Optimal constraint projection for hyperbolic systems

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1. Idea
2. Example – Scalar wave
3. Results

Constraint projection

Constraint violations are one of the most pressing issues in numerical relativity

- General evolution system

$$\partial_t u^\alpha = \ldots$$

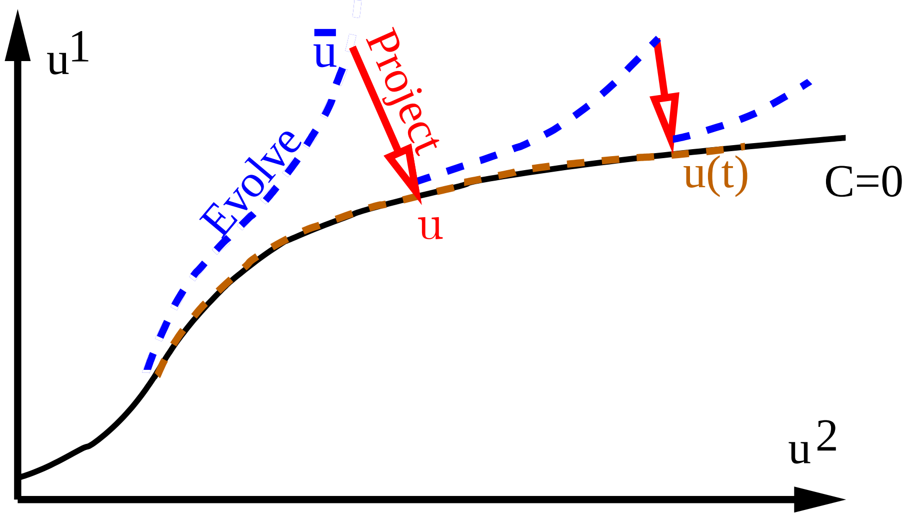
- Constraints

$$C^A[u^\alpha(t)] = 0$$

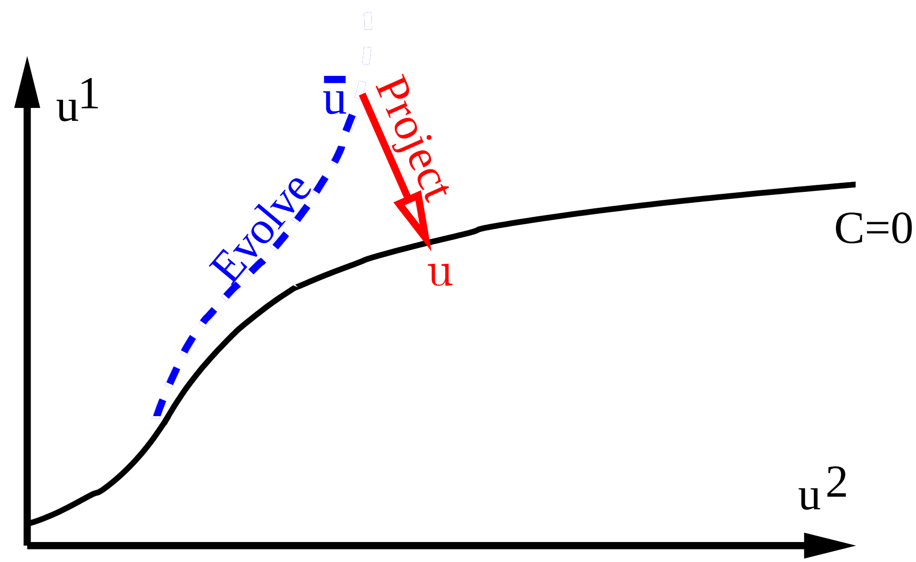
- $C^A = 0$ preserved
- $C^A = \varepsilon \neq 0$ often grows exponentially (ε, e.g., due to roundoff error)

- Idea: Whenever C^A too large, *project* into C=0 – manifold

- Problem: projection not unique; given $\bar{u}^\alpha$, how to define u^α?



Optimal constraint projection



Minimize distance between $\bar{u}^\alpha$ and u^α by insisting that the **Lagrangian**

$$\boldsymbol{\mathcal{L} = S_{\alpha\beta}(u^\alpha - \bar{u}^\alpha)(u^\beta - \bar{u}^\beta) + \lambda_A C^A}$$

be stationary under variations in the fields u^α and the Lagrange multipliers λ_A

Natural choice for $S_{\alpha\beta}$: Symmetrizer of symmetric hyperbolic evolution system.

Scalar wave $\Box\psi = 0$ (curved space)

- First order form $u^\alpha = \{\psi, \Pi \equiv -\partial_t\psi, \Phi_i \equiv \partial_i\psi\}$

$$\partial_t\psi - N^k\partial_k\psi = -N\Pi$$

$$\partial_t\Pi - N^k\partial_k\Pi + g^{ij}\partial_i\Phi_j = NJ^i\Phi_i + NK\Pi$$

$$\partial_t\Phi_i - N^k\partial_k\Phi_i + N\partial_i\Pi = -\Pi\partial_iN + \Phi_j\partial_iN^j + \gamma C_i$$

Constraints

$$C_i = \partial_i\psi - \Phi_i$$

- $\partial_t C_i - \mathcal{L}_N C_i = -\gamma C_i$ $\Leftarrow$ *Exponential growth for* $\gamma < 0$ $\Leftarrow$ *model for GR*

- Symmetrizer $dS^2 = S_{\alpha\beta}du^\alpha du^\beta = \Lambda^2 d\psi^2 - 2\gamma d\psi d\Pi + d\Pi^2 + g^{ij}d\Phi_i d\Phi_j$

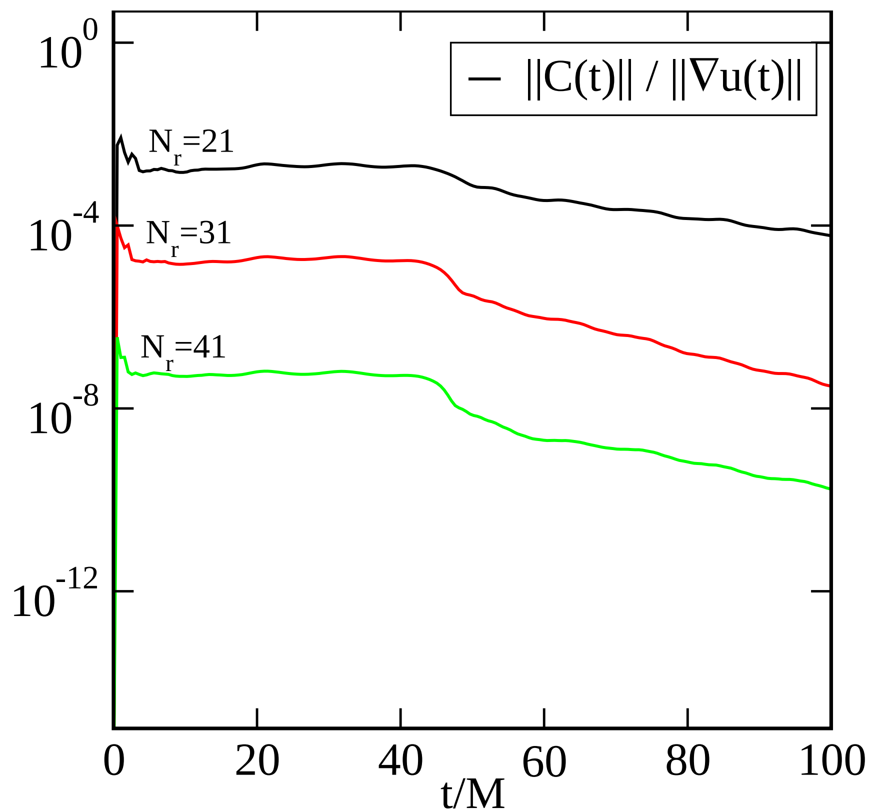
- **Optimal projection**:
Write down Lagrangian, work out variations, simplify results ...

$$\begin{aligned}\nabla^i\nabla_i\psi + (\Lambda^2 - \gamma^2)\psi &= \nabla^i\bar\Phi_i - (\Lambda^2-\gamma^2)\bar\psi\\ \Pi &= \bar\Pi + \gamma(\psi - \bar\psi)\\ \Phi_i &= \partial_i\psi\end{aligned}$$

Evolutions w/o constraint projection

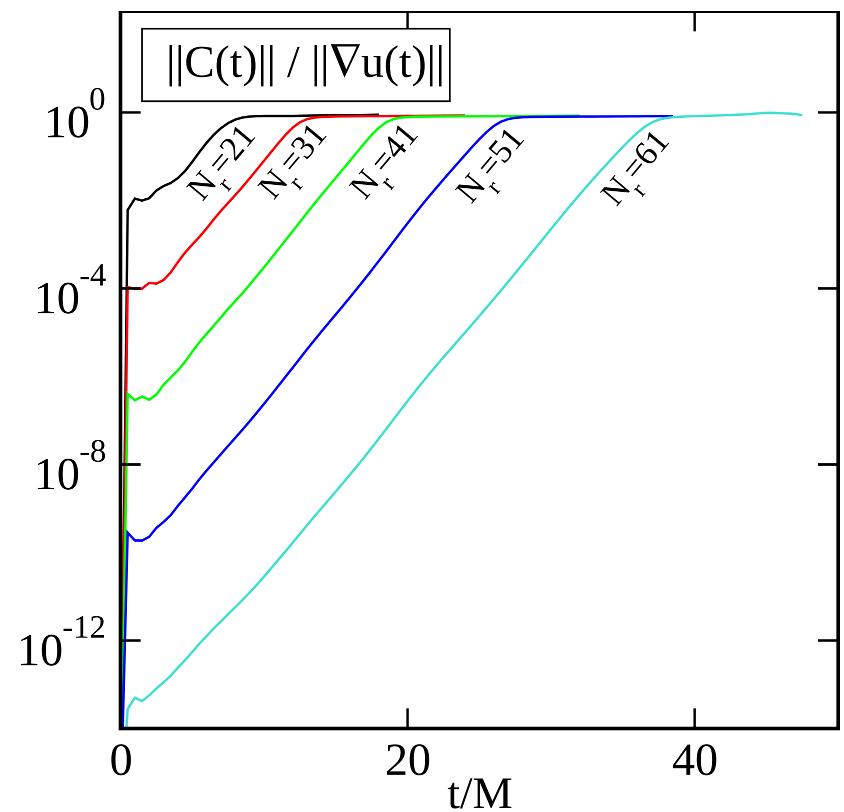
Spectral methods; Schwarzschild in Kerr-Schild coordinates.

$\gamma = 0$



constraints fine, runs perfect w/o projection

$\gamma = -1$

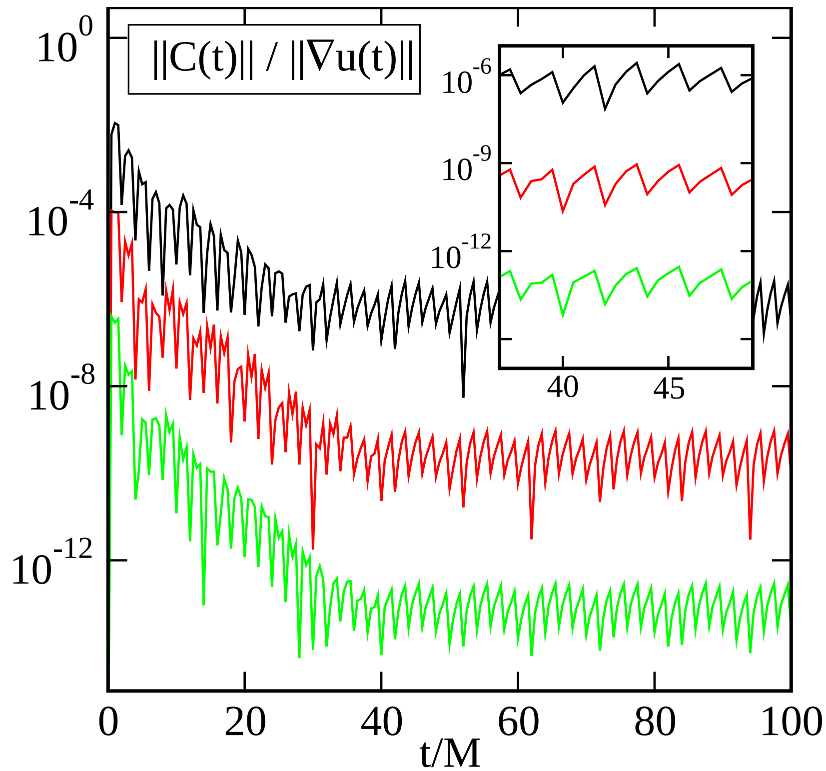


Constraints blow up exponentially

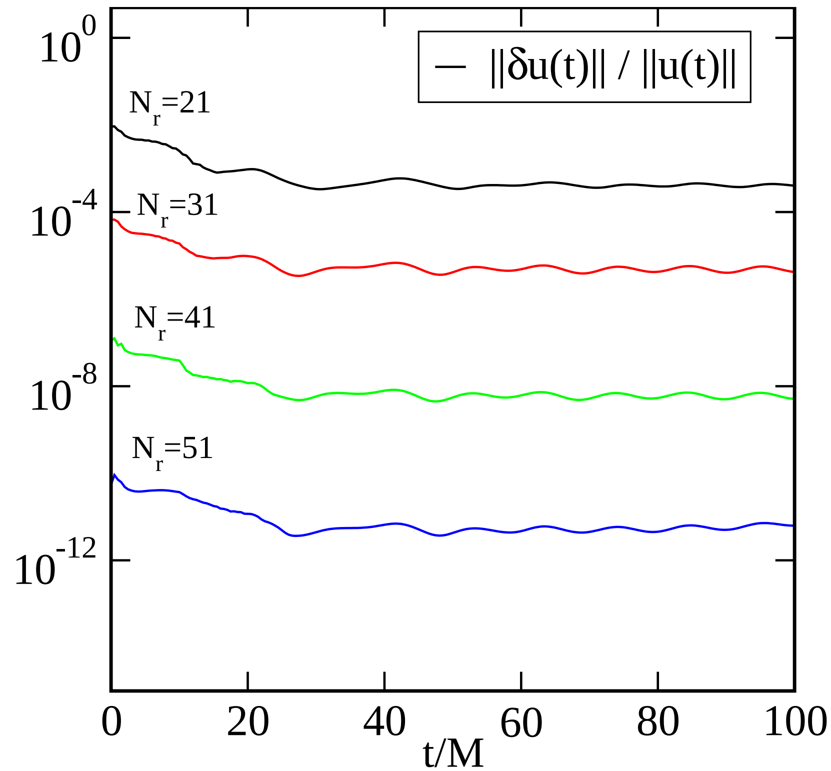
Evolutions with constraint projection

$\gamma = -1$, project every $\Delta T = 2M$

Constraint violations



Difference to reference run



Summary

- We introduced optimal constraint projection based on Lagrangian

$$\mathcal{L} = S_{\alpha\beta}(u^\alpha - \bar{u}^\alpha)(u^\beta - \bar{u}^\beta) + \lambda_A C^A$$

- Optimal constraint projection completely controls the “evil scalar wave”

- Work on GR in progress

No time to talk about...

1. Boundary conditions (must be *constraint preserving*)
2. Computational cost (insignificant)

$\Rightarrow$ see gr-qc/0407011