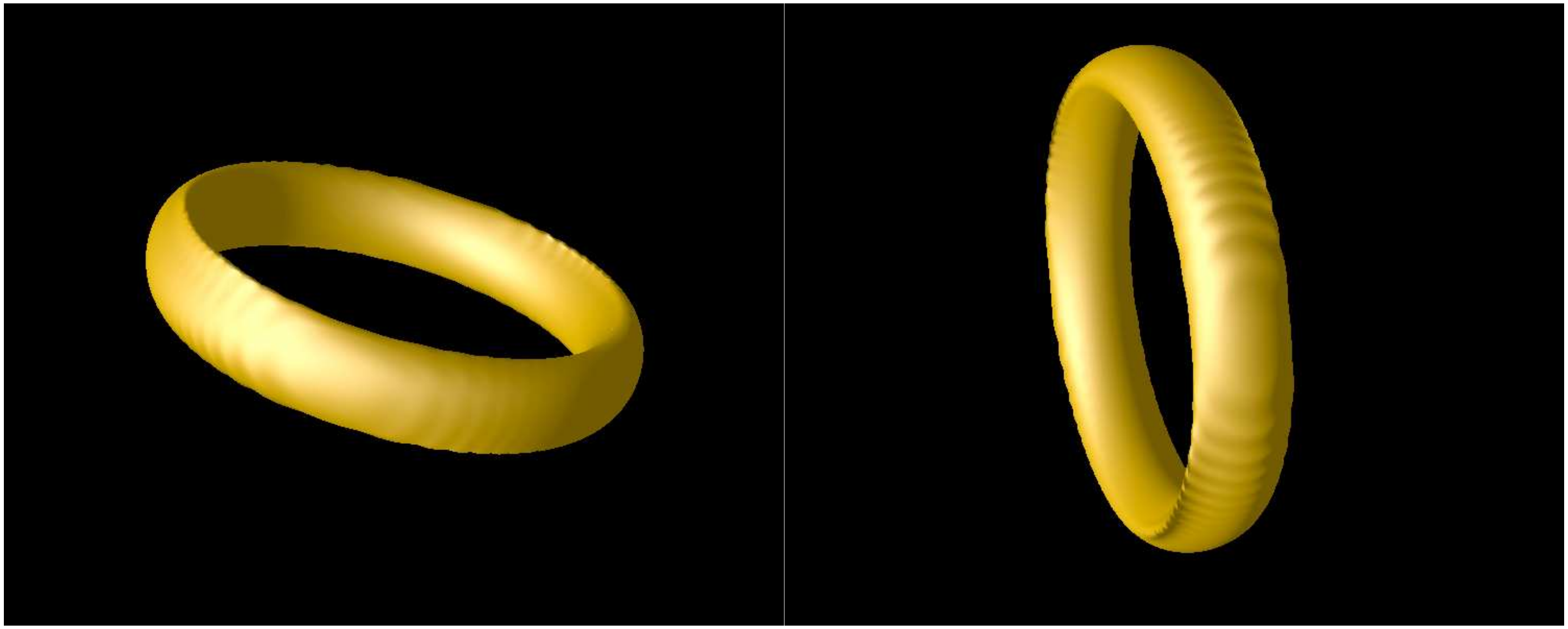
Binary black hole initial data

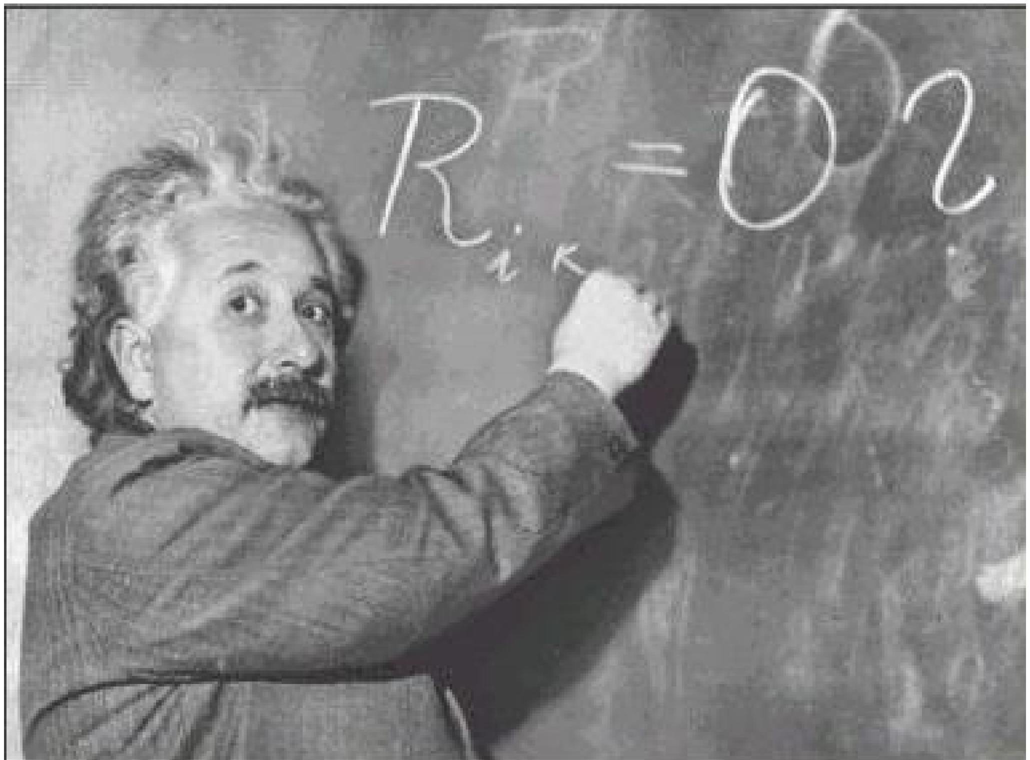
Harald P. Pfeiffer

Caltech



CGWA, University of Brownsville, Apr 9, 2004

World year of physics



Einstein and the Ricci Tensor

- 100 years of the “three papers”
- 90 years of general relativity
- **Two body problem** still unsolved

Gravitational wave detectors are rapidly improving

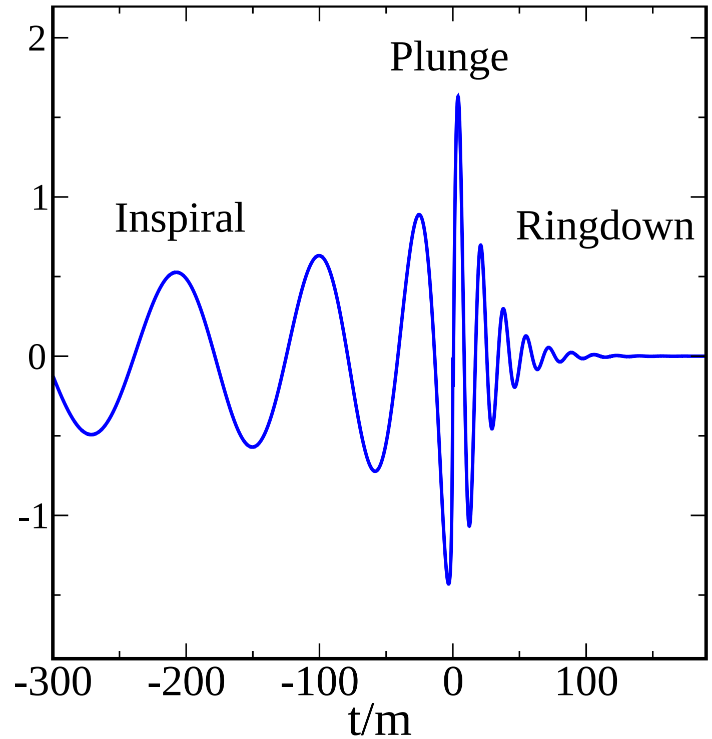
- GEO 600
- LIGO
- TAMA 300
- Virgo

Ligo Hanford site

One prime scientific target: Binary black hole coalescence

Binary black hole coalescence

- Inspiral – post-Newtonian expansions
- Late inspiral & plunge – **numerical relativity**
- Ringdown – perturbation theory



Waveforms from all three phases are important for...

1. Event detection
2. Parameter extraction
3. Testing general relativity

numerical relativity
$\longrightarrow$ **Initial data**
$\longrightarrow$ Evolutions (Lee Lindblom last week)

Outline of talk

1. *Construction* **any** *initial data* – Conformal method
2. *Construction* **BBH** *initial data* – Quasi-equilibrium method
3. *Surprising properties of initial data* – Non-uniqueness

Conformal method

Problem: Find solutions (g_{ij}, K_{ij}) of the constraint equations

$$R + K^2 - K_{ij}K^{ij} = 0$$

$$\nabla_j \left(K^{ij} - g^{ij}K \right) = 0$$

Σ spacelike hypersurface
g_{ij} induced metric on Σ
K_{ij} extrinsic curvature of Σ
$K = g_{ij}K^{ij}$ trace of ex. curvature
N lapse, β^i shift
$ds^2 = -N^2dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$

Strategy: Split g_{ij} and K_{ij} into smaller pieces, such that some are *freely specifiable*, and the rest completely determined

Extrinsic curvature decomposition (HP, York 2003)

Task: Find
g_{ij} and $K^{ij} = A^{ij} + 1/3\; g^{ij} K$
which satisfy

$$R + A_{ij}A^{ij} + \frac{2}{3}K^2 = 0$$

$$\nabla_j \left(A^{ij} - \frac{2}{3} g^{ij} K \right) = 0$$

$\tilde{g}_{ij}$, K, $\tilde{A}^{ij}_{TT}$, $\tilde{N}$ *freely specifiable*

ψ, V^i determined by *elliptic eqns*

$$\tilde{\nabla}^2 \psi + \ldots = 0$$

$$\tilde{\Delta}_{L,\tilde{N}} V^i + \ldots = 0$$

Decompose A^{ij}

$$g_{ij}$$

$$A^{ij} = \frac{1}{2N} (\mathbb{L}V)^{ij} + A^{ij}_{TT}$$

Conformally **rescale**

$$g_{ij} = \psi^4 \tilde{g}_{ij}$$

$$A^{ij}_{TT} = \psi^{-10} \tilde{A}^{ij}_{TT}$$

$$V^i = V^i$$

$$N = \psi^6 \tilde{N}$$

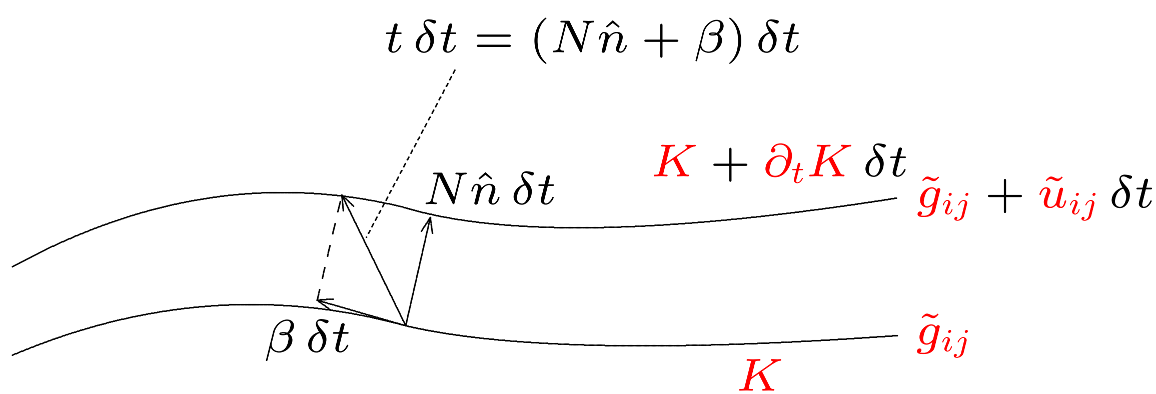
Conformal quantities

$$\tilde{g}_{ij}$$

$$\tilde{A}^{ij} = \frac{1}{2\tilde{N}} (\tilde{\mathbb{L}}V)^{ij} + \tilde{A}^{ij}_{TT}$$

Conformal thin sandwich

York 1999 – No decomposition; "just" say which free data you want



Specify these free data:

- conformal 3-metric $\tilde{g}_{ij}$
- its time derivative $\partial_t \tilde{g}_{ij} = \tilde{u}_{ij}$ (tracefree)
- trace of ex. curvature K
- its time derivative $\partial_t K$

Elliptic equations follow:

$$\tilde{\nabla}^2\psi + \ldots = 0$$

$$\tilde{\nabla}_i\left(\frac{1}{2\tilde{N}}(\tilde{\mathbb{L}}\beta)^{ij}\right) + \ldots = 0$$

$$\tilde{\nabla}^2\tilde{N} + \ldots = 0$$

Comparison

Extrinsic curvature decomposition (Hamiltonian viewpoint)

Two old versions:
"Conformal TT" & "Physical TT"
various disadvantages
widely used, b/c they were around

Final version w/ weight-function
HP, York 2003
+ equivalent to standard CTS
+ Avoids disadvantages of old versions
+ conformally covariant
+ Kerr has $A^{ij}_{\mathrm{TT}} = 0$

– Choice of $\tilde{M}^{ij}$ difficult
(HP, Cook, Teukolsky, 2002)

Conformal thin sandwich (Lagrangian viewpoint)

— Extended system —
HP, York 2003
Five eqns. with free data

$$(\tilde{g}_{ij}, \partial_t \tilde{g}_{ij}; K, \partial_t K)$$

+Time-derivatives more intuitive
+often natural choice exists
+obtain N, β^i

— Standard system —
York, 1999
Four eqns. with free data

$$(\tilde{g}_{ij}, \partial_t \tilde{g}_{ij}, K, \tilde{N})$$

Conformal lapse $\tilde{N} = \psi^{-6} N$
+ equivalent to "extrinsic curvature decomp."
– Choice of $\tilde{N}$ difficult

Solution procedure

1. **Choose** formalism
2. **Choose** free data:
   (a) $\tilde{g}_{ij}$, K, $\tilde{M}^{ij}$, $\tilde{N}$ (extrinsic curvature decomposition)
   $\tilde{g}_{ij}, K, \partial_t\tilde{g}_{ij}, \partial_t K$ (conformal thin sandwich)
   (b) topology of Σ, boundary conditions
3. **Solve** elliptic equations
4. **Assemble** physical initial data (g_{ij}, K_{ij})

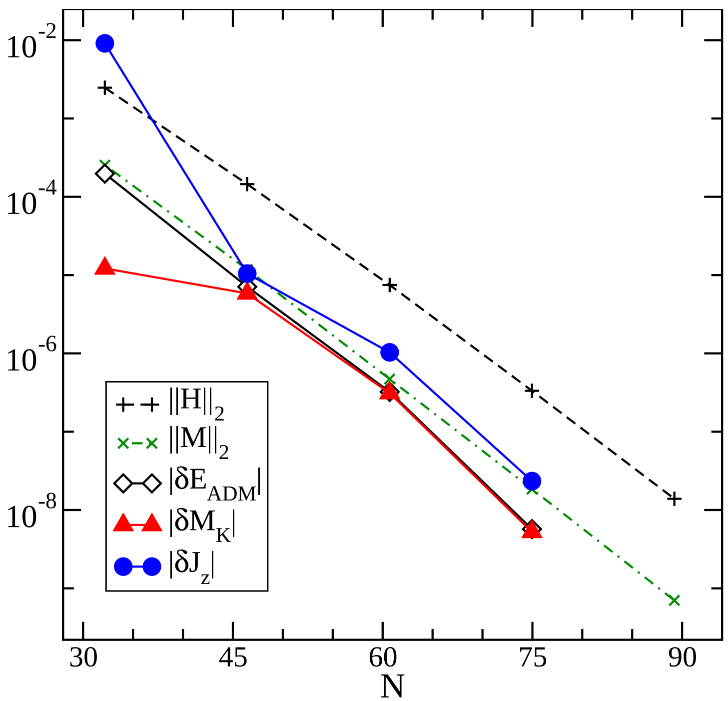
Formalism finished. Next...

– **numerics**: Solving the elliptic equations
– **physics**: Choosing the free data

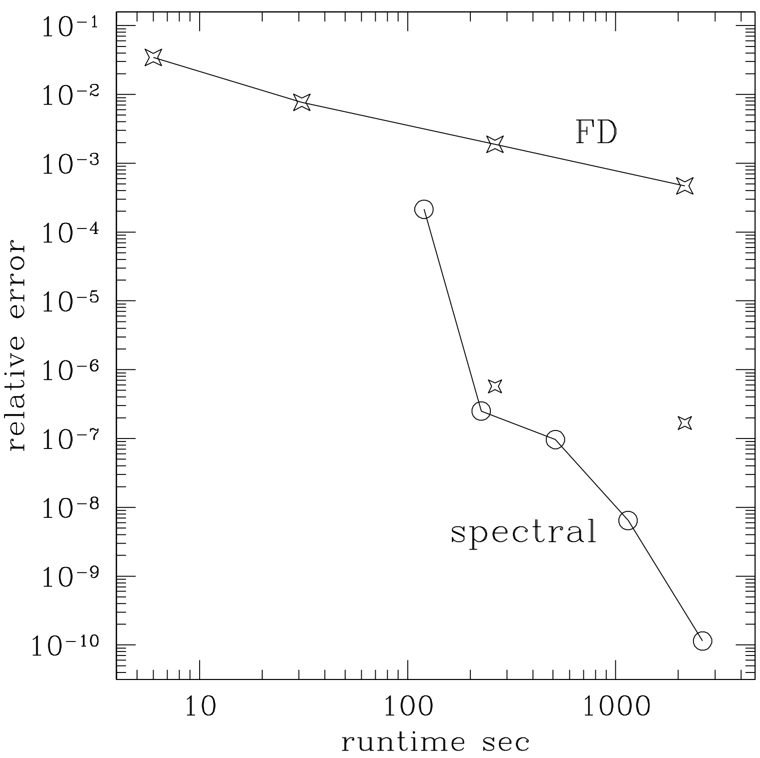
Spectral elliptic solver

Expand solution in basis-functions & solve for expansion-coefficients
Smooth solutions ⇒ exponential convergence

- *Superior accuracy:* Numerical errors ≪ physical effects
- *Superior efficiency:* Permits large parameter studies



Cook & HP, 2004



HP *et al*, 2003

Conformally flat Bowen-York data

How to choose the free data?

1. Maximal slice $K = 0 \Rightarrow$ Hamiltonian & momentum constraints decouple
2. Conformal flatness $\Rightarrow$ equations simplify
3. Analytic solution for momentum constraint (Bowen-York 1980)
   $\Rightarrow$ only Hamiltonian constraint left
4. Use puncture method or inversion symmetry
   to get boundary conditions

... Fairly simple to implement numerically. But convenience $\not\Rightarrow$quality.

Shortcomings of conformally flat Bowen-York data

Let me count the ways...

1. Single BH Bowen-York initial data do not represent stationary spacetimes
   (a) Kerr is not conformally flat (Kroon, 2004)
   (b) spinning BY is Kerr + gravitational waves
   (c) boosted BY is boosted Schwarzschild + gravitational waves

2. Binary compact objects are *NOT* conformally flat at 2-PN (Rieth, 1997)

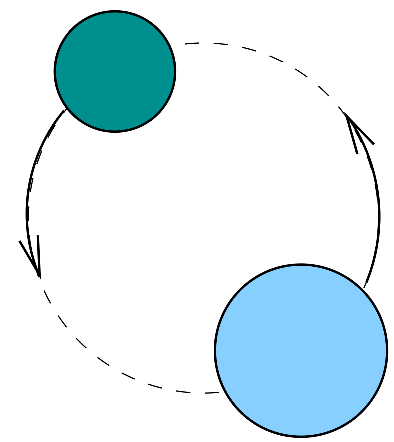
3. BBH initial data constructed from BY seems fishy
   (a) ISCO disagrees with PN calculations
   (b) ISCO disappears for corotating BH's (HP et al. 2000)
   (c) ISCO is wrong even in the test-mass limit (HP, 2003, thesis)
   (d) Evolutions from BY data find plunging BH's rather than orbiting ones (gr-qc/0411149)

4. Superposition of boosted single BH's is not an orbiting binary BH
   (a) Boosted electron + boosted positron $\neq$ positronium
   (b) General relativity is nonlinear, so effects of superposition are even less predictable

From Bowen-York toward astrophysical initial data

What means "astrophysical"?

ultimate goal	today's goal
hypersurface through inspiral as it occurs in nature	as little "junk radiation" as possible
contains embedded wavetrain of earlier inspiral	reduce initial burst of GW
BBH slowly inspiraling (i.e. $\dot{r}$ small but nonzero)	get remotely circular orbit at all
no BH quasi-normal ringing early in evolution	



How to judge "astrophysical relevance" of initial data?

- Ultimately, by evolutions.
- Robustness – How sensitive are results to the arbitrary choices inherent in all (current) methods?
- Consistency – Compute sequences of quasi-circular orbits, ISCO, take limits of large separation, and large mass ratio. Are various results consistent?

Recent approaches to BBH initial data

(old) extrinsic curvature decomposition

- Superposed Kerr-Schild (Matzner *et al* 1998, Marronetti & Matzner, 2000)

$$\tilde{g}_{ij} = \delta_{ij} + 2\,[Hl_il_j]^A + 2\,[Hl_il_j]^B\,, \qquad \tilde{M}^{ij} \approx K_A^{ij} + K_B^{ij}$$

  questionable until prooven otherwise (HP *et al* 2002)

- Incorporate PN information (Tichy *et al* 2003, Yunes *et al* 2004)

$$\tilde{g}_{ij} \approx g_{ij}^{\rm PN}, \qquad \tilde{M}^{ij} \approx K_{\rm PN}^{ij}$$

  very promising, looking forward to further results

Conformal thin sandwich equations

- Gourgoulhon, Grandclement, Bonazzola, 2002, 2002
  "Helical Killing vector approximation" (+other assumptions)
  basically right, but various deficiencies
  Laid some fundations for Cook & HP

- Cook & HP, 2002, 2003, 2004
  Quasi-equilibrium method with isolated horizon BCs

Quasi-equilibrium method

- $T_{\text{inspiral}} \gg T_{\text{orbit}}$

  Corotating coordinates $\Rightarrow \partial_t \approx 1/T_{\text{inspiral}} \approx 0$

  $\Rightarrow$ natural choice: vanishing time derivatives

N.B. Essentially equivalent:
- Helical Killing vector
- Quasi-equilibrium
- Time independence in corotating coordinates

Depending on context, different pictures are useful.

- **Conformal thin sandwich formalism**
  1. $\partial_t \tilde{g}_{ij} = 0 = \partial_t K$
  2. Need not choose $\tilde{M}^{ij}$
  3. $\tilde{g}_{ij}$ and K still undetermined

- **Boundary conditions at infinity** from asymptotic flatness & corotation:

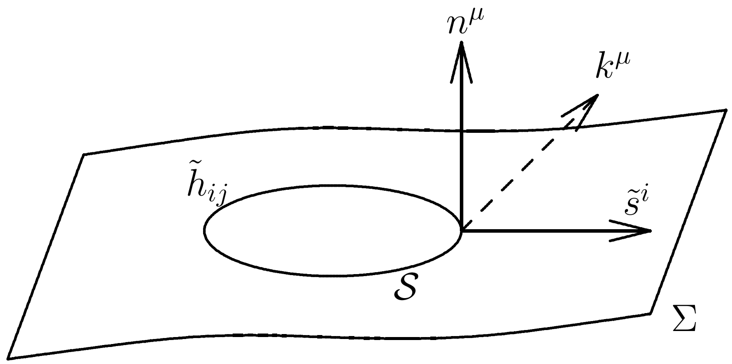
$$\psi = 1$$

$$\beta^i = (\vec{\Omega}_{\text{orbital}} \times \vec{r})^i$$

$$N = 1$$

- **New contribution**: *inner boundary conditions* (next slide...)

Quasi-equilibrium excision boundary conditions



n^μ – normal to hypersurface
k^μ – outward pointing null normal to $\mathcal{S}$
$\tilde{s}^i$ – (conformal) spatial normal to $\mathcal{S}$
$\tilde{h}_{ij}$ – conformal induced metric of $\mathcal{S}$

- **Excise** topological sphere(s) $\mathcal{S}$
- **Require**
  1. $\mathcal{S}$ be apparent horizon(s)
  2. The shear $\sigma_{\mu\nu}$ of k^μ vanishes
  3. When evolved, the coordinate locations of the AH's remain stationary
- Item **2** is an **isolated horizon condition**. It implies for the expansion

$$\mathcal{L}_k\theta = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} = 0 \quad \text{on } \mathcal{S}$$

$\Rightarrow$ AH moves along k^μ, and its area is constant (initially)

Quasi-equilibrium excision boundary conditions cont'd

- **Rewrite** in variables of conformal thin sandwich

$$\tilde{s}^k\tilde{\nabla}_k \ln\psi = -\frac{1}{4}\tilde{h}^{ij}\tilde{\nabla}_i\tilde{s}_j + \frac{1}{6}\psi^2 K - \frac{\psi^2}{8N}\tilde{s}_i\tilde{s}_j(\tilde{\mathbb{L}}\beta)^{ij} \qquad \text{on } \mathcal{S} \qquad (1)$$

$$\beta^i = \psi^2 N\tilde{s}^i + \beta^i_{\parallel} \qquad \text{on } \mathcal{S} \qquad (2)$$

  moderately complicated boundary conditions

- **Rotating black holes**
  Vanishing shear $\Leftrightarrow$ $\beta^i_{\parallel}$ is conformal Killing vector of $\mathcal{S}$.
  Those exist for *general* rotation axis, conformal metric and shape of $\mathcal{S}$:
  – any 2-sphere is conformally flat
  – *Any* rotation through the center of an Euclidean sphere is a Killing vector...
  – ... and is therefore a conformal Killing vector of $\mathcal{S}$

- Lapse boundary condition not fixed by IH (also Jaramillo *et al*, 2004)

Numerical solutions: Single black holes I

- Quasi-equlibrium method works for any choice of $\tilde{g}_{ij}$, K, $\mathcal{S}$ and lapse-BC.
- For now arbitrary choices: Conformal flatness, (mostly) maximal slicing, $\mathcal{S}$ =sphere
- **Do not** use knowledge of single BH solutions – use single BHs to **test** method

Spherical symmetry

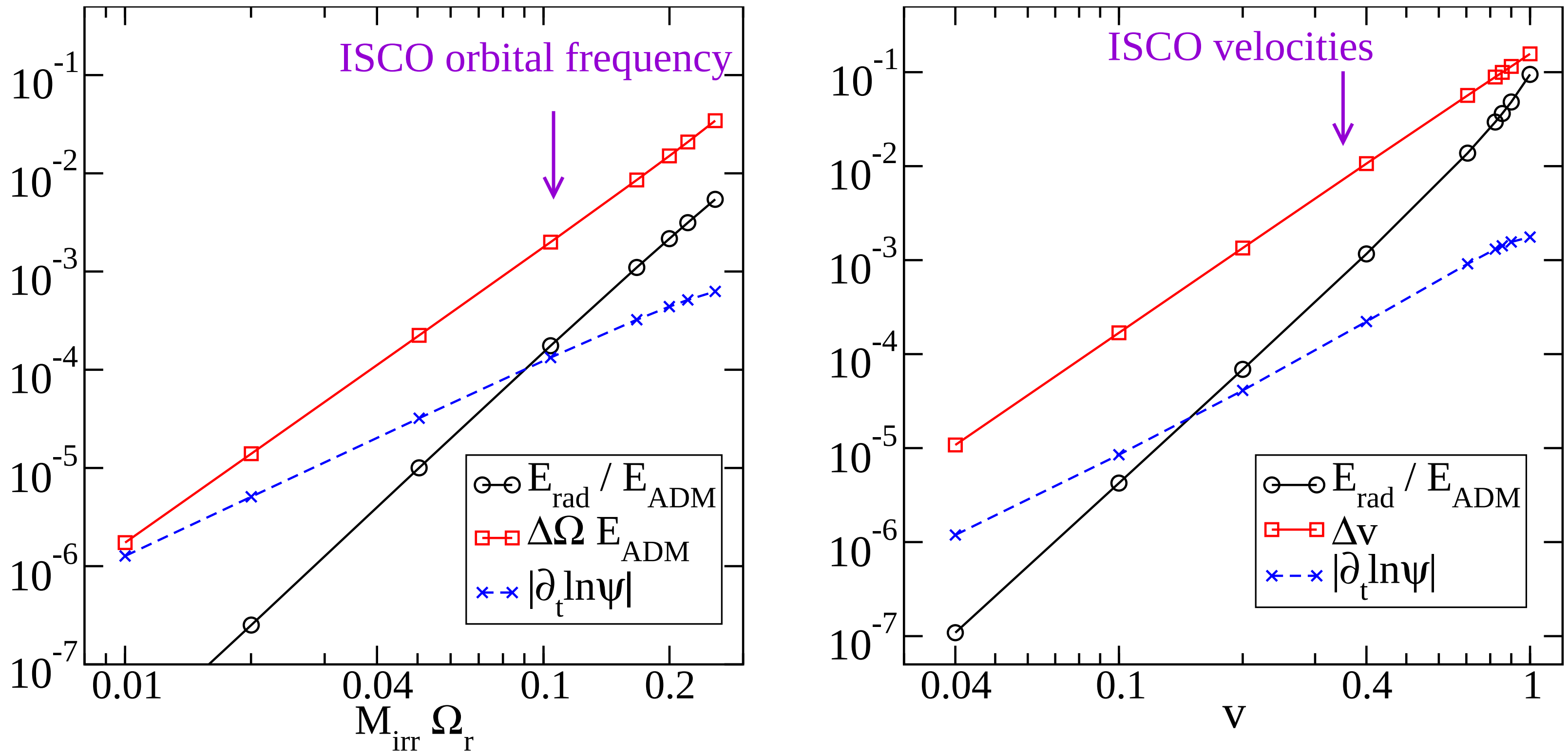
1. w.l.o.g. conformllly flat
2. Try different choices for K and lapse boundary condition
3. *any* spherically symmetric K and *any* spherically symmetric lapse-BC yield:
   – exact slice through Schwarzschild
   – totally vanishing time-derivatives $\partial_t g_{ij} = \partial_t K_{ij} = 0$
4. **Full success**: Recover Schwarzschild independent of arbitrary choices.

Numerical solutions: Single black holes II

- **Spinning/Boosted black holes**
  Compute quantities that vanish for Kerr:

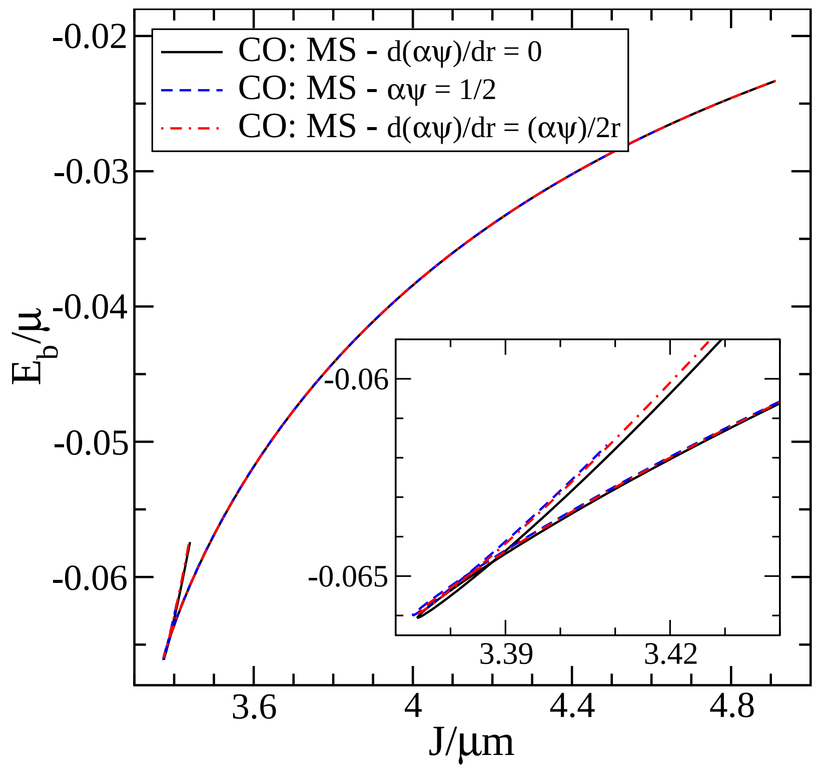
$$E_{rad} \equiv \sqrt{E_{\mathrm{ADM}}^2 - P_{\mathrm{ADM}}^2} - \sqrt{M_{\mathrm{irr}}^2 + J_{\mathrm{ADM}}^2/(4M_{\mathrm{irr}}^2)} \tag{3}$$

$$\Delta\Omega \equiv \Omega_r - \frac{J_{\mathrm{ADM}}/E_{\mathrm{ADM}}^3}{2 + 2\sqrt{1 - J_{\mathrm{ADM}}^2/E_{\mathrm{ADM}}^4}}, \qquad \Delta v \equiv v - \frac{P_{\mathrm{ADM}}}{E_{\mathrm{ADM}}} \tag{4}$$



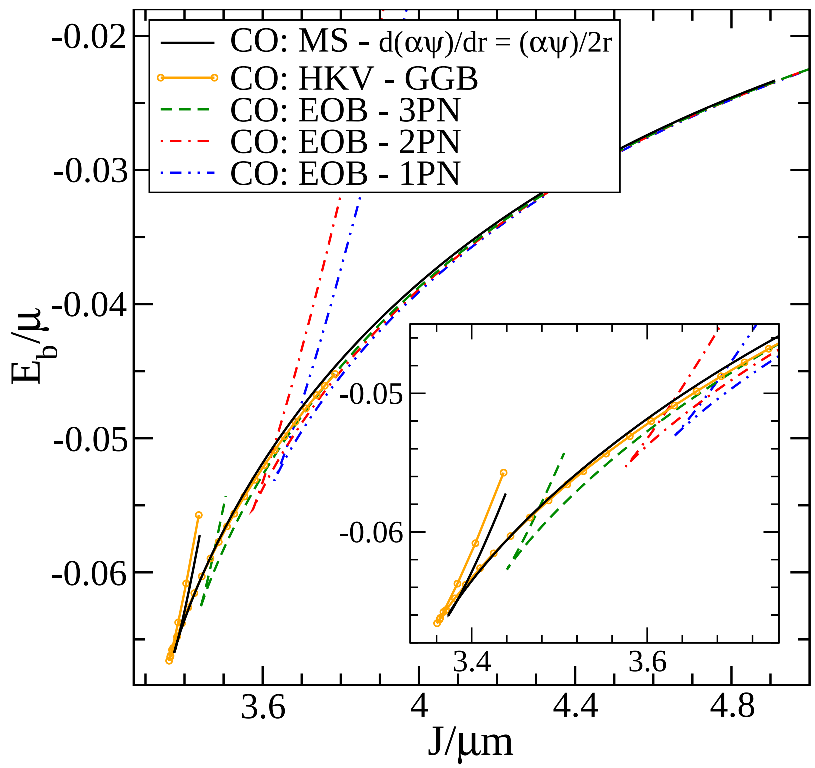
Binary black hole solutions (corotating, $K = 0$)

Test three different lapse boundary conditions



No difference – solution robust

Compare to GGB and post-Newtonian results

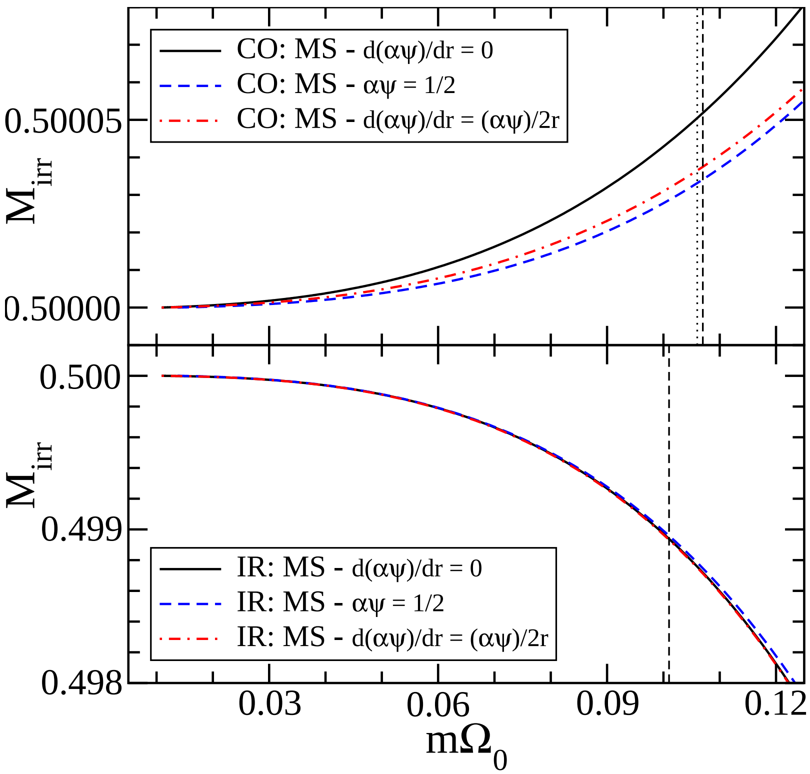


Excellent agreement

Testing the 2nd law

Normalize sequences such that $dE_{\mathrm{ADM}} = \Omega_0 \, dJ_{\mathrm{ADM}}$

Irreducible mass along these sequences



$\Leftarrow$ corotating sequences
(three different lapse BC's)
M_{irr} slightly increasing during inspiral – ok

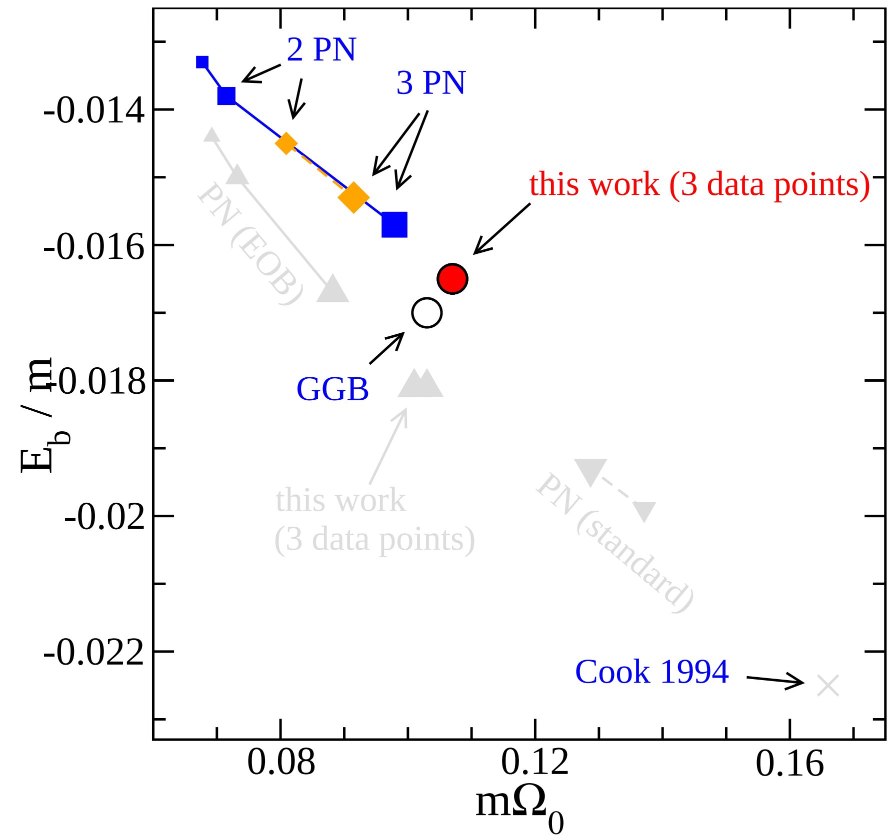
$\Leftarrow$ irrotational sequences
(three different lapse BC's)
M_{irr} decreasing during inspiral
– Normalization of sequences wrong?

ISCO location

Caution: ISCO is not a sharp, well-defined concept! Anyway...

Color: Corotating BH's

Grey: Irrotational BH's



- Excellent agreement between NR and PN
- GGB close; deviation due to their regularization?
- Bowen-York w/ effective potential is history

Summary of QE method

- **Framework** for BBH initial data in a kinematical setting (helical Killing vector)

- Explicitly displays the **remaining choices** $\tilde{g}_{ij}, K, \mathcal{S}$, Lapse-BC

- Close in spirit to GGB, but greatly improved:
  1. Constraints are satisfied
  2. Incorporates isolated horizon boundary conditions
  3. General spins possible
  4. Retains freedom to choose any $\tilde{g}_{ij}, K, \mathcal{S}$.
  5. Lapse is positive on horizon

- **Agrees** very well with **PN** (even with simple choices)

- Future: **physically motivated** choices for $\tilde{g}_{ij}, K, \mathcal{S}$ and Lapse-BC

- Far future: Replace "$\partial_t = 0$" by radiation reaction

And now to something totally different...

Solutions to the extended system have been found numerically in quite a few situations.

Was this luck? Or can one solve for all choices of free data and boundary conditions?

- **Existence?**
  - ⋆ Standard system (4 eqns): Many mathematical results
  - ⋆ Extended system (5 eqns): terra incognita

- **Uniqueness?**
  - ⋆ Standard system (4 eqns): Unique (in all known cases)
  - ⋆ Extended system (5 eqns): terra incognita

Some results on the standard system

Mathematics:

1. asymptotically flat
2. no inner boundaries
3. maximal slice $K = 0$

$\longrightarrow$ Yamabe constant $\mathcal{Y}[\tilde g_{ij}]$:

$$\boxed{\mathcal{Y}[\tilde g_{ij}] > 0 \Leftrightarrow \text{existence \& uniqueness}}$$

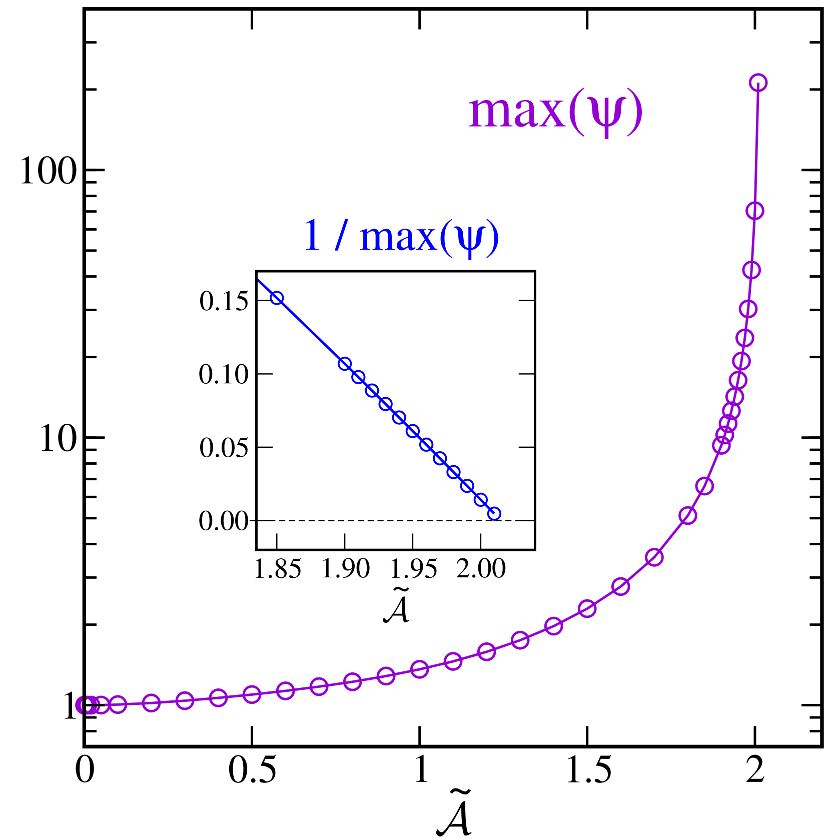
(Cantor 1977, Murray & Cantor 1981, Maxwell 2005)

$\tilde u_{ij}$ and $\tilde N$ irrelevant

Def'n of $\mathcal{Y}$ not useful for numerical work

Free data based on "Teukolsky wave"
ingoing, $M=0$, odd parity, centered at $r=20$

$$\tilde g_{ij} = \delta_{ij} + \tilde{\mathcal{A}}\, h_{ij}$$
$$\tilde u_{ij} = \tilde{\mathcal{A}}\, \partial_t h_{ij}$$
$$K = 0, \quad \tilde N = 1$$



Extended system

$$\tilde{g}_{ij} = \delta_{ij} + \tilde{\mathcal{A}}\, h_{ij}$$

$$\tilde{u}_{ij} = \tilde{\mathcal{A}}\, \partial_t h_{ij}$$

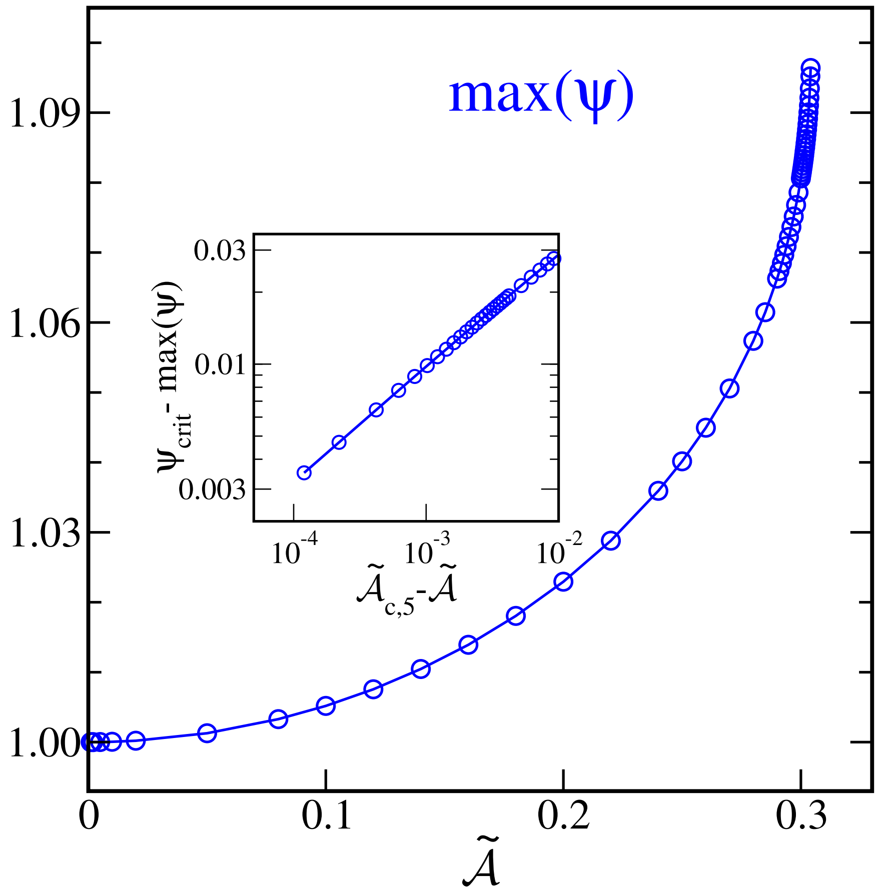
$$K = 0, \quad \partial_t K = 0$$

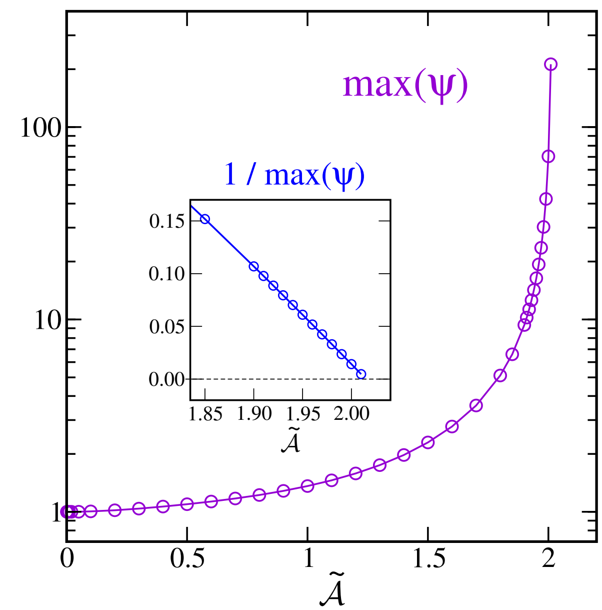
Smaller $\tilde{\mathcal{A}}_c$

finite ψ as $\tilde{\mathcal{A}} \to \tilde{\mathcal{A}}_c$

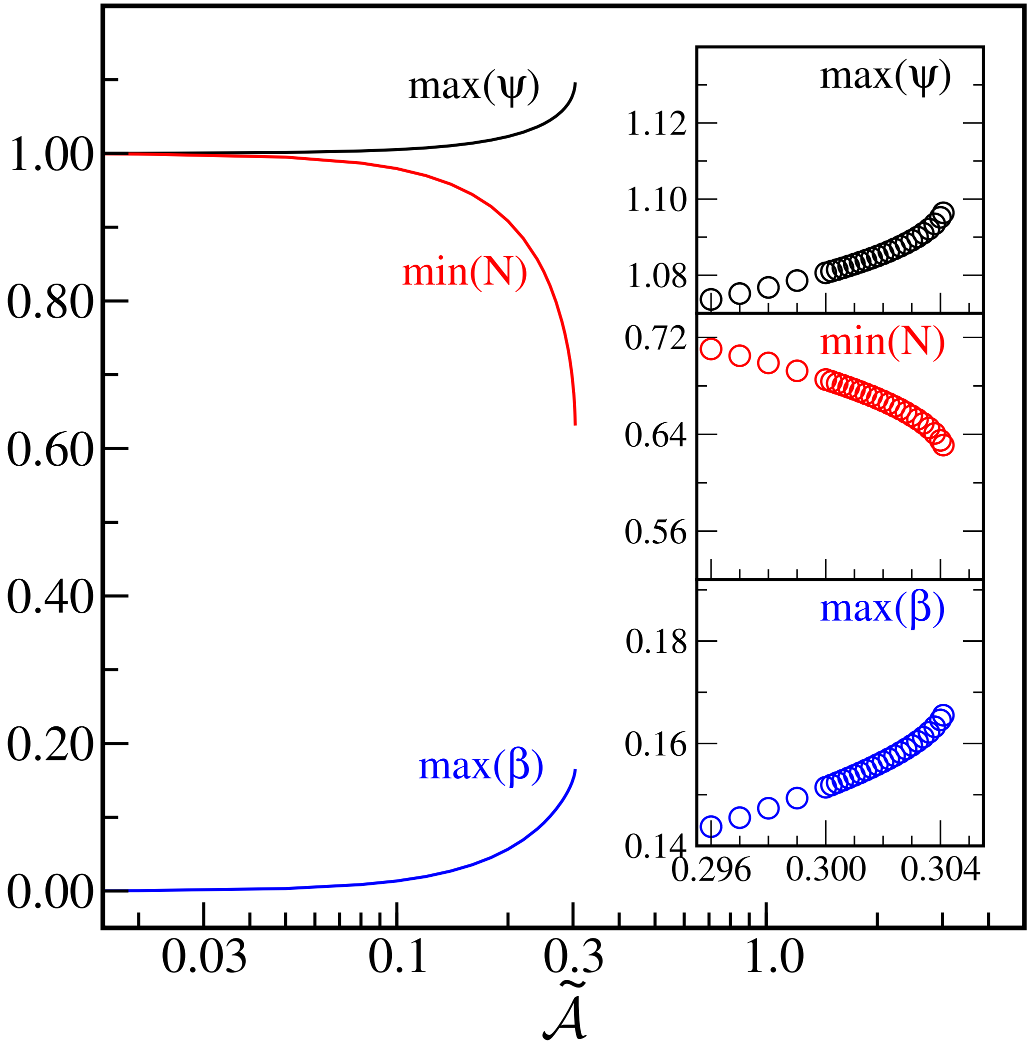
Parabolic behavior

$\psi \approx \psi_c - \text{const.}(\tilde{\mathcal{A}}_c - \tilde{\mathcal{A}})^{1/2}$

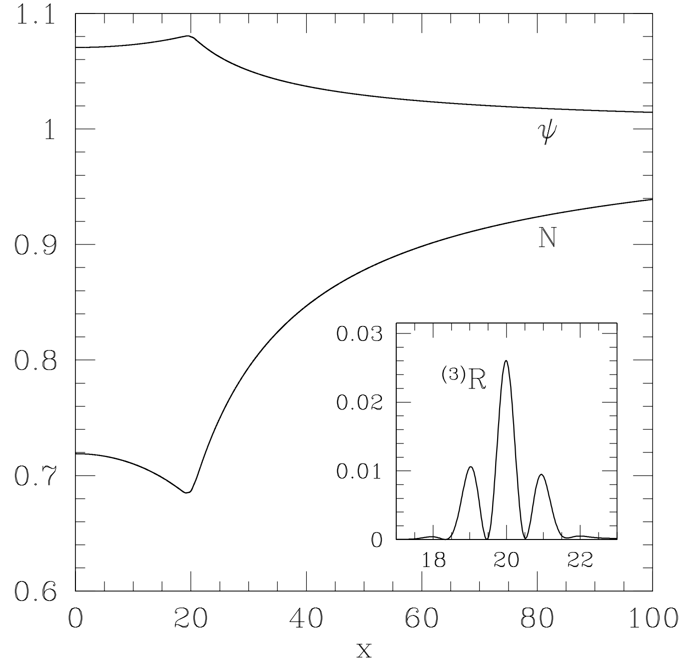




A more comprehensive look



Equatorial plane, $\tilde{\mathcal{A}} = 0.3$



A second branch

- For $\delta\tilde{\mathcal{A}} \equiv \tilde{\mathcal{A}}_c - \tilde{\mathcal{A}} \ll 1$:

$$\mathbf{u}(\tilde{\mathcal{A}}, \vec{x}) = \mathbf{u}_c(\vec{x}) - \mathbf{v}_c(\vec{x})\sqrt{\delta\tilde{\mathcal{A}}}, \qquad \mathbf{u} = \left(\psi, \beta^i, N\right)$$

- *Two* branches??

$$\mathbf{u}_\pm(\tilde{\mathcal{A}}, \vec{x}) = \mathbf{u}_c(\vec{x}) \pm \mathbf{v}_c(\vec{x})\sqrt{\delta\tilde{\mathcal{A}}}$$

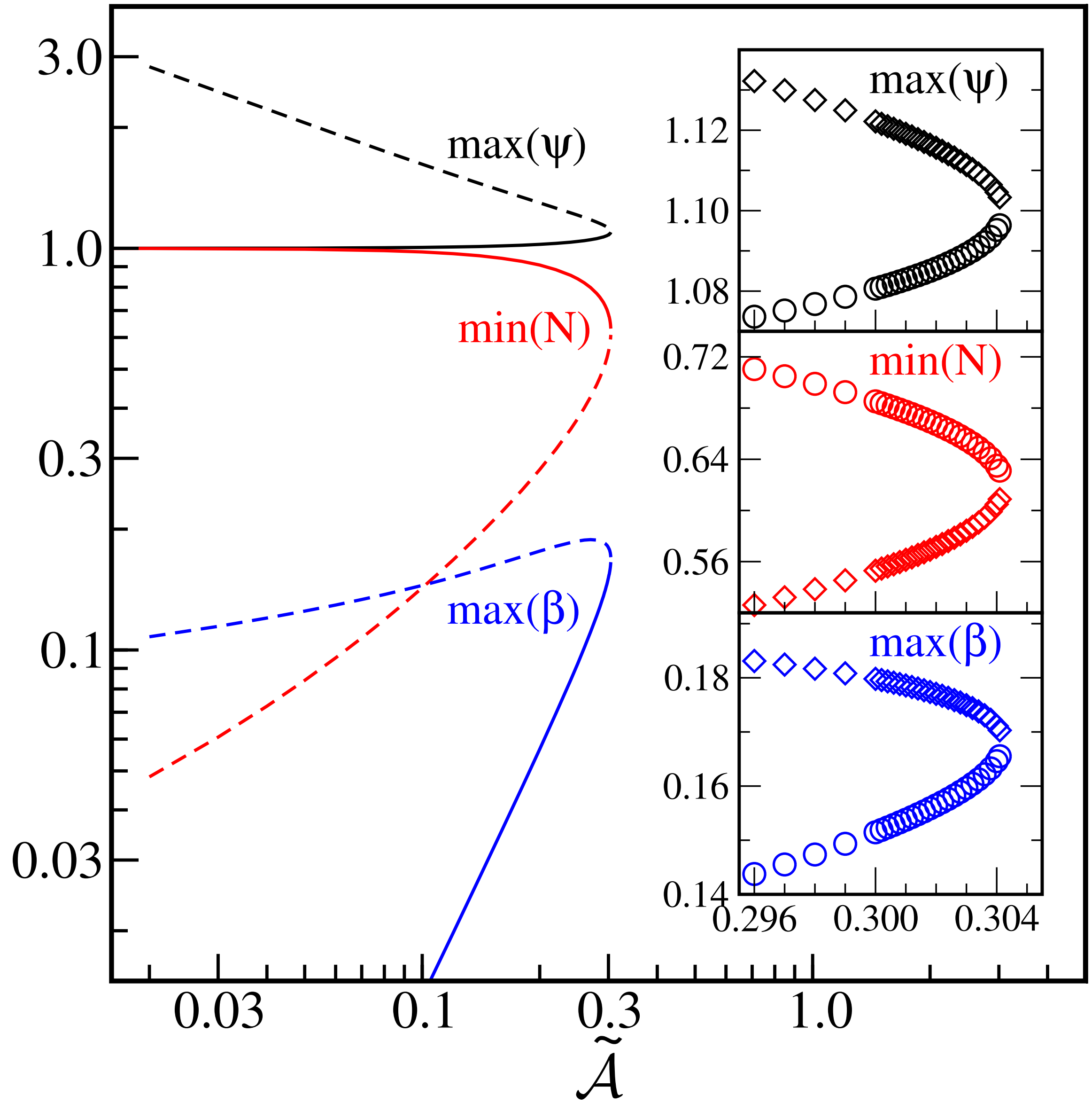
- Problem: With "simple" initial guess, elliptic solver converges always to $\mathbf{u}_-$; need good guess to converge to $\mathbf{u}_+$.

$$\frac{\mathrm{d}\mathbf{u}_-(\tilde{\mathcal{A}}, \tilde{\mathbf{x}})}{\mathrm{d}\tilde{\mathcal{A}}} = \frac{1}{2\sqrt{\delta\tilde{\mathcal{A}}}}\mathbf{v}_c(\vec{x})$$

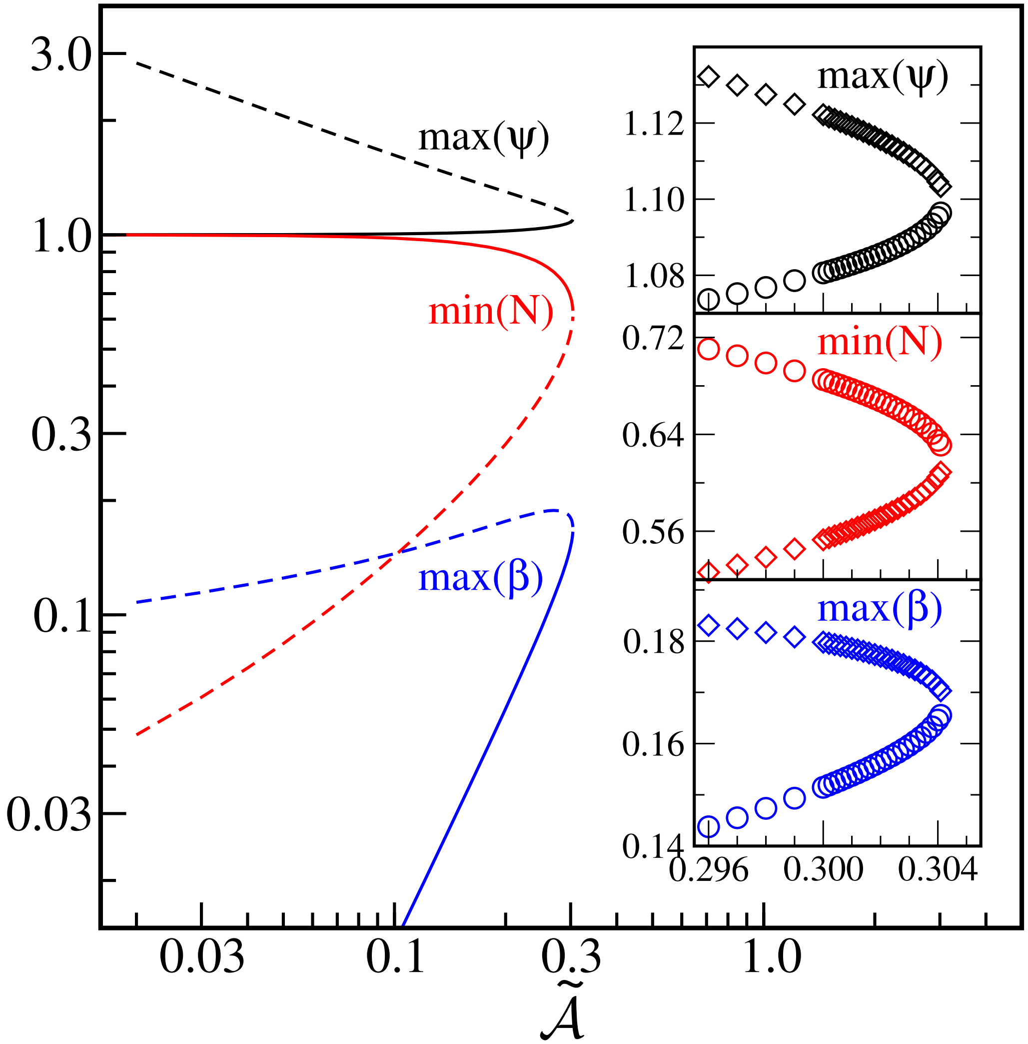
$$\mathbf{u}_+(\tilde{\mathcal{A}}, \vec{x}) \approx \mathbf{u}_-(\tilde{\mathcal{A}}, \vec{x}) + 4\,\delta\tilde{\mathcal{A}}\,\frac{\mathrm{d}\mathbf{u}_-(\tilde{\mathcal{A}}, \tilde{\mathbf{x}})}{\mathrm{d}\tilde{\mathcal{A}}}$$

- Take two numeric solutions $\mathbf{u}_-$ of five coupled 3-D nonlinear elliptic equations, and **finite-difference** them to obtain $\mathrm{d}\mathbf{u}_-/\mathrm{d}\tilde{\mathcal{A}}$!!

Constructing the upper branch u_+

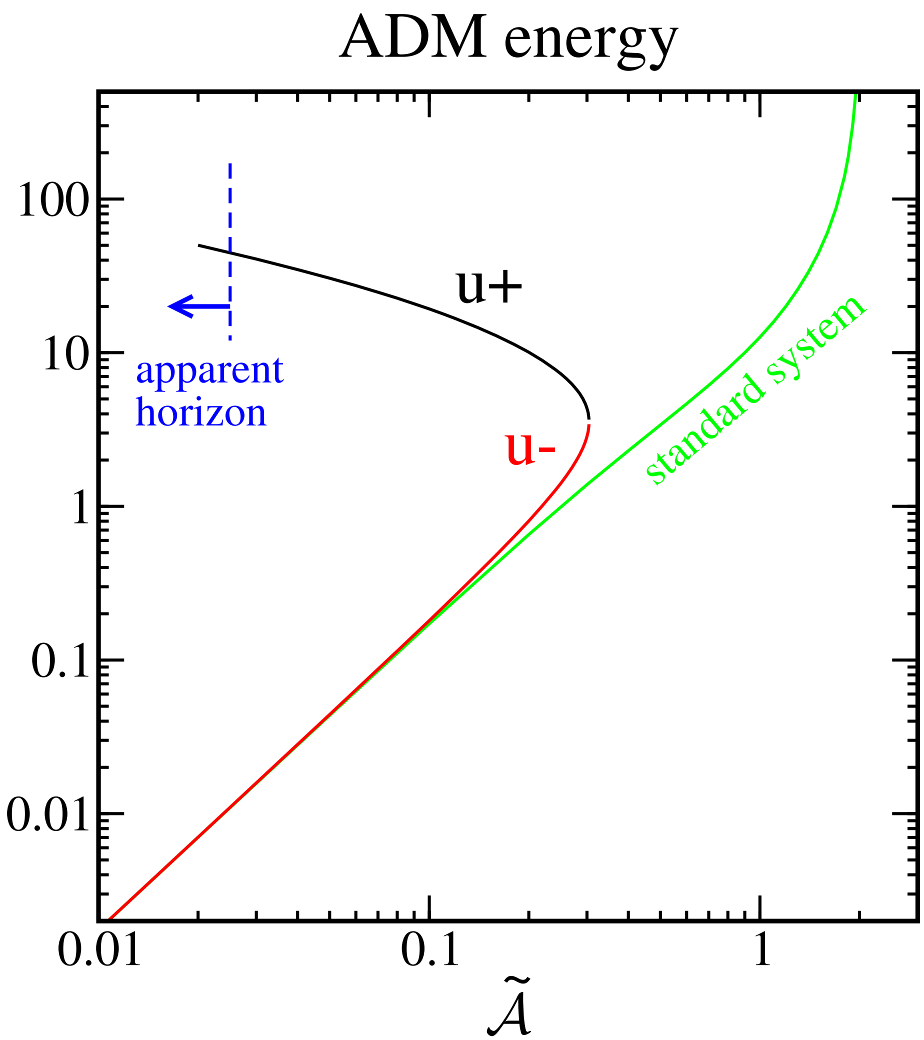


Two branches

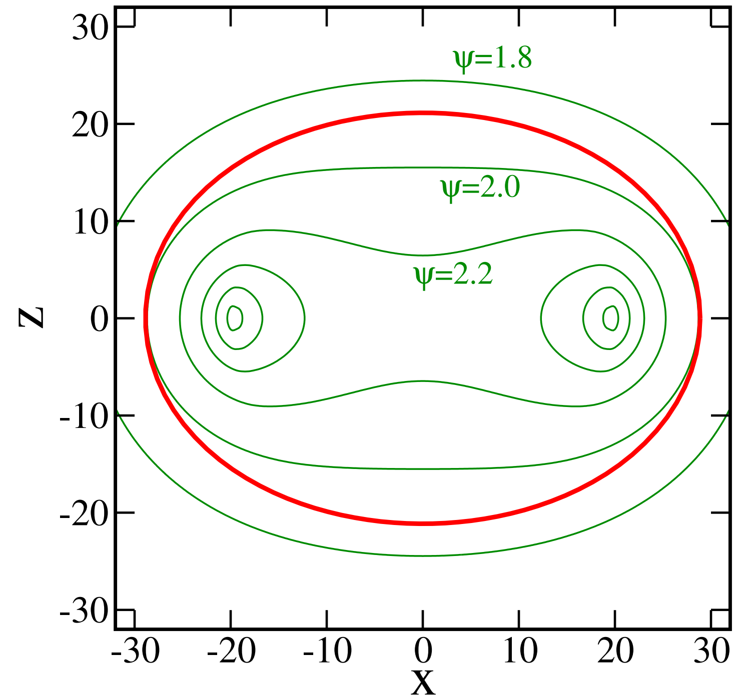


- Parabolic close to critical point
- $\mathbf{u}_+$ and $\mathbf{u}_-$ meet at $\tilde{\mathcal{A}}_c$
- $\mathbf{u}_+$ extents to small $\tilde{\mathcal{A}}$
- $\mathbf{u}_+$ deviates strongly from Minkowski at small $\tilde{\mathcal{A}}$
- No indication that $\mathbf{u}_+$ terminates
- Apparently two solutions for arbitrary small $\tilde{\mathcal{A}}$!!

Energy & Apparent horizon

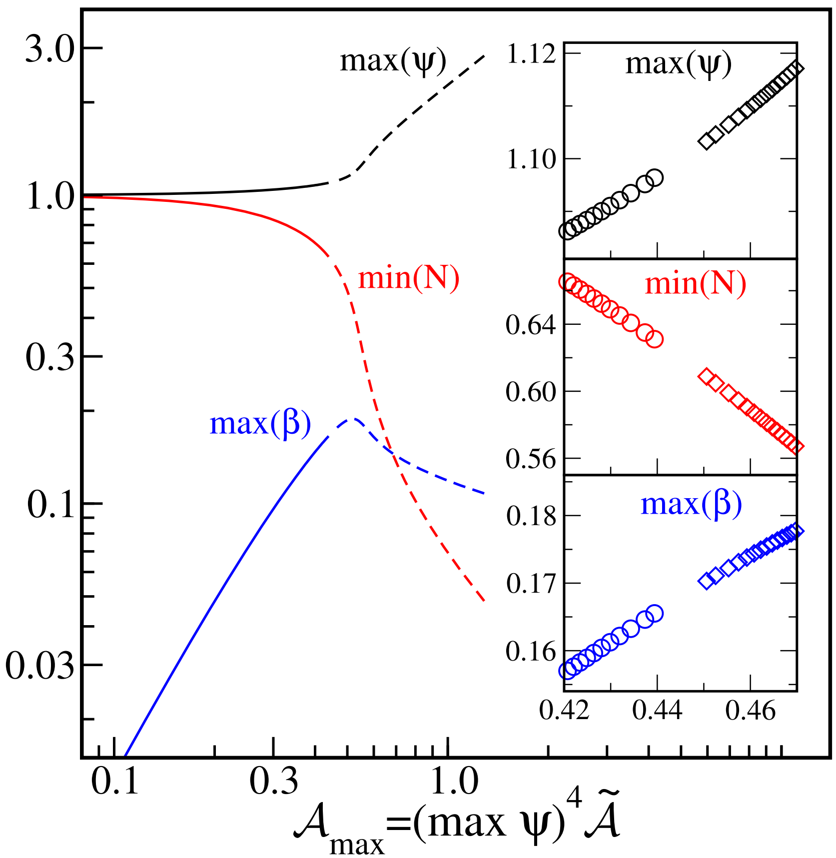


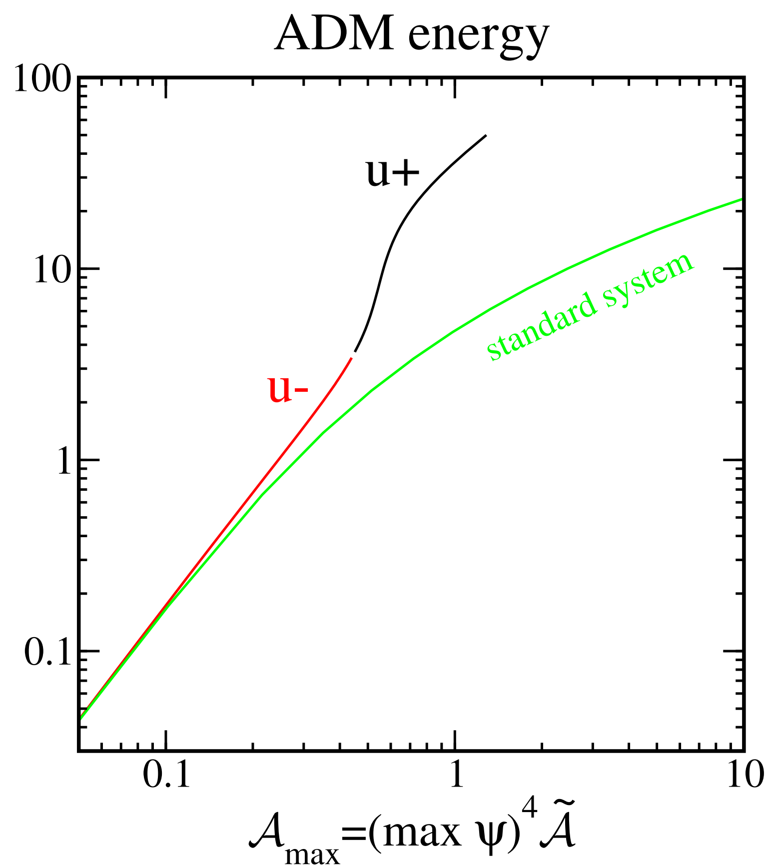
Apparent



Unique solutions, nevertheless?

- *Physics* is determined by g_{ij}, K_{ij}. For example $g_{ij} = \psi^4 \tilde{g}_{ij} = \psi^4 \delta_{ij} + \psi^4 \tilde{\mathcal{A}}\ h_{ij}$
- *Physical* amplitude of perturbation is $\mathcal{A} = \psi^4 \tilde{\mathcal{A}}$
- Question: For given *physical* amplitude $\mathcal{A}$, how many solutions exist?





Summary

- The conformal method is complete and self-consistent

  Conformal thin sandwich $\leftrightarrow$ *extrinsic curvature decomposition*

  ... forget York 1973, Ó Murchadha & York 1974 ...

- Quasi-equilibrium initial data is state-of-the-art

  Built-in potential for the next round of improvements ($\tilde{g}_{ij}$)

- Extended conformal thin sandwich harbors surprises:

  — Non-uniqueness —