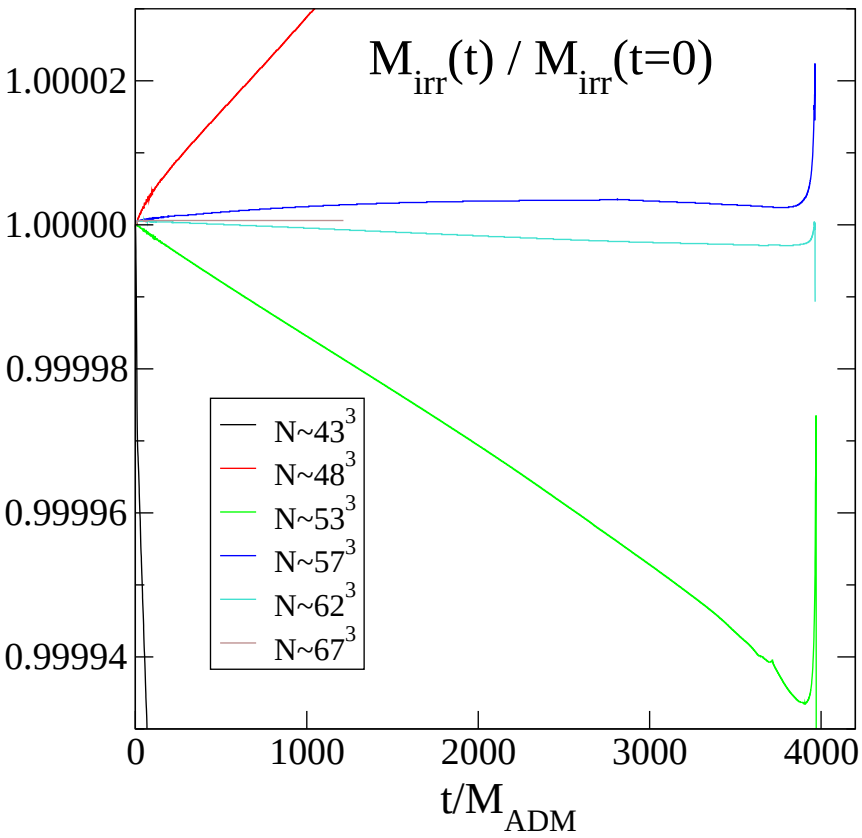
Irreducible Mass

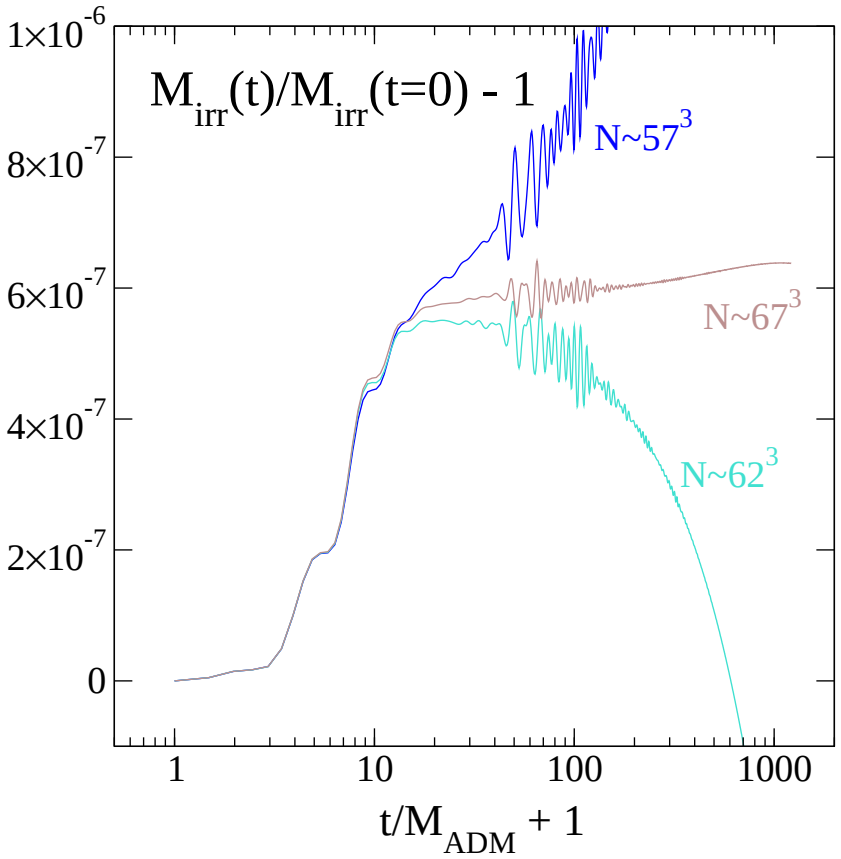


Irreducible Mass

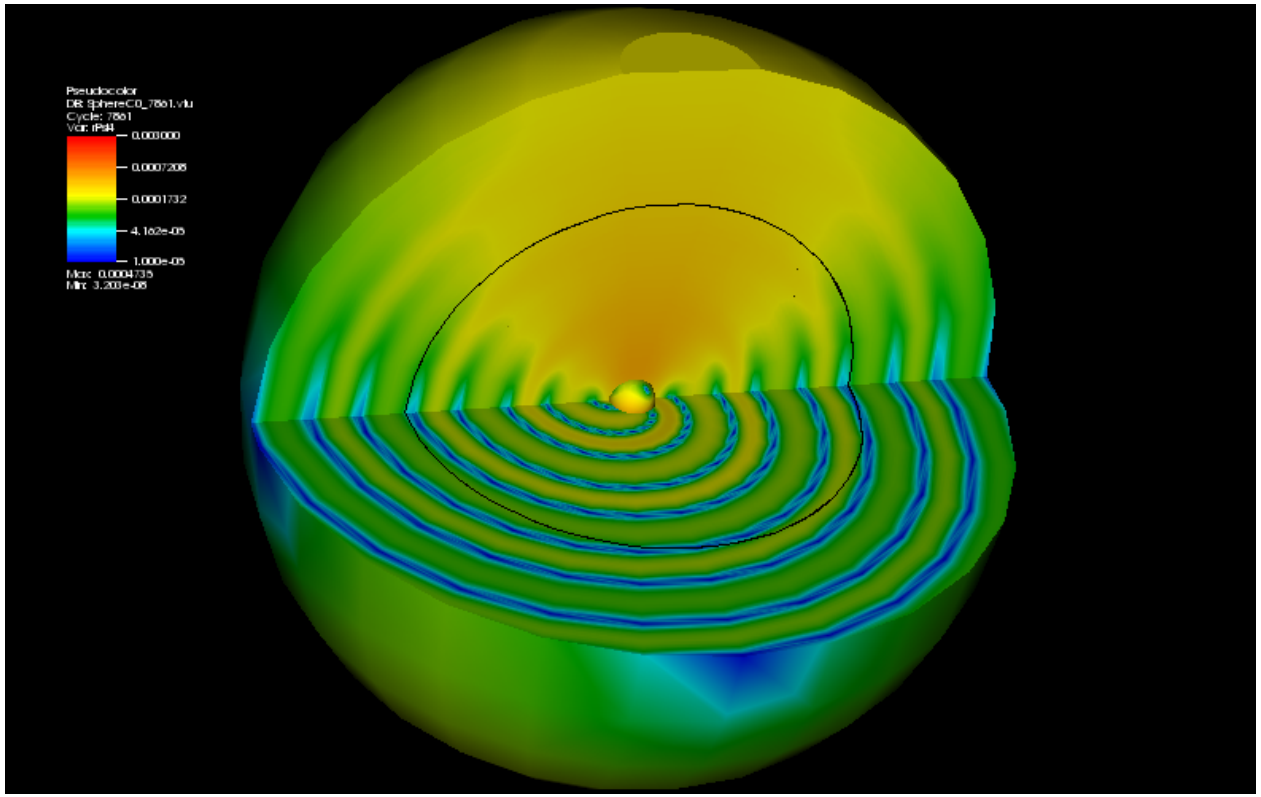


Irreducible Mass

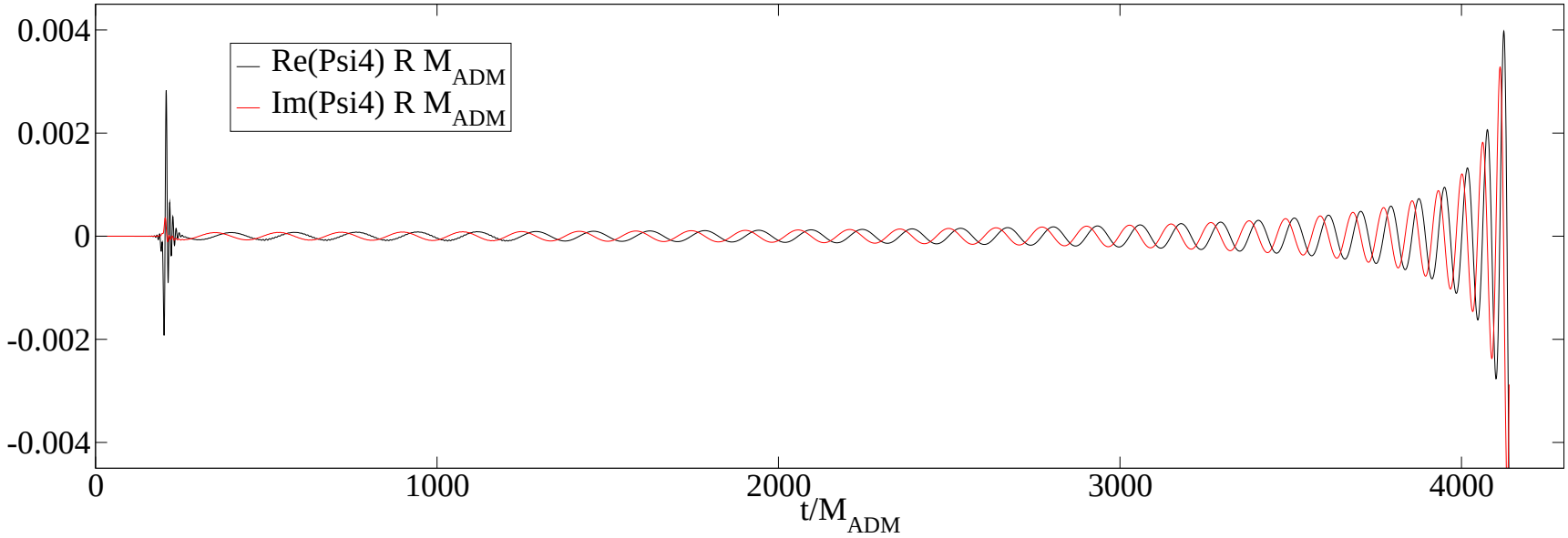


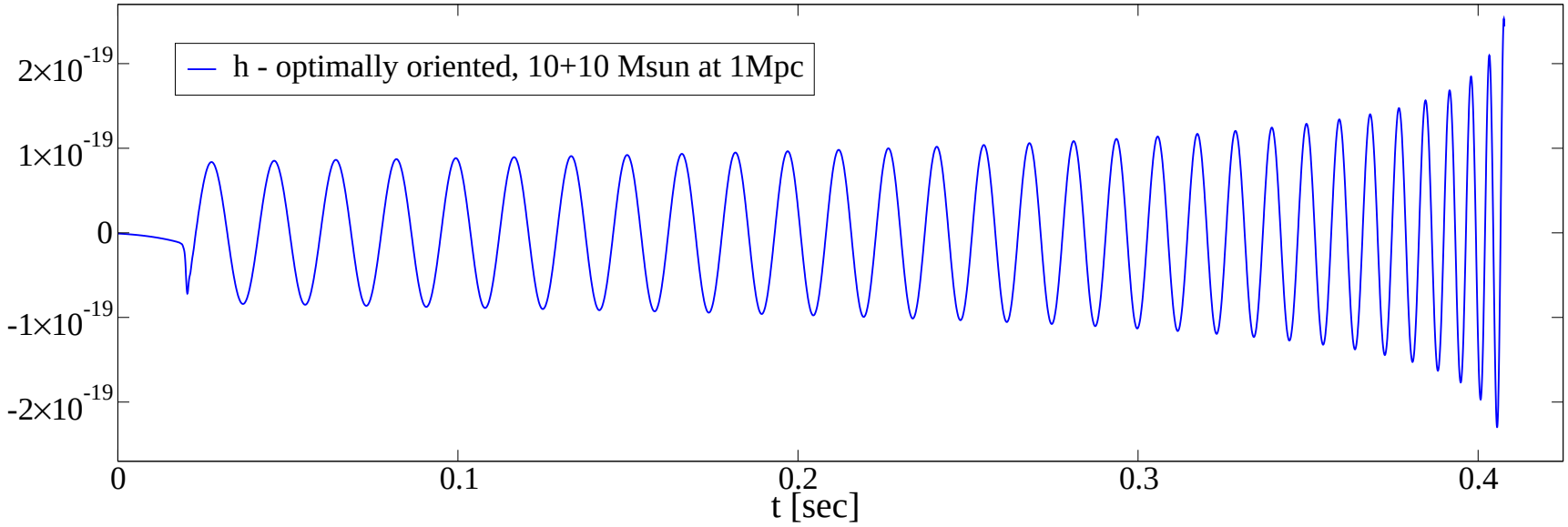


- Initial increase of $M_{\rm irr}$ by $6 \cdot 10^{-6}$.
- $|\delta M_{\rm irr}| < 10^{-7}$ in the next 3.5 orbits


Pseudocolor
DB: Sphere.C0_7861.vtu
Cycle: 7861
Var: rPsi4
0.000000
0.0007208
0.0001732
4.162e-05
1.000e-05
Max: 0.0004735
Min: 3.200e-08

Waveforms


Re(Psi4) R M_ADM
Im(Psi4) R M_ADM
0.004
0.002
0
-0.002
-0.004
0
1000
2000
3000
4000
t/M_ADM


h - optimally oriented, 10+10 Msun at 1Mpc
2×10^-19
1×10^-19
0
-1×10^-19
-2×10^-19
0
0.1
0.2
0.3
0.4
t [sec]

Comparison against post-Newtonian

What to compare?

- Orbital phase?
  - Is gauge dependent
- Waveform h
  - For NR, $h \sim \int\int \Psi_4 + A + Bt$.
    Integration constants are problematic.
- Newman-Penrose scalar Ψ_4
  - Choose optimal orientation (direction normal to orbital plane)
  - Decompose and compare magnitude and phase: $\Psi_4 = |\Psi_4| \exp(-i\varphi)$.
- Procedure
  - Set $M_1 = M_2 = M_{\mathrm{irr,NR}}$.
  - Two-parameter fit in t_c and ϕ_0 over the 2nd quarter of the evolution.
  - Examine how closely PN and NR track each other at other times.

Digging through post-Newtonian

- Equation of motion (EoM):

$$a_1^i = \frac{G m_2 n_{12}^i}{r_{12}^2} + \frac{1}{c^2}(\ \dots\ ) + \dots + \frac{1}{c^7}(\ \dots\ ) + \mathcal{O}\Big(\frac{1}{c^8}\Big)$$

  Radiation reaction enters at $1/c^{5/2}$; thus, corrections to radiation reaction only known to $\mathcal{O}(1/c^2)$ beyond leading order.

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  Radiation reaction enters at $1/c^{5/2}$; thus, corrections to radiation reaction only known to $\mathcal{O}(1/c^2)$ beyond leading order.

- EoM imply energy

$$E = -\frac{\mu c^2 x}{2}\left\{1 + (\ldots)x + \ldots + (\ldots)x^3 + (\ldots)x^{7/2}\right\} + \mathcal{O}\left(\frac{1}{c^8}\right)$$

  where $x = (Gm\omega/c^3)^{2/3} = \mathcal{O}(1/c^2)$

- For circular orbits, multipole expansions give energy flux

$$\mathcal{F} = \frac{32 c^5}{5G}\nu^2 x^5\left\{1 + (\ldots)x + \ldots + (\ldots)x^{7/2}\right\} + \mathcal{O}\left(\frac{1}{c^8}\right)$$

Digging through post-Newtonian

- Assume energy balance $E/dt = -\mathcal{F}$

$$\Rightarrow \quad \frac{dx}{dt} = -\frac{\mathcal{F}}{dE/dx} = \frac{32c^3\nu}{5Gm}x^4\left\{1 + (\ldots)x + \ldots + (\ldots)x^{7/2}\right\}$$

- Integration yields orbital phase

$$\phi = \frac{1}{4}\left(\frac{5Gm}{c^3\nu}\right)^{1/4}(t_c - t)^{-1/4}\left\{1 + \ldots + (\ldots)(t_c - t)^{-7/8} + \mathcal{O}(c^{-8})\right\}$$

Alternative: use $\mathcal{F}/(dE/dx)$ as is without series expansion.

Digging through post-Newtonian

- Waveforms

$$h_{+/x} = \frac{2G\mu x}{c^2 R}\left\{H^{(0)}_{+/x}(\imath,\psi) + \ldots + x^{5/2} H_{+/x}(\imath,\psi)^{(5/2)} + \mathcal{O}(c^{-6})\right\}$$

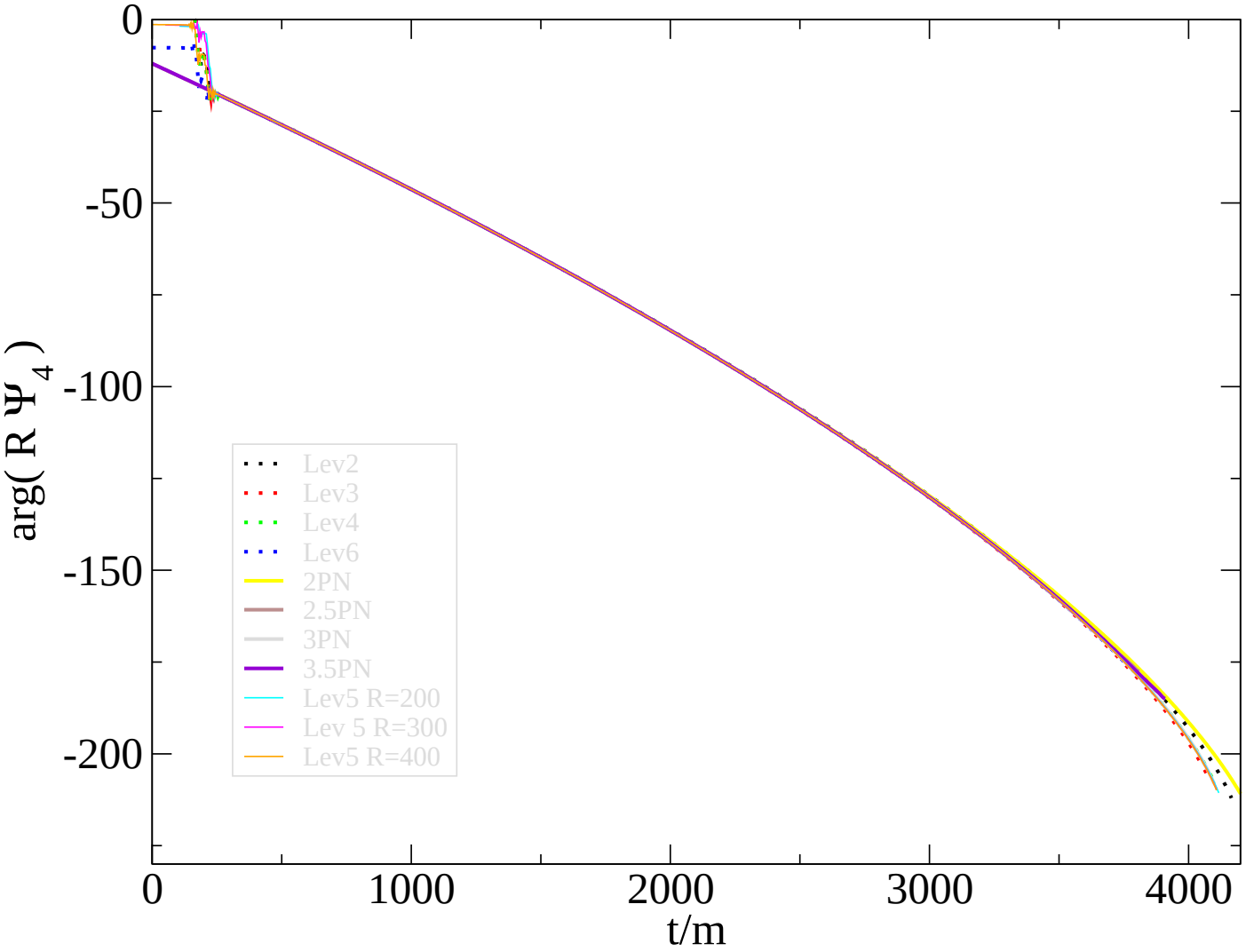
with $\psi = \phi - \frac{2GM_{\rm ADM}\omega}{c^3}\ln\frac{\omega}{\omega_0}$

- Newman-Penrose scalar

$$\Psi_4 = \ddot{h}_+ - i\ddot{h}_x = A_{\rm PN}\exp\left(i\varphi_{\rm PN}\right)$$

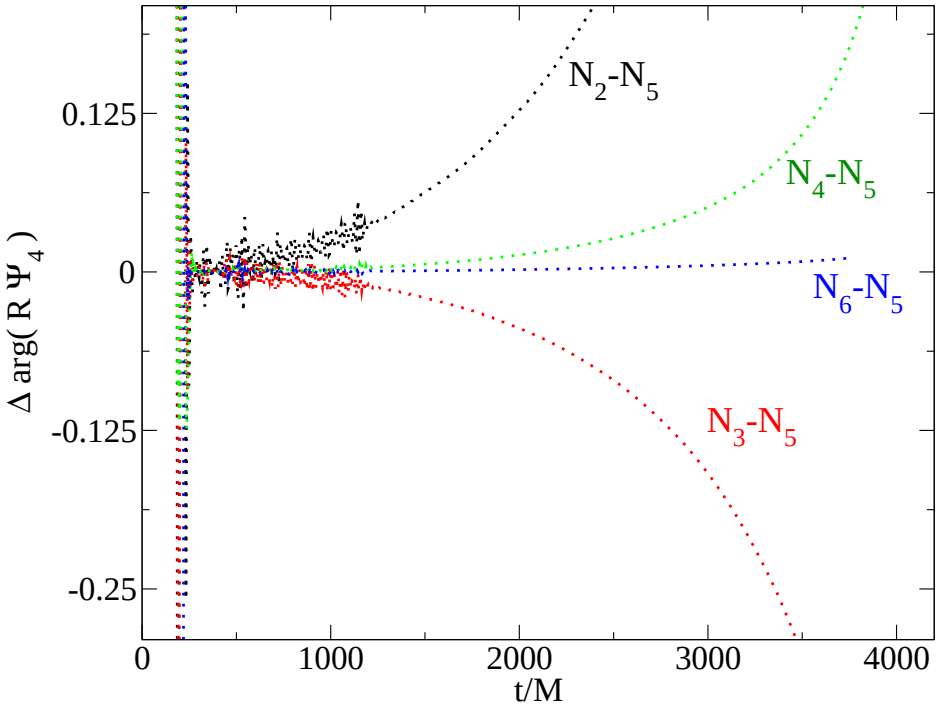
- For "n-th order comparison", use ϕ up to order n, and use $H_{+/x}$ up to order $\min(n, 5/2)$.

Comparison of Phase



PN vs NR – Phase

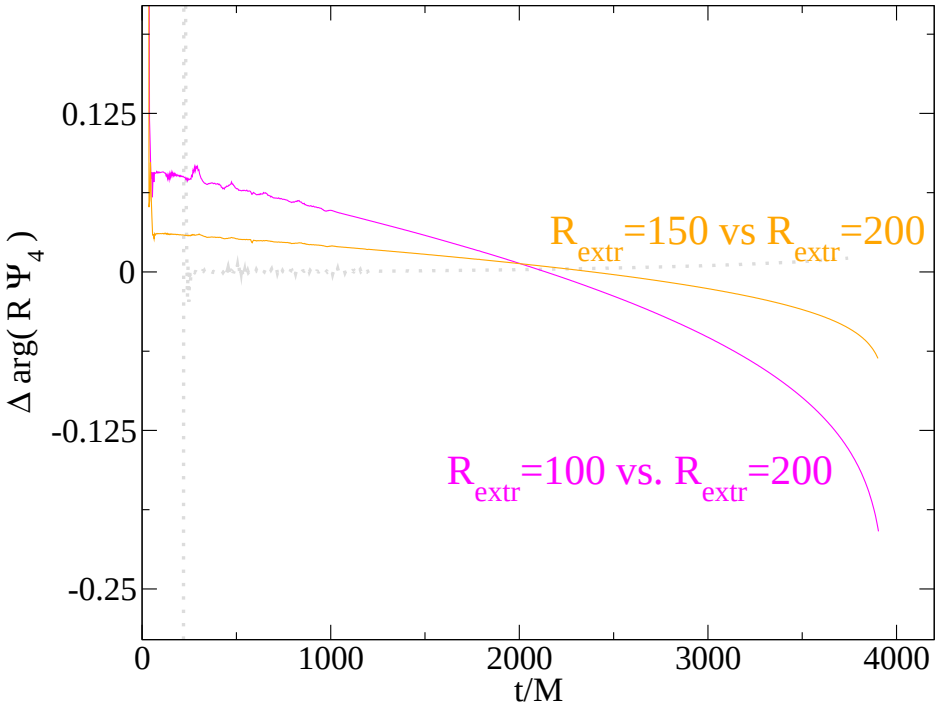
Error budget in simulation



- Spatial truncation error $\sim$ 0.02rad

PN vs NR – Phase

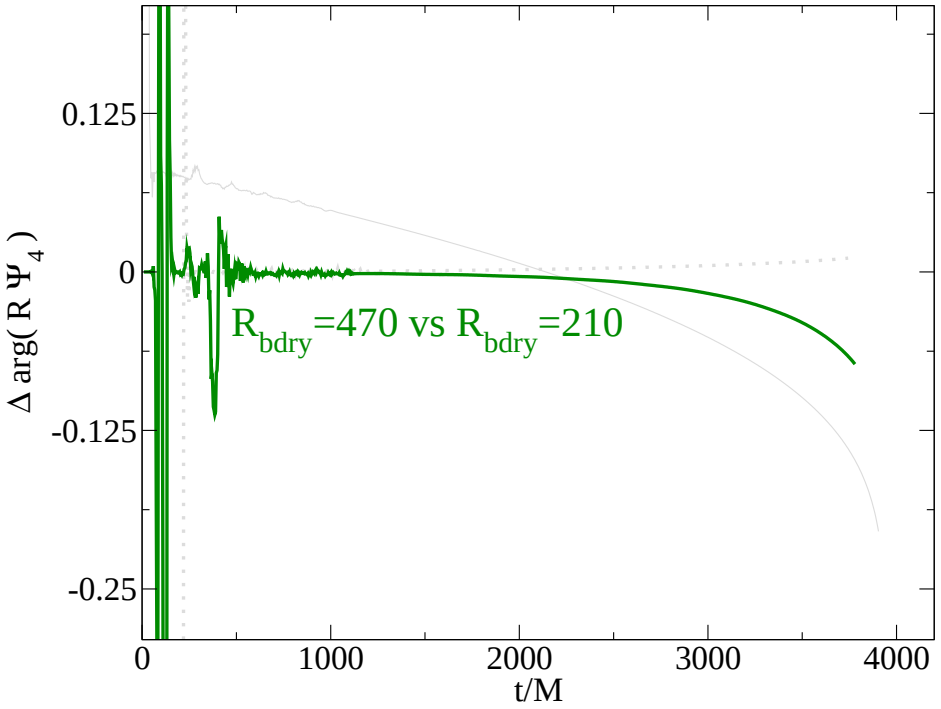
Error budget in simulation



- Spatial truncation error $\sim$ 0.02rad
- Finite extraction radius $\sim$ 0.2rad
  (Extrapolation $R_{\text{extr}} \to \infty$ underway)

PN vs NR – Phase

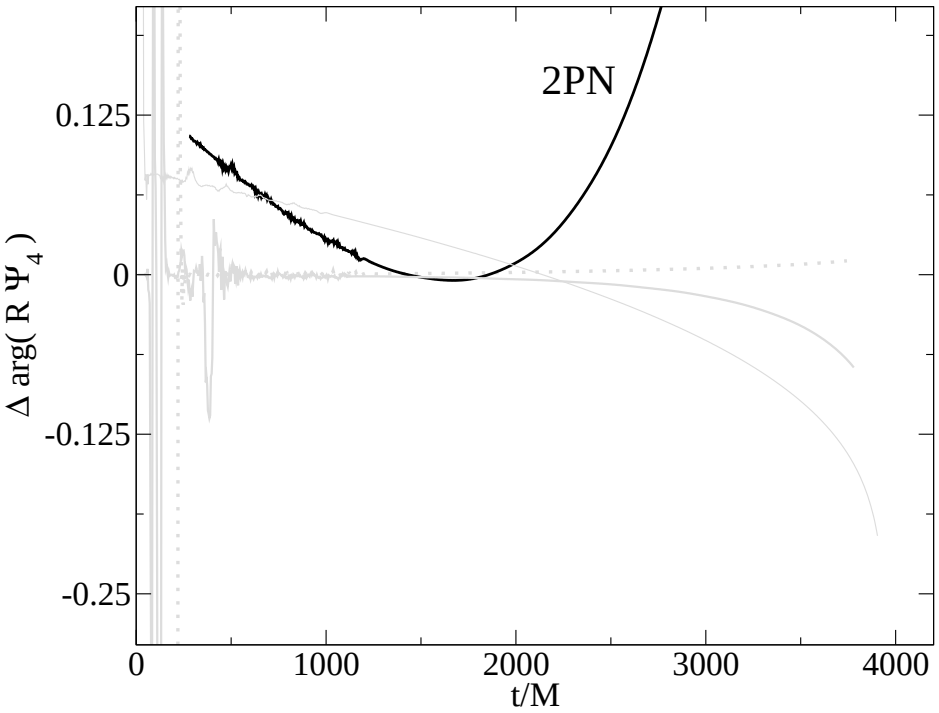
Error budget in simulation



- Spatial truncation error $\sim$ 0.02rad
- Finite extraction radius $\sim$ 0.2rad
  (Extrapolation $R_{\mathrm{extr}} \to \infty$ underway)
- Outer boundary $\sim$ 0.1rad

PN vs NR – Phase

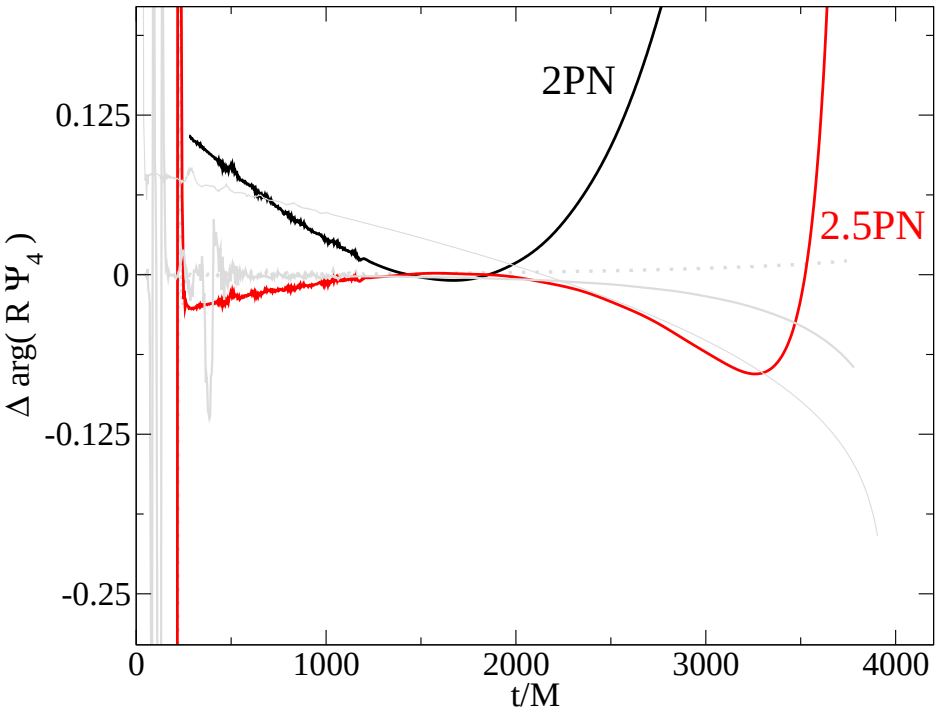
Differences PN-NR



- 2PN errors significant throught run

PN vs NR – Phase

Differences PN-NR



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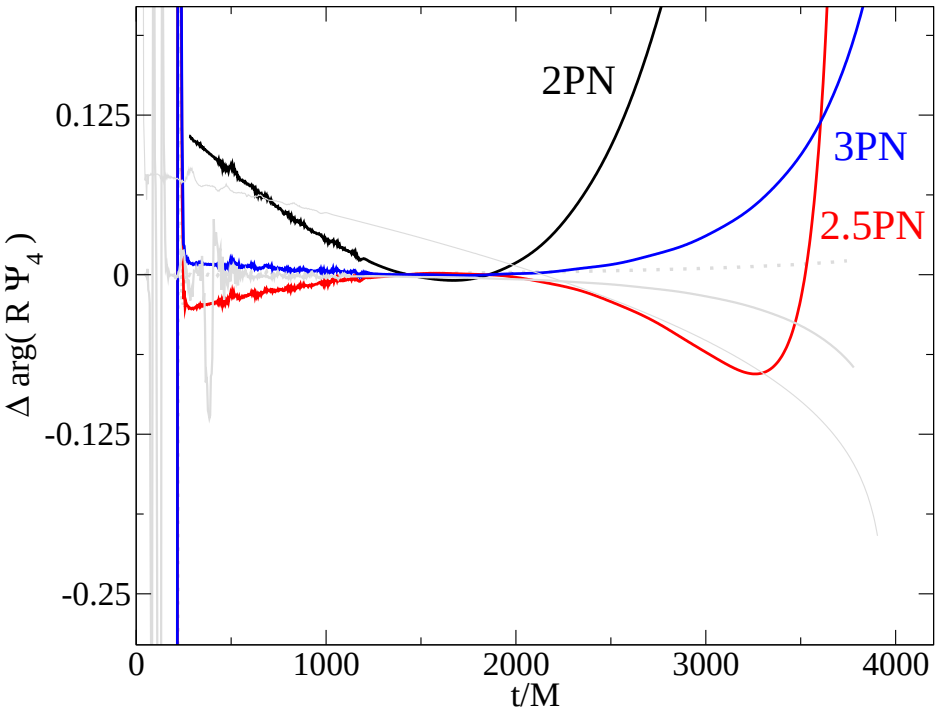
Differences PN-NR



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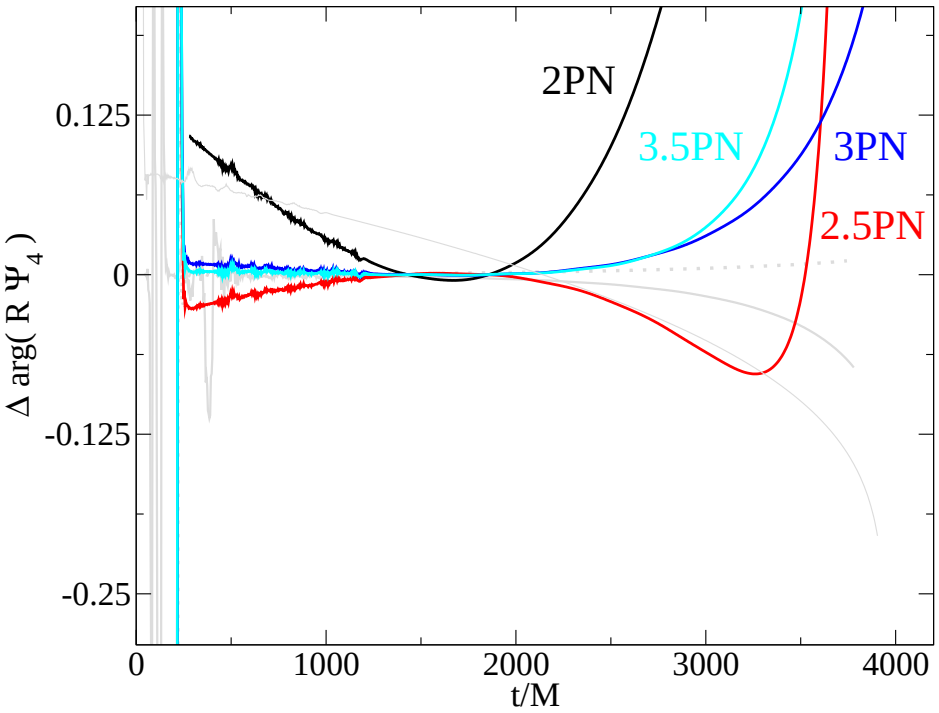
PN vs NR – Phase

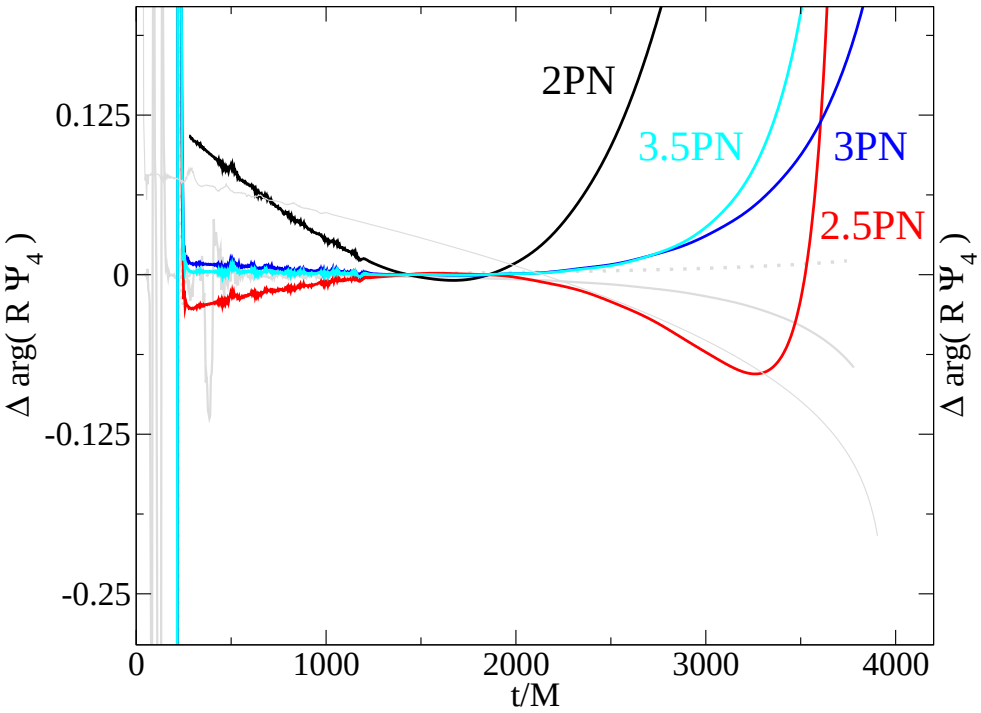
Differences PN-NR

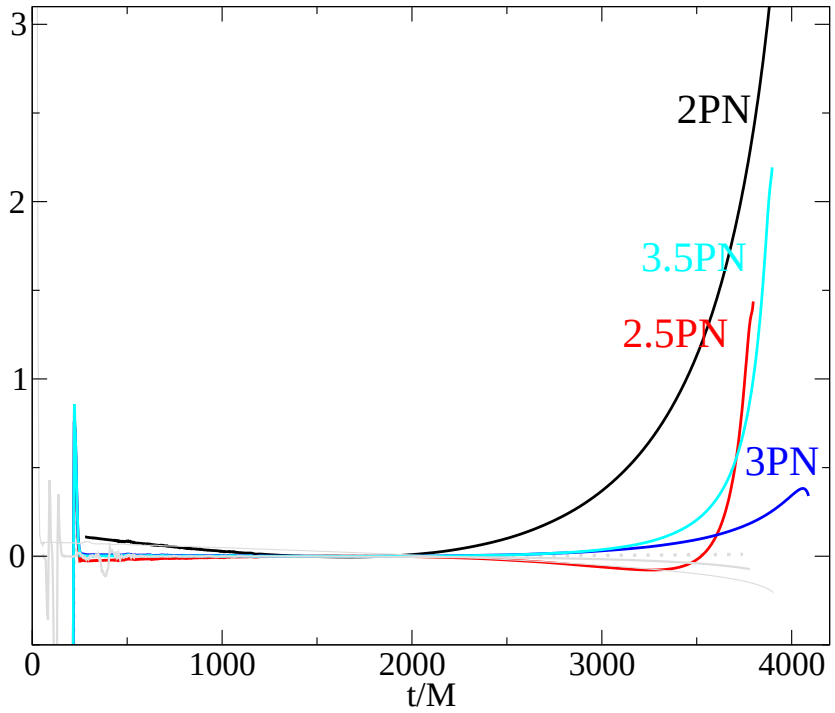


- 2PN errors significant throught run
- Good convergence of PN toward NR for $t \lesssim 2500M$
  - Verification of NR

PN vs NR – Phase

Differences PN-NR

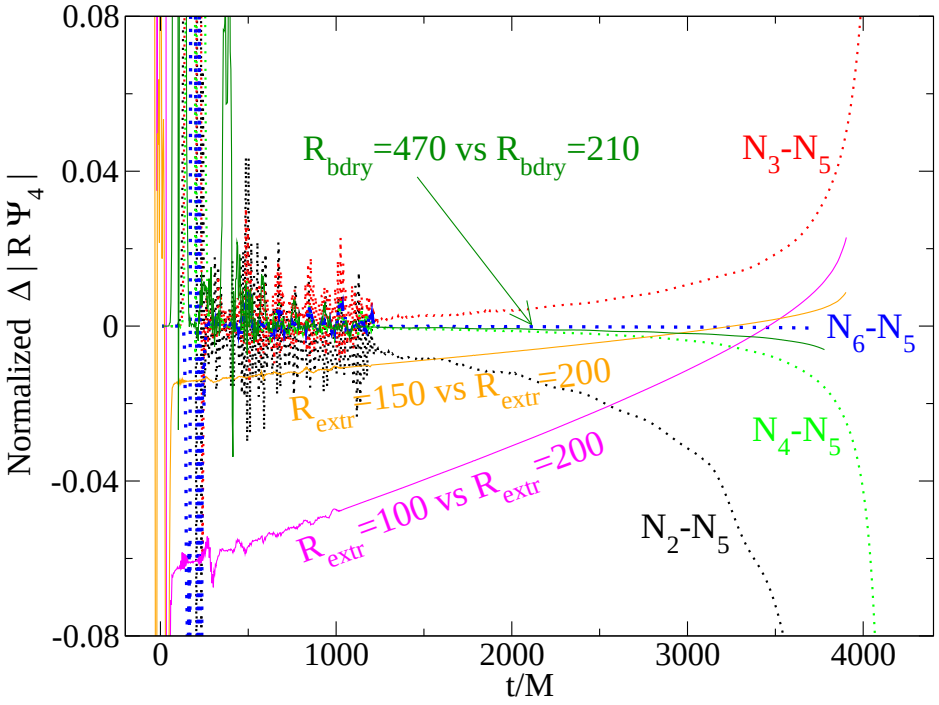




- 2PN errors significant throught run
- Good convergence of PN toward NR for $t \lesssim 2500M$
  - Verification of NR
- half-PN orders diverge a few 100M before coalescence
- 3PN–NR $\lesssim$ 0.5rad

Pn vs NR – Amplitude

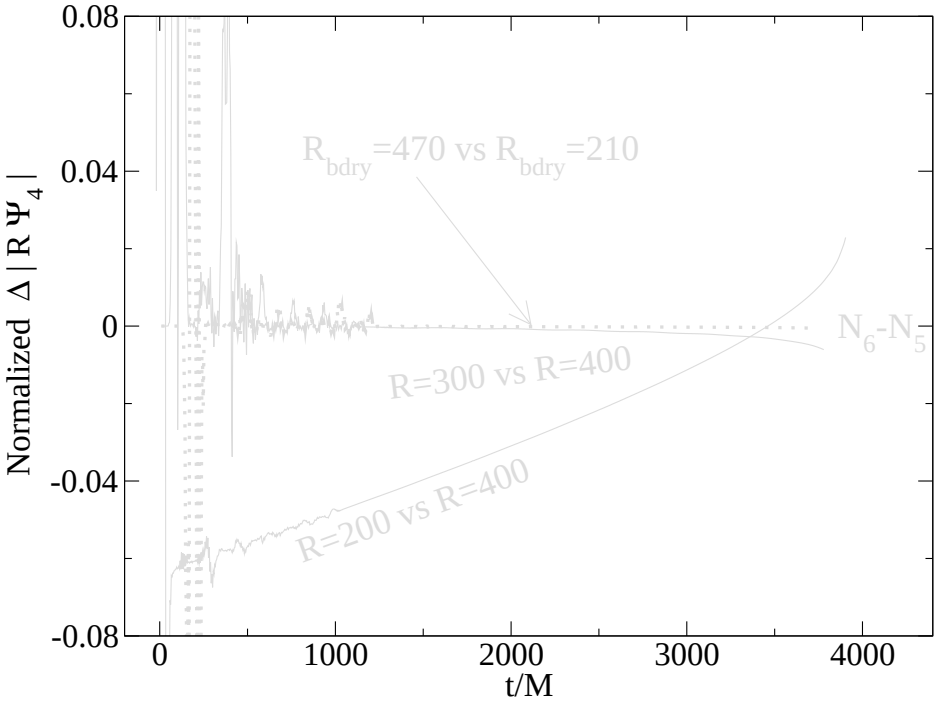
Error budget in simulation



- Spatial truncation error $\sim$ 0.1%
- Extraction radius $\sim$ 5%
- Outer boundary $\sim$ 1%

Pn vs NR – Amplitude

Differences PN - NR



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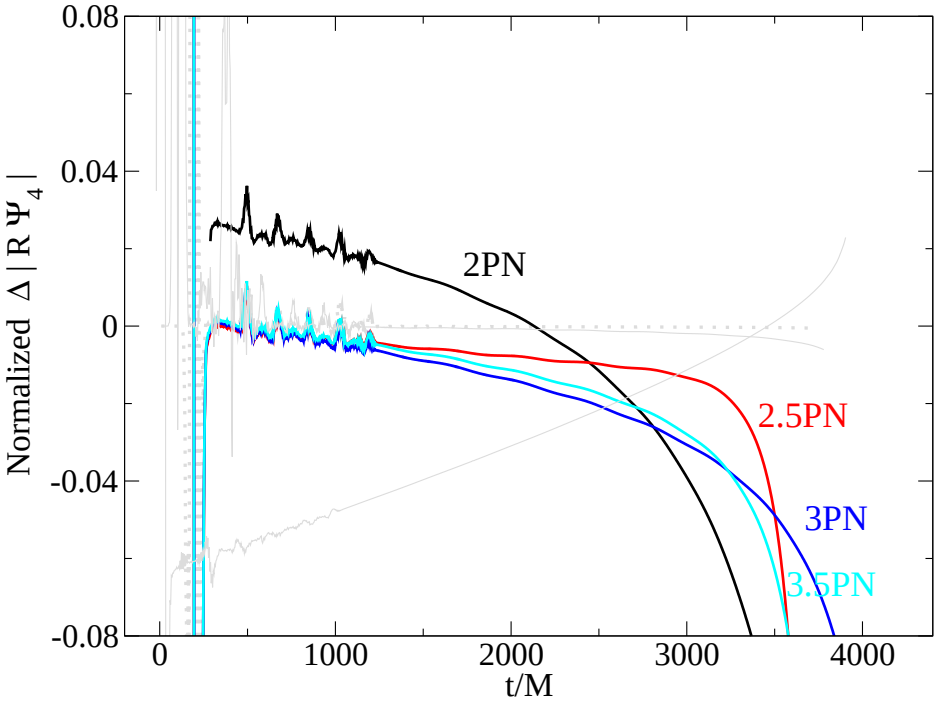
Differences PN - NR



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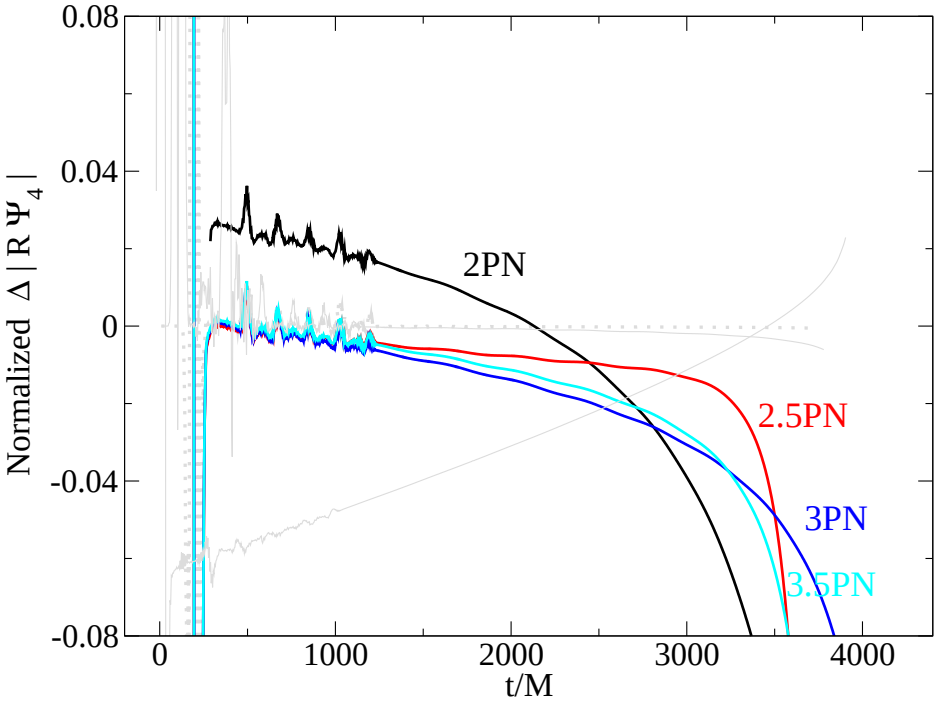
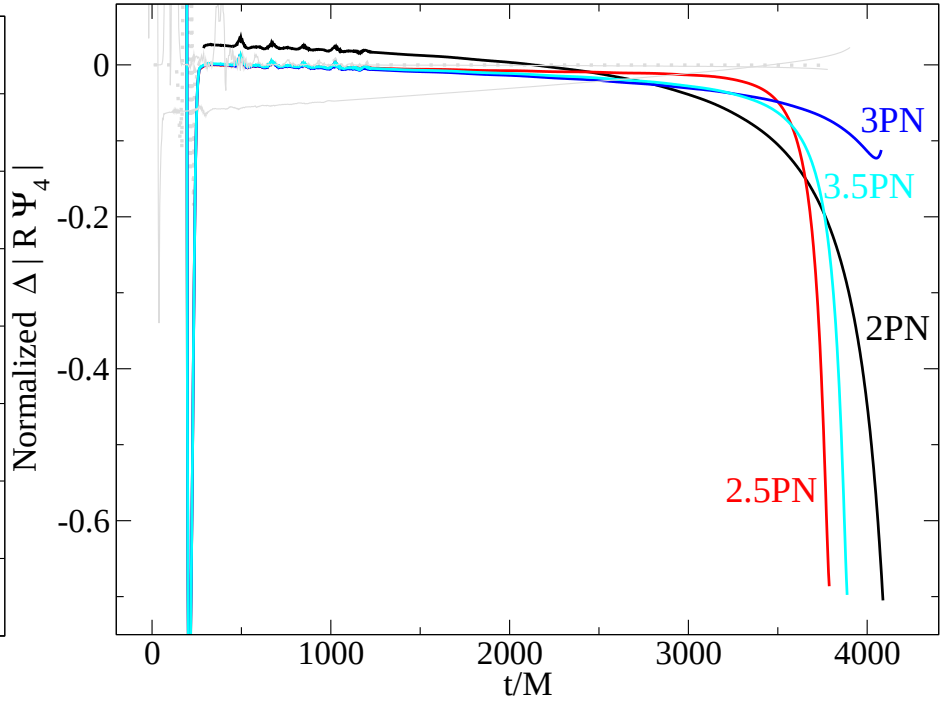
Pn vs NR – Amplitude

Differences PN - NR

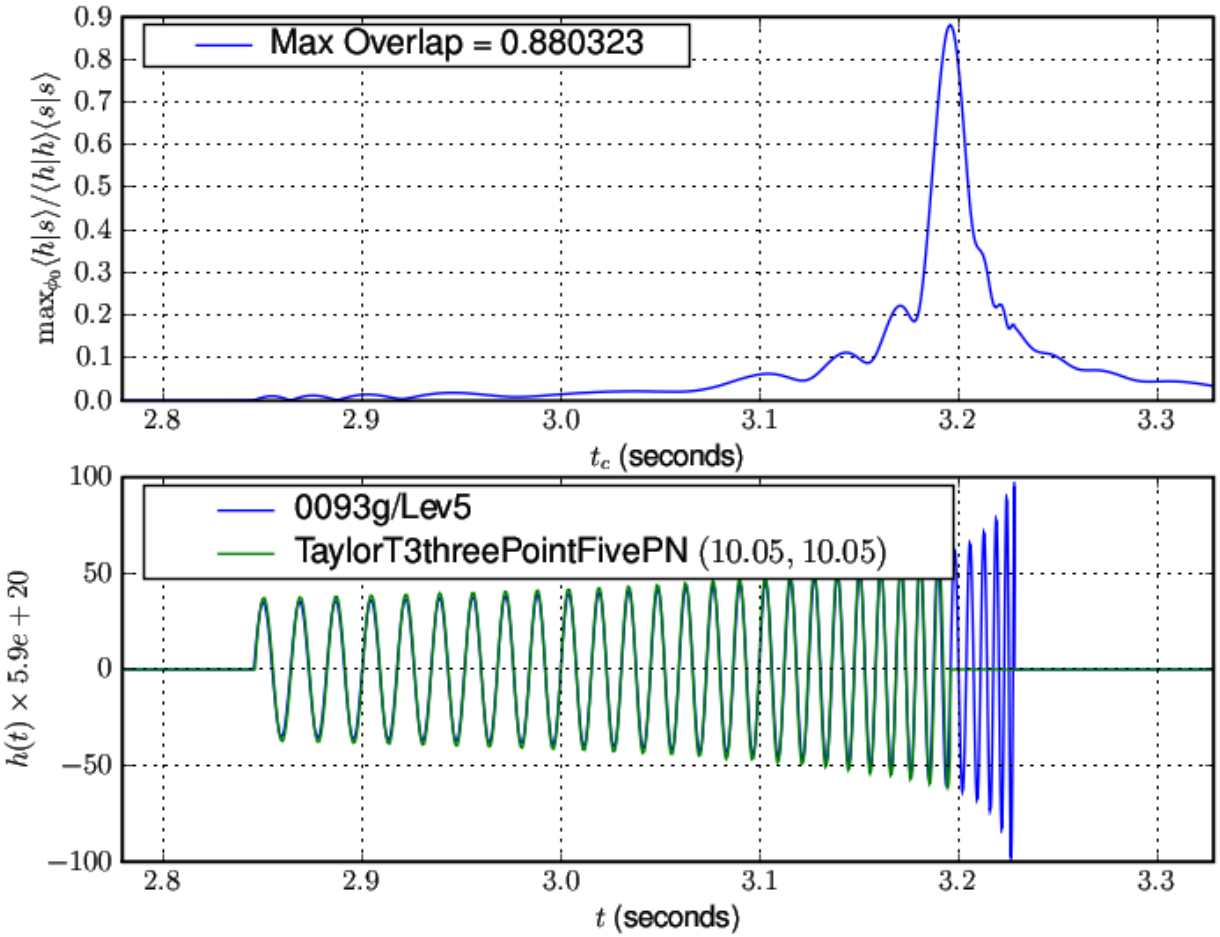




- No significant differences for $t \lesssim 3000M$
- 3PN–NR $\lesssim 10\%$ throught run
- other PN-orders abysmal at late times

Making contact with data-analysis

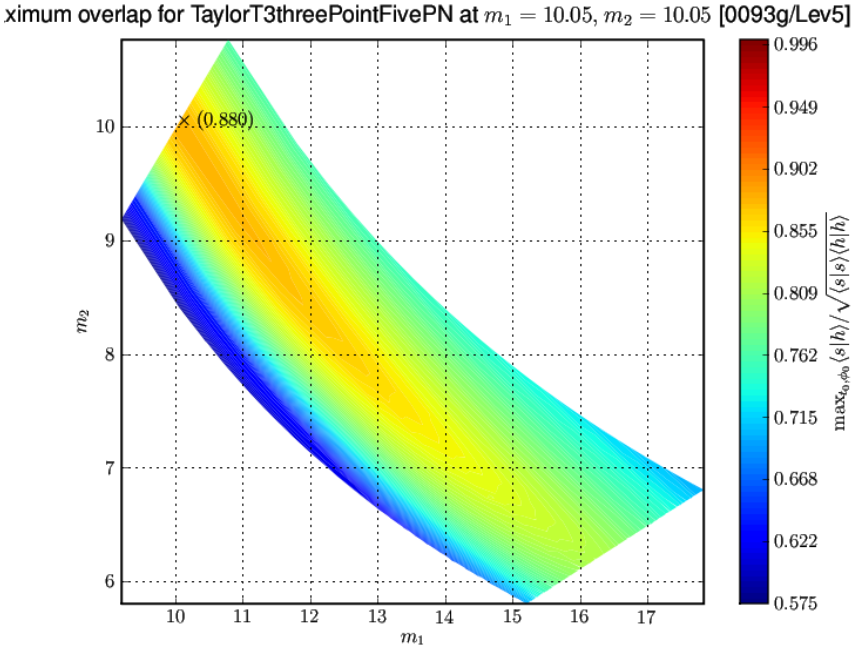
Overlap with detection templates

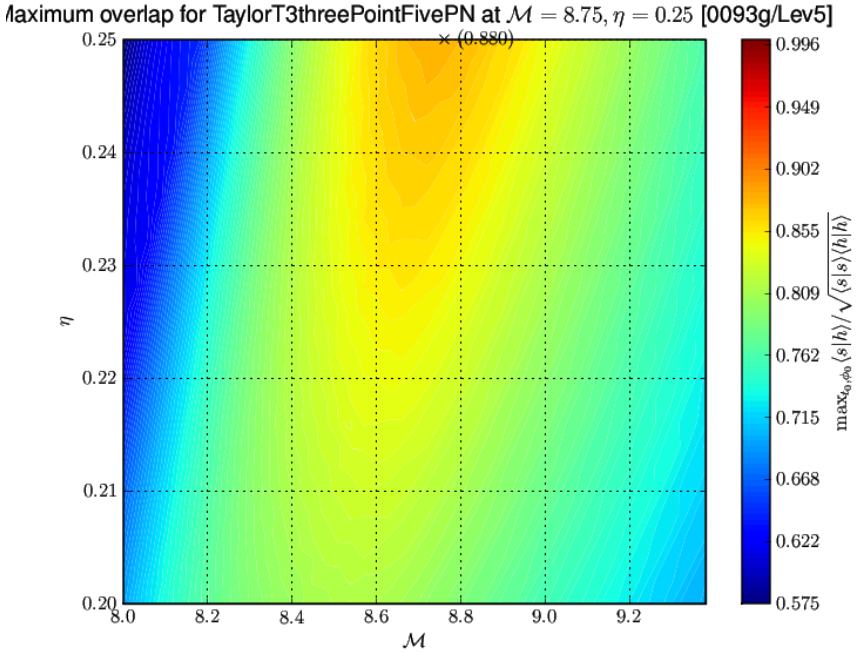


- In collaboration with D. Brown
- Template too short, causes loss in overlap (3.5PN turnover?)

Making contact with data-analysis

Parameter estimation





- Masses in simulation: 10.07Msun. Recovered 10.05Msun.

Summary

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- Accuracy sufficient to investigate breakdown of PN
  (With extrapolation $R_{\rm extr} \to \infty$ even for 3PN.)
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  - Agreement between NR and PN at early times
  - Taylor-3PN agrees very well throughout simulation
    - $\delta\phi \lesssim 0.5$rad
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