# Constraint control in hyperbolic evolution systems

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# Gravitational wave detectors are rapidly improving

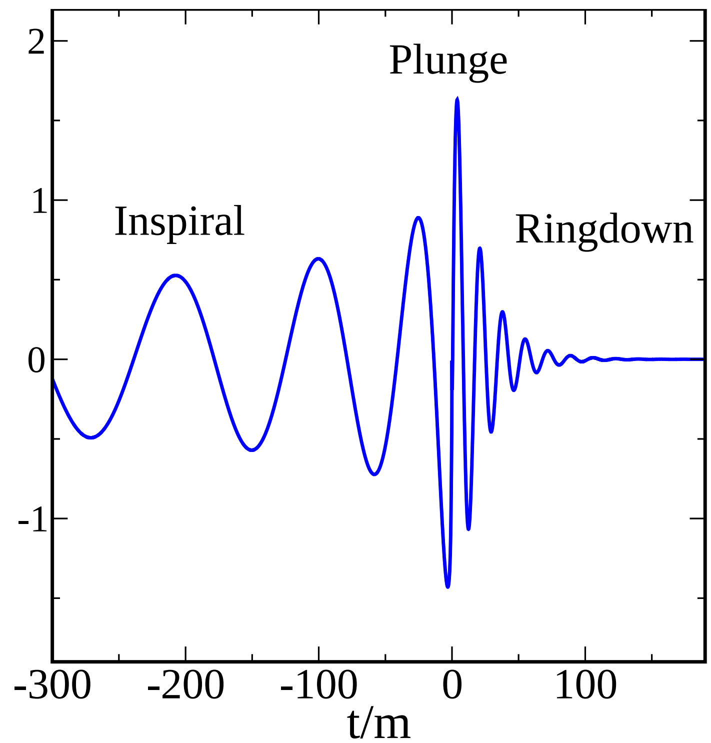
- GEO 600
- LIGO
- TAMA 300
- Virgo

Ligo Hanford site

One prime scientific target: Binary black hole coalescence

# Binary black hole coalescence

- Inspiral – post-Newtonian expansions
- Late inspiral & plunge – **numerical relativity**
- Ringdown – perturbation theory

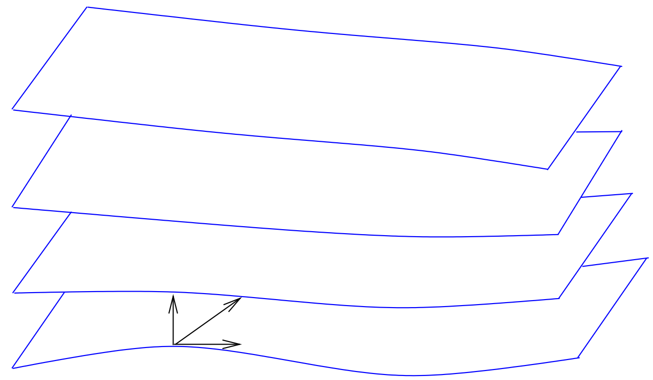



Waveforms from all three phases are important for...

1. Event detection
2. Parameter extraction
3. Testing general relativity

# Numerical relativity: Solving $R_{\mu\nu} = 0$

**Foliate** spacetime by $t =$ const. surfaces



**Split** 4-dimensional quantities into:

1. 3-dim quantities *within* each surface
   - $g_{ij}$ metric within surface
   - $K_{ij}$ extrinsic curvature ("momentum" of $g_{ij}$)
2. quantities that connect neighboring surfaces

**evolution equations**

$$\partial_t g_{ij} = \ldots$$

$$\partial_t K_{ij} = N R_{ij} + \ldots$$

**constraint equations**

$$R + K^2 - K_{ij}K^{ij} = 0$$

$$\nabla_j \left( K^{ij} - g^{ij} K \right) = 0$$

Maxwell-equations

$$\partial_t \vec{E} = \nabla \times \vec{B}$$

$$\partial_t \vec{B} = -\nabla \times \vec{E}$$

$$\nabla \vec{E} = 0$$

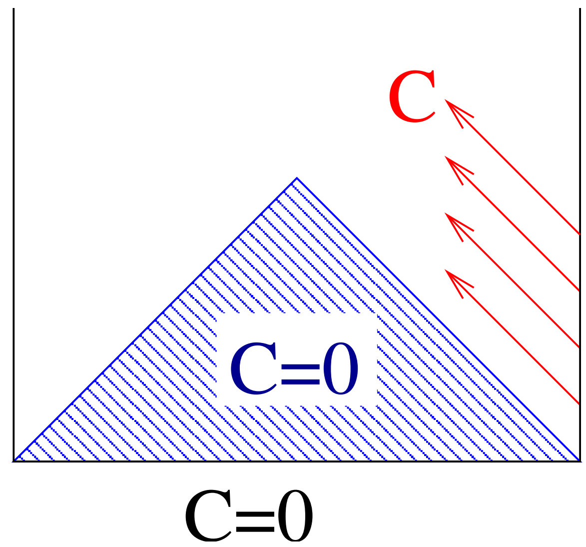
$$\nabla \vec{B} = 0$$

# Topic of this talk: *Constraints* "$C = 0$"

Analytically,

- If $C = 0$ for initial data, then $C = 0$ in domain of dependence
- BUT... $C \neq 0$ can enter through timelike boundaries
- AND... Small, but non-zero $C$ may grow exponentially fast,

$$\partial_t C = \gamma C, \qquad \text{for } \gamma \in \mathbf{R}$$




**Outline**

1. With toy-problem illustrate
   – both problems
   – and their solutions
2. For Einstein's equations
   – present C-preserving BC's

## Massless scalar field $\Box\psi = 0$

- Reduction to first order system — define $\Pi \equiv -\partial_t\psi$, $\Phi_i \equiv \partial_i\psi$

$$\partial_t\psi + \Pi = 0$$

$$\partial_t\Pi + \delta^{ij}\partial_i\Phi_j = 0$$

$$C_i \equiv \partial_i\psi - \Phi_i = 0 \quad \text{Constraint}$$

$$\partial_t\Phi_i + \partial_i\Pi = 0 \quad \Leftarrow \partial_t C_i = 0$$

- In curved spacetime:

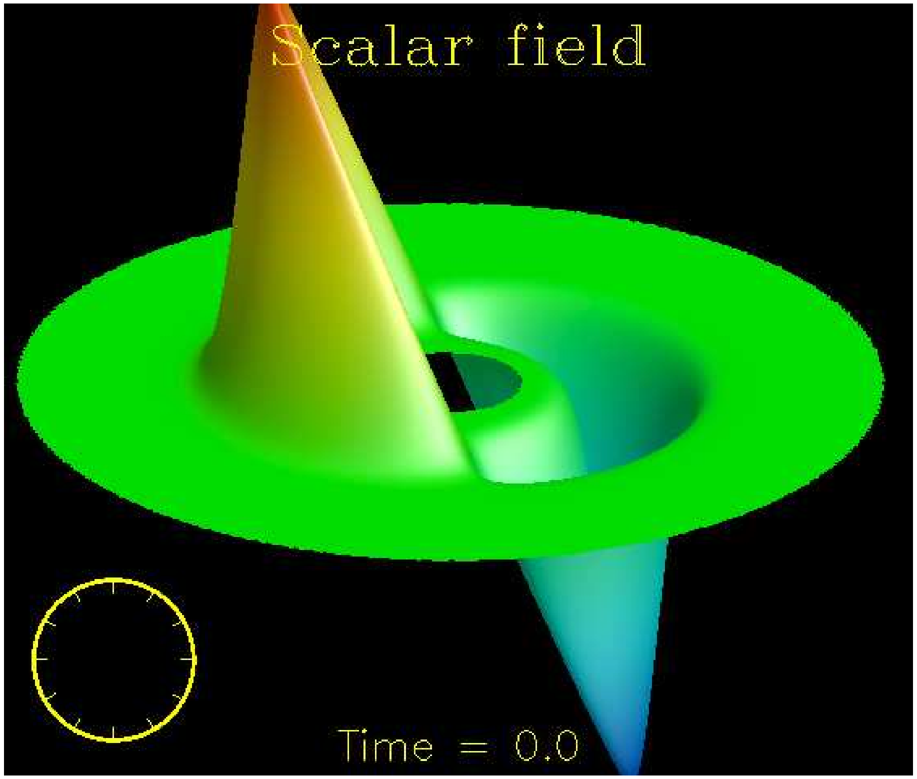
$$\partial_t\psi - \beta^k\partial_k\psi = -N\Pi$$

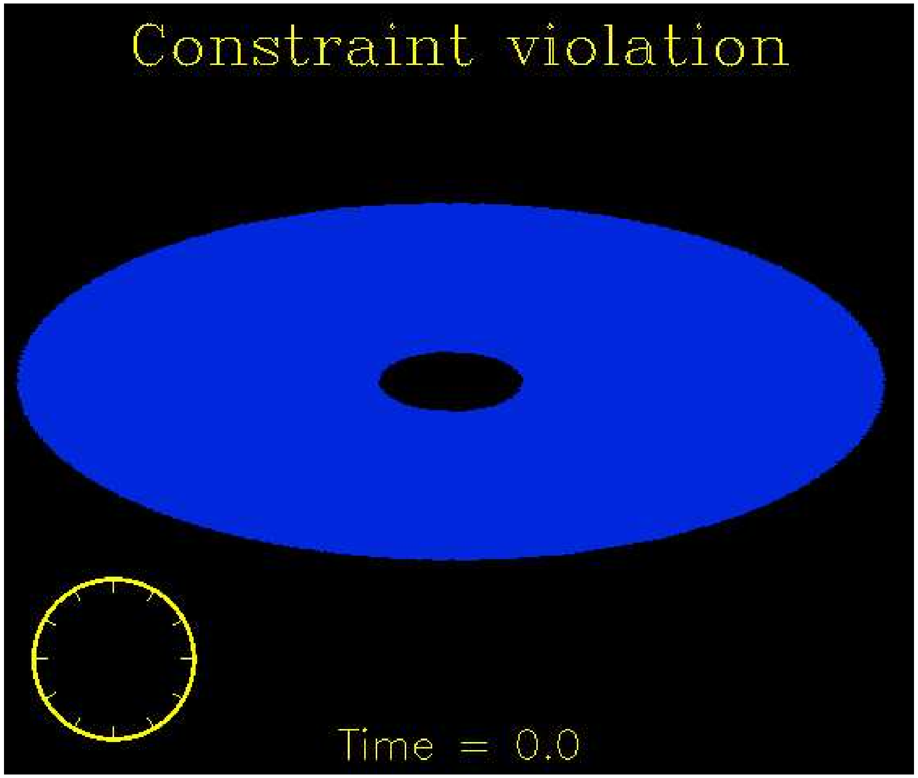
$$\partial_t\Pi - \beta^k\partial_k\Pi + Ng^{ikj}\partial_i\Phi_j = NJ^i\Phi_i + NK\Pi$$

$$\partial_t\Phi_i - \beta^k\partial_k\Phi_i + N\partial_i\Pi = -\Pi\partial_i N + \Phi_j\partial_i\beta^j$$

$$\text{Constraint} \quad C_i \equiv \partial_i\psi - \Phi_i = 0$$

# Scalar field around Schwarzschild BH




**Spectral method:**

– expand $u(\vec{x}, t) = \sum \tilde{u}_{klm}(t) T_k(r) Y_{lm}(\theta, \phi)$

– evolve $u(x_i, t)$ by "method of lines" ($x_i$ collocation points)

# Boundary conditions

- **Abstract form of evolution equations** with $u^\alpha = \{\psi, \Pi, \Phi_i\}$)

$$\partial_t u^\alpha + A^{k\alpha}{}_\beta[u]\partial_k u^\beta = F^\alpha[u]$$

- **Characteristic decomposition**
  1. eigenvectors $e^{\hat\alpha}{}_\beta$ & eigenvalues $v_{(\hat\alpha)}$ ($n^i$ outward pointing unit-normal)

$$e^{\hat\alpha}{}_\alpha\, n_k A^{k\alpha}{}_\beta = v_{(\hat\alpha)} e^{\hat\alpha}{}_\beta$$

  2. Characteristic fields

$$u^{\hat\alpha} \equiv a^{\hat\alpha}{}_\beta\, u^\beta$$

- **(strong) hyperbolicity**
  $\longrightarrow$ complete set of eigenvectors, and all real eigenvalues $v_{(\hat\alpha)}$
  $\longrightarrow$ Apply BC's precisely to *incoming* $u^{\hat\alpha}$ (those with $v_{(\hat\alpha)} < 0$)

- On last slide: **freezing BC's**: $u^{\hat\alpha}_{\text{incoming}} = \text{const}$
  $\Rightarrow$ freezing BC's are insufficient

Inside BH, ALL $u^{\hat\alpha}$ outgoing
NO BC's needed,
NONE applied

# Constraint preserving BC's for the scalar field

$$\partial_t\psi - \beta^k\partial_k\psi = -N\Pi$$

$$\partial_t\Pi - \beta^k\partial_k\Pi + N g^{ikj}\partial_i\Phi_j = N J^i\Phi_i + N K\Pi$$

$$\partial_t\Phi_i - \beta^k\partial_k\Phi_i + N\partial_t\Pi = -\Pi\partial_i N + \Phi_j\partial_i\beta^j$$

Constraint $\quad C_i \equiv \partial_i\psi - \Phi_i = 0$

**Characteristic fields $u^{\hat\alpha}$ and speeds $v_{(\hat\alpha)}$**

$$U^\pm = \Pi \pm n^k\Phi_k \qquad v_\pm = \pm 1 - n_k\beta^k$$

$$Z^1 = \psi \qquad v = -n_k\beta^k < 0$$

$$Z^2_i = P^k{}_i\Phi_k \equiv (\delta^k{}_i - n^k n_i)\Phi_k \qquad v = -n_k\beta^k < 0$$

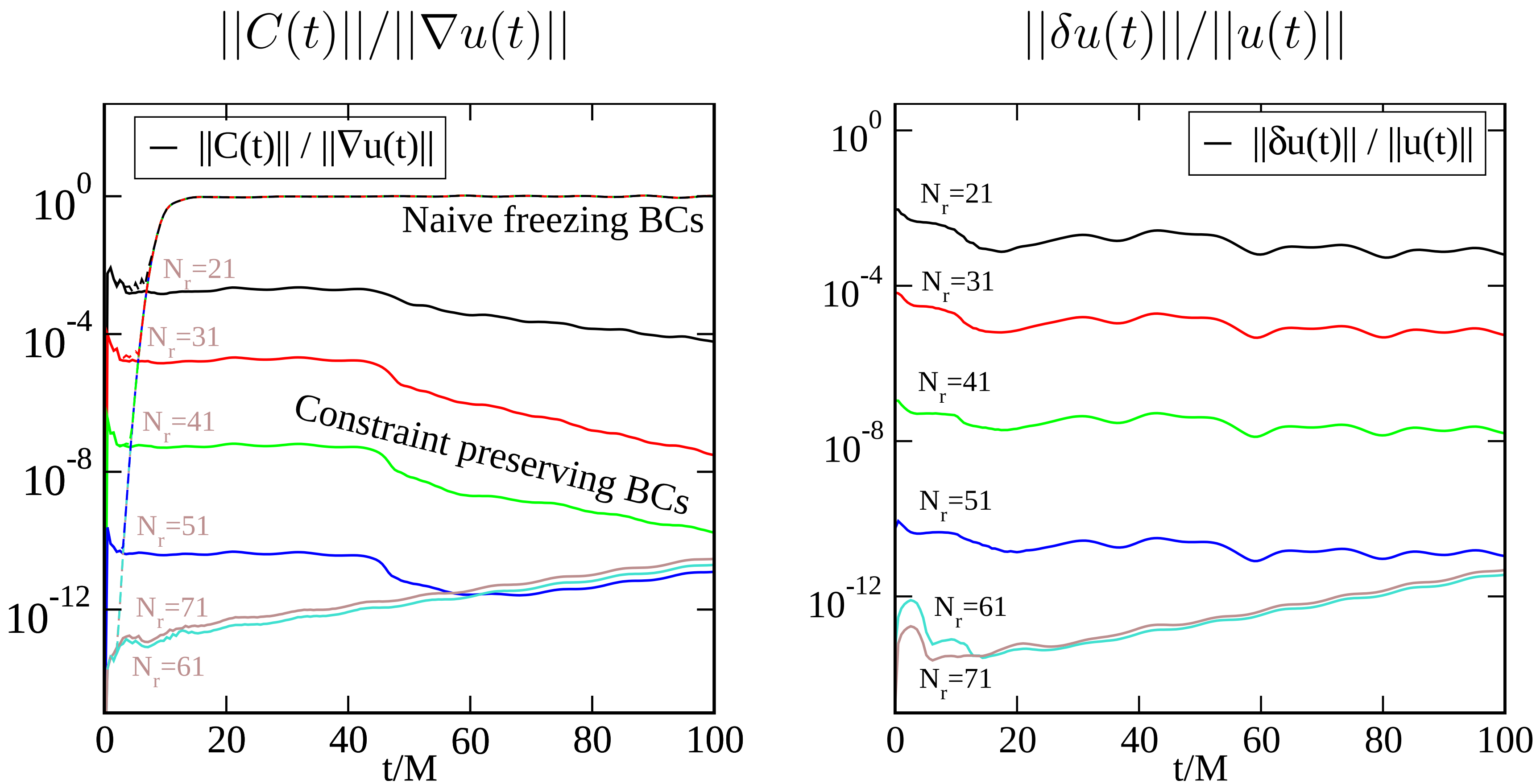
**Boundary conditions** (at outer boundary)

$\partial_t U^- = 0$ — *physics*: approx. outgoing wave BC

$\partial_t Z^2_i = P^k{}_i\partial_t\partial_k\psi$ — enforce $P^k{}_iC_k = 0$ by choice of $P^k{}_i\Phi_k$

$\partial_t Z^1 = \beta^k\Phi_i - N\Pi$ — enforce $n^iC_i = 0$ by choice of $\psi$

# Scalar field with constraint preserving BC's




N.B. Exponential convergence of spectral method apparent.

# Bulk constraint violations — evil scalar wave system

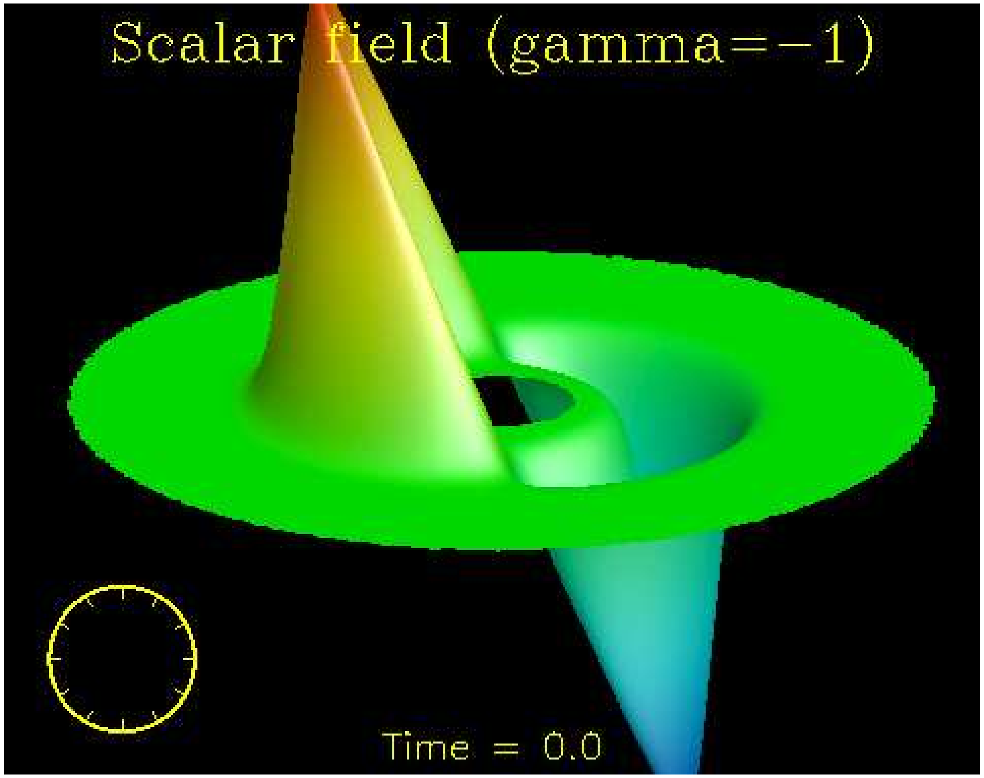
- The scalar field system presented so far does not exhibit bulk-constraint violations.

- Introduce parameter $\gamma$ and require $\partial_t C_i = -\gamma C_i$

$$\partial_t\psi + \Pi = 0$$

$$\partial_t\Pi + \delta^{ij}\partial_i\Phi_j = 0$$

$$\partial_t\Phi_i + \partial_t\Pi = \gamma\partial_i\psi - \gamma\Phi_i$$

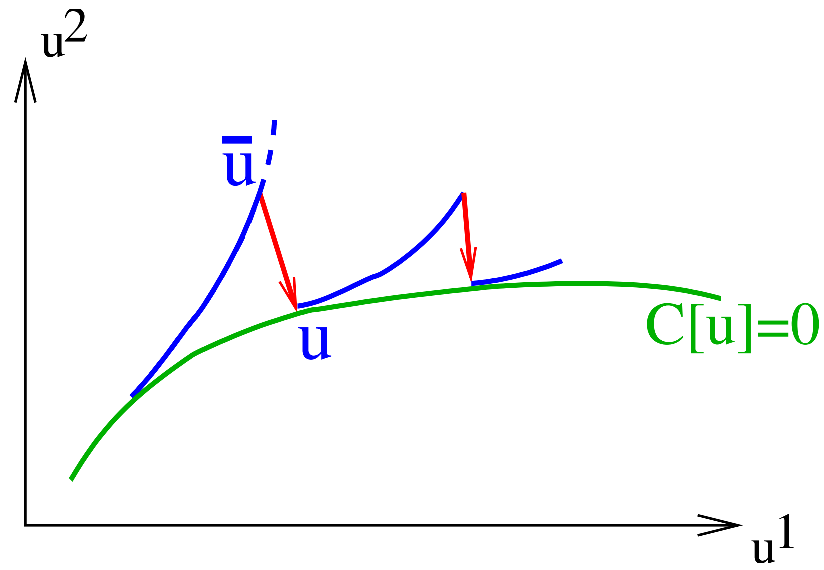



- Modified system still hyperbolic
- $\gamma \geq 0 \Rightarrow$ stable
- **$\gamma < 0 \Rightarrow C_i$ exponentially growing**

# Constraint projection

**Idea:**
– Use free evolution until constraints become too large
– *Project* the current configuration $\bar{u}$ back into the constraint-satisfying submanifold
– Continue with free evolution
– Repeat

**Problem:** Projection is not unique



# Optimal constraint projection

1. Go to **closest** constraint-satisfying point $u$, measured by some positive definite metric $S_{\alpha\beta}$
2. Incorporate constraints by Lagrangian multipliers $\lambda$
3. Minimize Lagrangian

$$\mathcal{L} = \int \left\{ (u^\alpha - \bar{u}^\alpha) S_{\alpha\beta} (u^\beta - \bar{u}^\beta) + \lambda^i C_i \right\}$$

   with respect to $u^\alpha$ and $\lambda^i$

4. For *symmetric hyperbolic* evolution systems, there is a natural choice for $S_{\alpha\beta}$, the symmetrizer satisfying

$$S_{\alpha\beta} A^{k\alpha}{}_{\gamma} = S_{\alpha\gamma} A^{k\alpha}{}_{\beta}$$

   NB: Most interesting evolution systems are symmetric hyperbolic

# Optimal constraint projection for the evil scalar wave

**Symmetrizer** $S_{\alpha\beta} \mathrm{d}u^\alpha \mathrm{d}u^\beta = \Lambda^2 \mathrm{d}\psi^2 - 2\gamma \mathrm{d}\psi \mathrm{d}\Pi + \mathrm{d}\Pi^2 + g^{ij} \mathrm{d}\Phi_i \mathrm{d}\Phi_j$ with $\Lambda^2 > \gamma^2$

**Lagrangian density**

$$\mathcal{L} = g^{1/2}\Big[\Lambda^2(\psi - \bar\psi)^2 - 2\gamma(\psi - \bar\psi)(\Pi - \bar\Pi) + (\Pi - \bar\Pi)^2 + g^{ij}(\Phi_i - \bar\Phi_i)(\Phi_j - \bar\Phi_j) + \lambda^i(\partial_i\psi - \Phi_i)\Big]$$

**Variations** (after integration by parts)

$$\frac{\delta\mathcal{L}}{\delta\psi} = 2g^{1/2}\left[\Lambda^2(\psi - \bar\psi) - \gamma(\Pi - \bar\Pi)\right]\delta\psi - \partial_i(g^{1/2}\lambda^i) \tag{1}$$

$$\frac{\delta\mathcal{L}}{\delta\Pi} = 2g^{1/2}\left[\Pi - \bar\Pi - \gamma(\psi - \bar\psi)\right] \quad \Rightarrow \quad \Pi = \bar\Pi + \gamma(\psi - \bar\psi) \tag{2}$$

$$\frac{\delta\mathcal{L}}{\delta\Phi_i} = \left[2g^{1/2}g^{ij}(\Phi_i - \bar\Phi_i) - g^{1/2}\lambda^j\right] \tag{3}$$

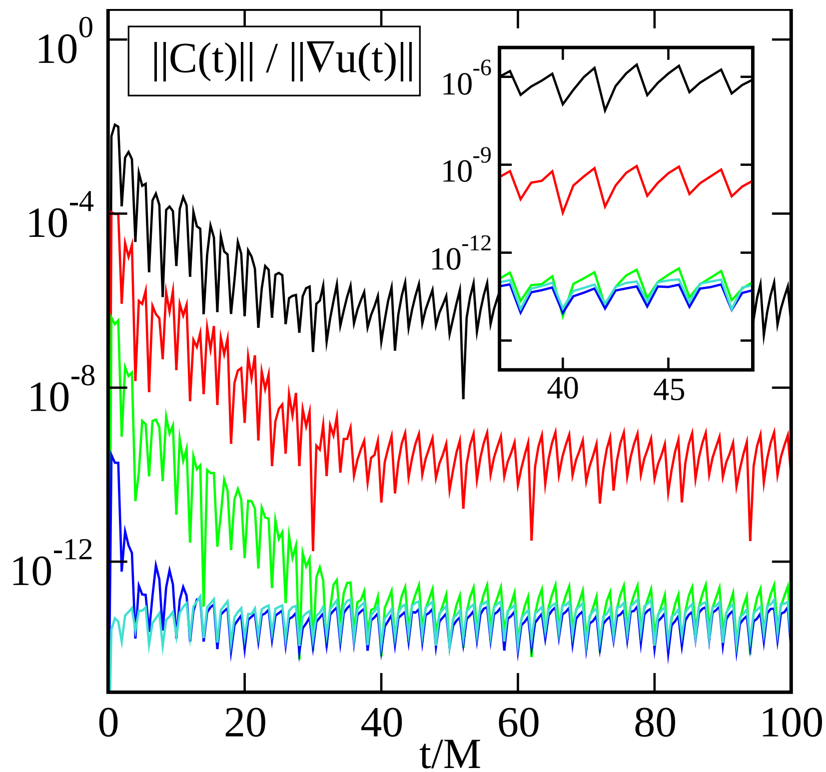
$$\frac{\delta\mathcal{L}}{\delta\lambda^i} = g^{1/2}(\partial_i\psi - \Phi_i) \quad \Rightarrow \quad \Phi_i = \partial_i\psi \tag{4}$$

Substitute (3) into (1), use (2) and (3)

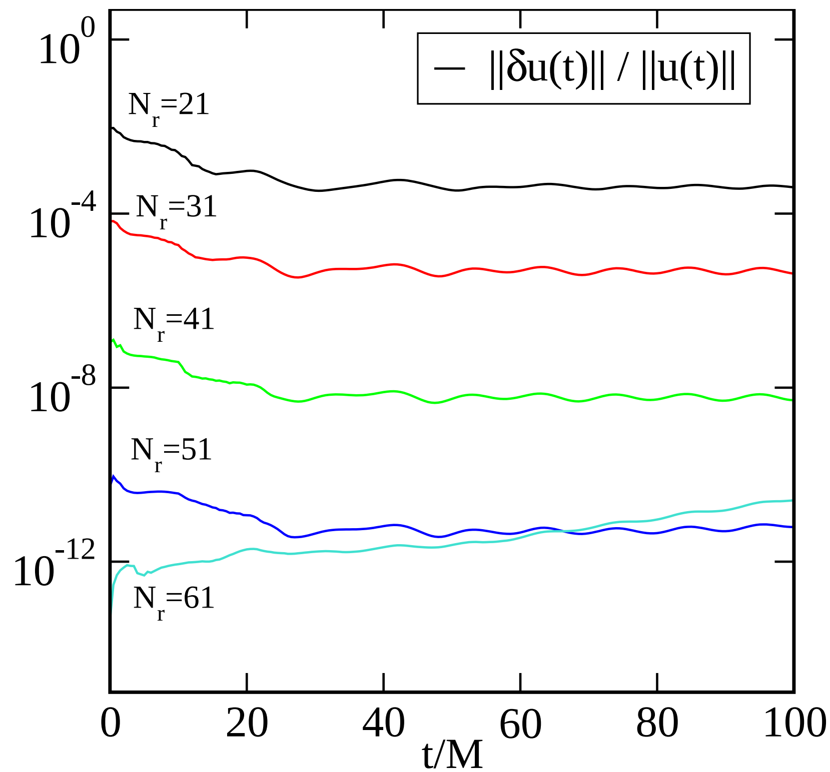
$$\nabla^2\psi - (\Lambda^2 - \gamma^2)\psi = \nabla^i\bar\Phi_i - (\Lambda^2 - \gamma^2)\bar\psi$$

# Constraint projected evil scalar wave

Constraints




Difference to $\gamma = 0$ – solution

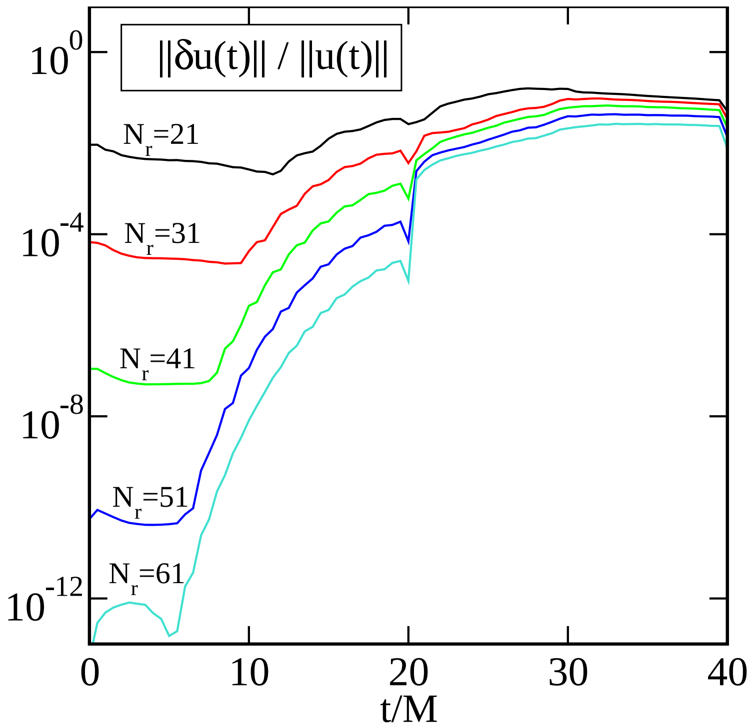



$\gamma = -M$ in evolution equations, constraint preserving BCs
$\Lambda = 2/M$ in symmetrizer, project every $T = 2M$

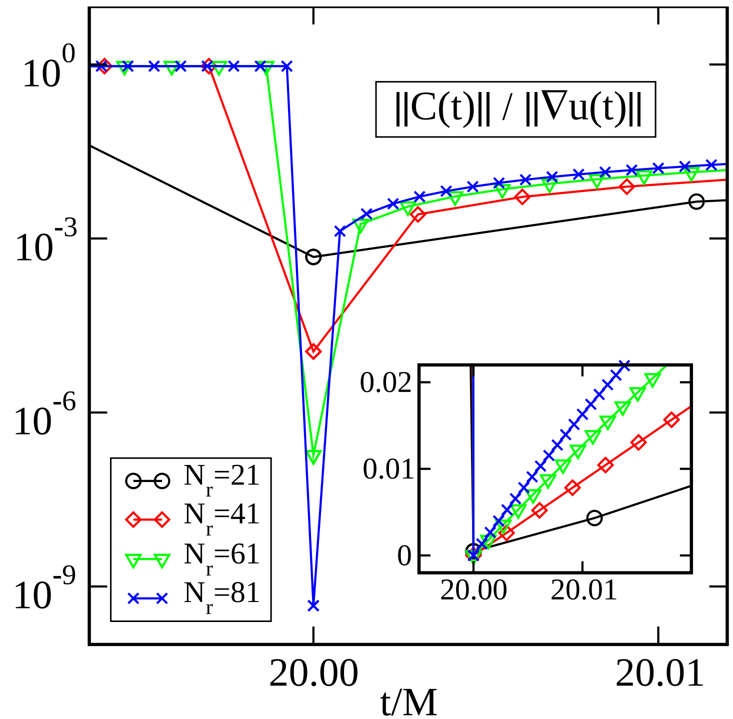
# Constraint projection as substitute for Constraint preserving BC's?

Evolve "nice" scalar wave system ($\gamma = 0$) with freezing boundary conditions
Perform a single constraint projection at $T = 20M$

Convergence




Constraints

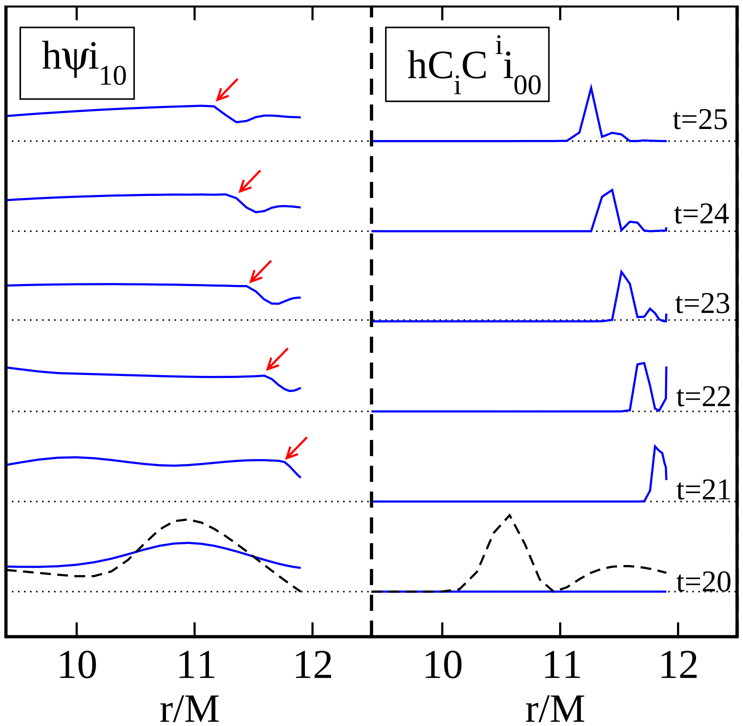



Spectral convergence is sigificantly reduced (lost?) after projection

Constraint-violations return within a *single* timestep

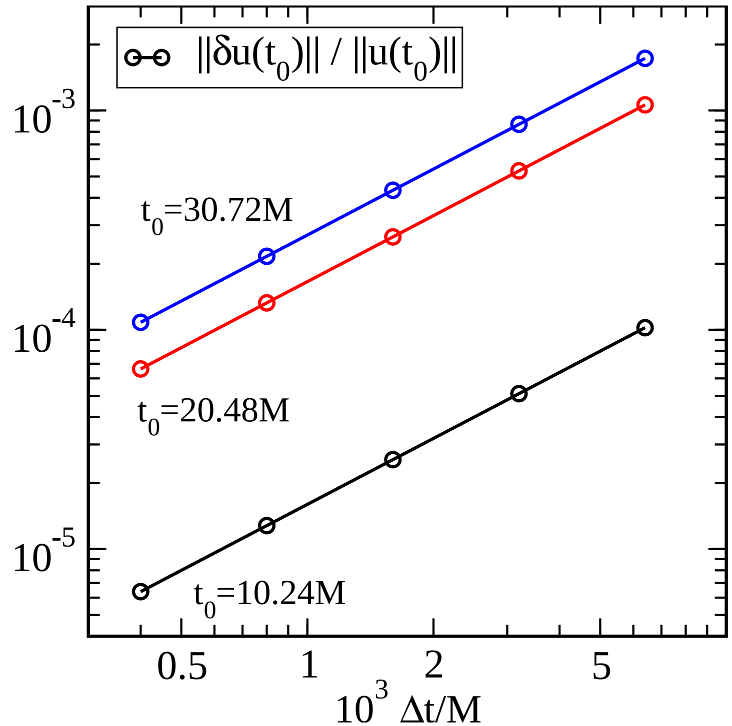
# Constraint projection as substitute for Constraint preserving BC's?

Radial profiles just after constraint projection




Very sharp constraint violating pulse enters

Project after *every* timestep convergence with timestep $\Delta t$




Converence only first order in $\Delta t$

# Summary of experiments

| | "nice" scalar field ($\gamma = 0$) freezing BC | "nice" scalar field ($\gamma = 0$) C-preserving BC | $\gamma = -1$ C-preserving BC |
|---|---|---|---|
| no projection | C-influx | ok! | bulk C-violations |
| projection | convergence $\mathcal{O}(\Delta t)$ (at best!) | ok! | **ok!** |

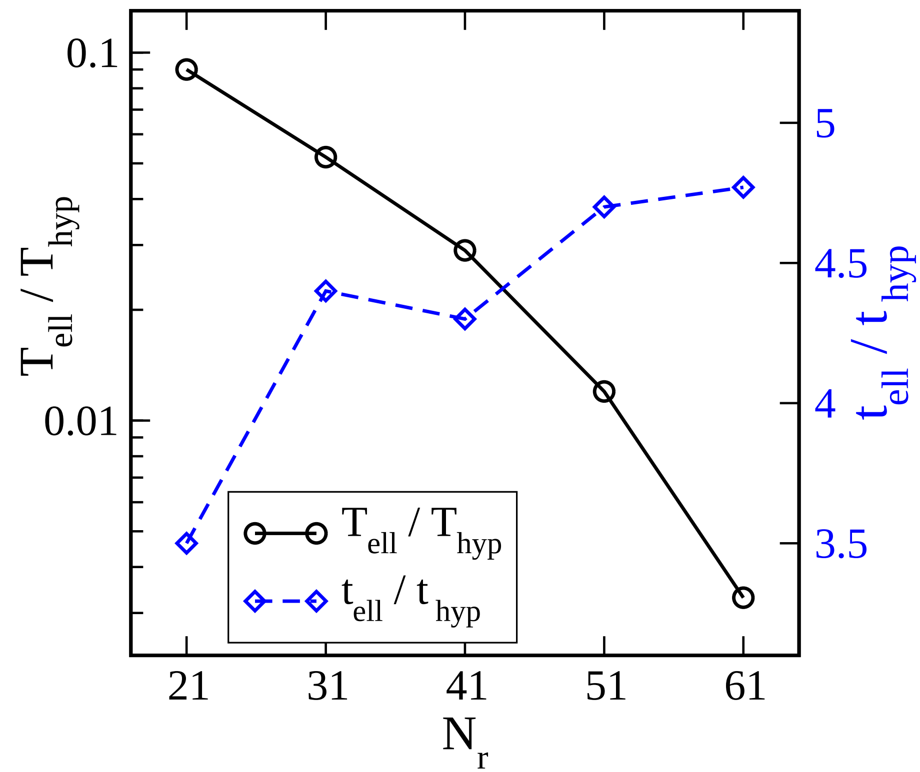
- Projection cures bulk-constraint violations.
- C-preserving BC's cure C-influx through boundaries.
- No other combination works.

# Computational cost of projection

Project on time-scale $T$ of constraint-growth ($\sim 1/\gamma$)

Evolution time-step $\Delta t \ll T$

$\Rightarrow$ computational cost of projection neglible




For Einstein's equations, bulk-C's grow slower, $T_{\rm growth} \gtrsim 100M$
Even fewer projections may suffice

# Einstein's equations – KST-system

- **Kidder-Scheel-Teukolsky** evolution system (PRD, 2001)

- 30 evolved variables $u^\alpha = \left\{ g_{ij},\ K_{ij},\ D_{kij} \equiv 1/2\,\partial_k g_{ij} \right\}$

- $g_{ij}$ spatial metric, $K_{ij}$ extrinsic curvature
  $D_{kij}$ brings system into first order form (cf. $\Phi_i$ for the scalar field)

- Evolution equations are the 3+1 evolution Eqs. with the replacement $\partial_k\partial_l g_{ij} \longrightarrow 2\partial_{(k}D_{l)ij}$ and with constraints added at several places

$$\partial_t g_{ij} \simeq \beta^n \partial_n g_{ij}$$

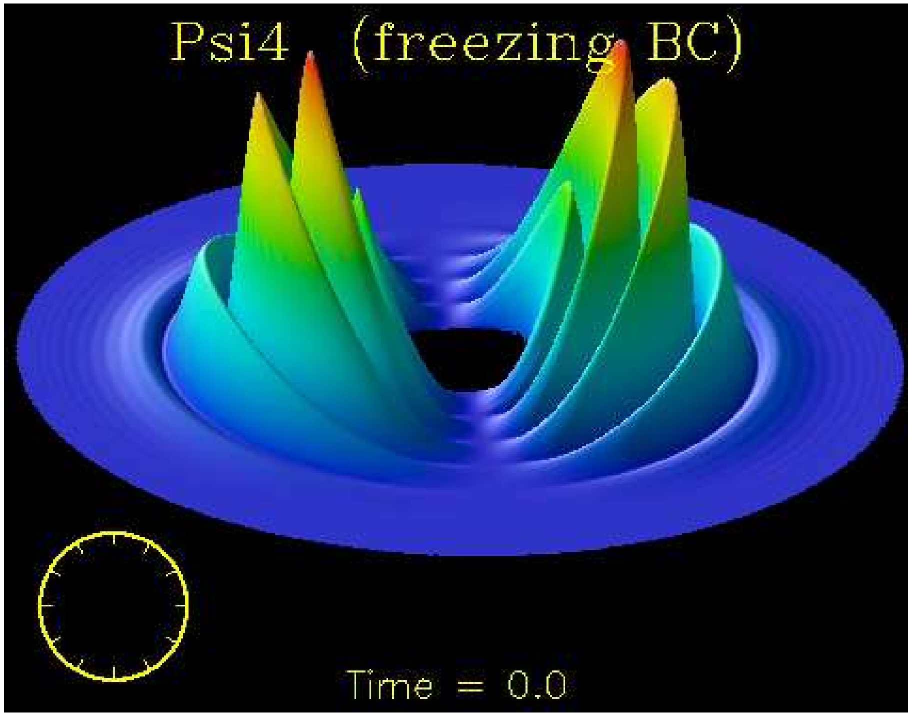
$$\partial_t K_{ij} \simeq \beta^n \partial_n K_{ij} - N\Big[(1+2\gamma_0)g^{cd}\delta^n{}_{(i}\delta^b{}_{j)} - (1+\gamma_2)g^{nd}\delta^b{}_{(i}\delta^c{}_{j)} - (1-\gamma_2)g^{bc}\delta^n{}_{(i}\delta^d{}_{j)} + g^{nb}\delta^c{}_i\delta^d{}_j + 2\gamma_1 g^{n[b}g^{d]c}g_{ij}\Big]\partial_n D_{bcd}$$

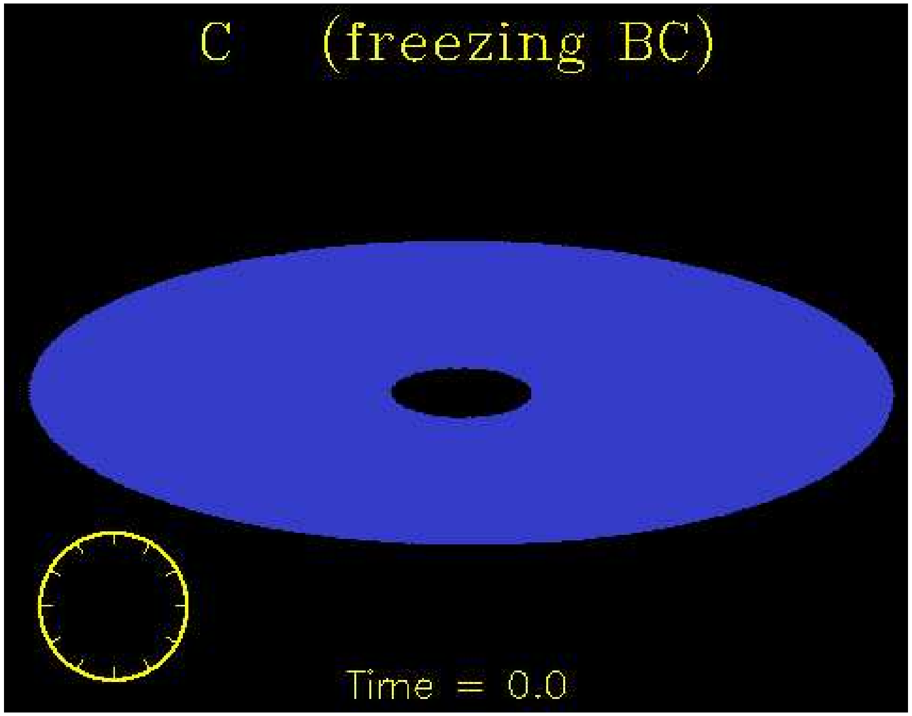
$$\partial_t D_{kij} \simeq \beta^n \partial_n D_{kij} - N\Big[\delta^n{}_k\delta^b{}_i\delta^c{}_j - \frac{1}{2}\gamma_3 g^{nb}g_{k(i}\delta^c{}_{j)} - \frac{1}{2}\gamma_4 g^{nb}g_{ij}\delta^c{}_k + \frac{1}{2}\gamma_3 g^{bc}g_{k(i}\delta^n{}_{j)} + \frac{1}{2}\gamma_4 g^{bc}g_{ij}\delta^n{}_k\Big]\partial_n K_{bc}$$

(lower order terms not shown)

# Evolutions with freezing boundary conditions

Kerr-Schild with superposed Teukolsky wave (low amplitude, $E_{\mathrm{ADM}}/M_{\mathrm{AH}} \approx 1 + 10^{-5}$)



# Constraints

$$\mathcal{C} = \frac{1}{2}\left({}^{(3)}R - K_{ij}K^{ij} + K^2\right)$$
Hamiltonian constraint

$$\mathcal{C}_i = \nabla_j K^j{}_i - \nabla_i K$$
Momentum constraint

$$\mathcal{C}_{kij} = \partial_k g_{ij} - 2D_{kij}$$
Def'n of $D_{kij}$

$$\mathcal{C}_{klij} = 2\partial_{[k}D_{l]ij}$$
2nd partial derivs commute

Movie showed $C \equiv \left(\mathcal{C}^2 + \mathcal{C}_i\mathcal{C}^i + \mathcal{C}_{ij}\mathcal{C}^{ij} + \mathcal{C}_{ijk}\mathcal{C}^{ijk} + \mathcal{C}_{ijkl}\mathcal{C}^{ijkl}\right)^{1/2}$

# Characteristic fields

$$Z^1 = \gamma_3 n^i D_i^1 - 2(1+\gamma_4) n^i D_i^2,$$

$$Z_i^2 = \gamma_4 P^j{}_i D_j^1 - (\gamma_3 + 2\gamma_4) P^j{}_i n^k n^l D_{jkl},$$

$$Z_i^3 = 3P^j{}_i D_j^1 - 2P^j{}_i D_j^2 - 4P^j{}_i n^k n^l D_{jkl},$$

$$Z_i^4 = +\,48 v_2^2 n^l P^j{}_i n^k D_{ljk} + 2\gamma_4(5-9\gamma_2) P^j{}_i D_j^2 - 2(6+\gamma_4)(5-9\gamma_2) P^j{}_i n^k n^l D_{jkl} + \ldots$$

$$Z_{ij}^5 = \left(P^a{}_i P^b{}_j - \frac{1}{2} P_{ij} P^{ab}\right) n^k D_{abk},$$

$$Z_{kij}^6 = P_{kij}^{cab} D_{cab},$$

$$U^{1\pm} = \pm\left[1 + 2v_1^2 + (1+2\gamma_1) q\right] n^i D_i^1 - v_1(1-q) P^{ij} K_{ij} + 2v_1 n^i n^j \left[K_{ij} \pm v_1 n^k D_{kij}\right] + \ldots$$

$$U_i^{2\pm} = \pm\, 2v_2 n^k P^j{}_i K_{jk} + (1+2\gamma_0) P^j{}_i D_j^1 - (1-\gamma_2) P^j{}_i D_j^2 + (2\gamma_0 - \gamma_2) P^j{}_i n^k n^l D_{jkl},$$

$$U^{3\pm} = \pm\,(1+2\gamma_1) n^i D_i^1 \mp (1+2\gamma_1+\gamma_2) n^i D_i^2 + v_3 P^{ij} K_{ij},$$

$$U_{ij}^{4\pm} = \left(P^a{}_i P^b{}_j - \frac{1}{2} P_{ij} P^{ab}\right) \left[K_{ab} \pm n^k D_{kab} \mp (1+\gamma_2) n^k D_{(ab)k}\right].$$

Only $U^4 \pm_{ij}$ has characteristic speed $\pm 1$ independent of parameter choices. **Physical mode!**

# Constraint preserving boundary conditions

General procedure to derive constraint preserving BCs (Steward, 1998, Calabese *et al.*. 2003):

1. Derive **constraint evolution system**

$$\partial_t C^A + A[u^\alpha]^{kA}{}_B \partial_k C^B = F[u^\alpha]^A{}_B C^B$$

   where $C^A = \{\mathcal{C}, \mathcal{C}_i, \mathcal{C}_{kij}, \mathcal{C}_{klij}\}$

2. For the KST-system, this is strongly hyperbolic whenever the KST-system is
3. Comptute **characteristic fields of the C-system**, $C^{\hat{A}}$
4. Require that the ***incoming*** **C-fields vanish**

$$C^{\hat{A}} \equiv 0 \qquad \text{for } v_{\hat{A}} < 0 \qquad (1)$$

5. The $C^{\hat{A}}$ are functions of $u^\alpha$ and therefore of $u^{\hat{\alpha}}$.
   Consequently, **(1) represents conditions on $u^{\hat{\alpha}}$** $\Leftarrow$ ***Constraint preserving BCs***

6. This procedure fixes many of the required boundary conditions on the $u^{\hat{\alpha}}$.

# Physical boundary conditions

C-preserving BCs ***cannot*** fix physical modes $U_{ij}^{4-}$

Follow the idea of Bardeen & Buchman, 2002:

- Consider **Weyl-tensor**, decomposed into electric and magnetic parts

$$E_{\mu\nu} = C_{\mu\sigma\nu\tau}\, n^\sigma n^\tau, \tag{5}$$

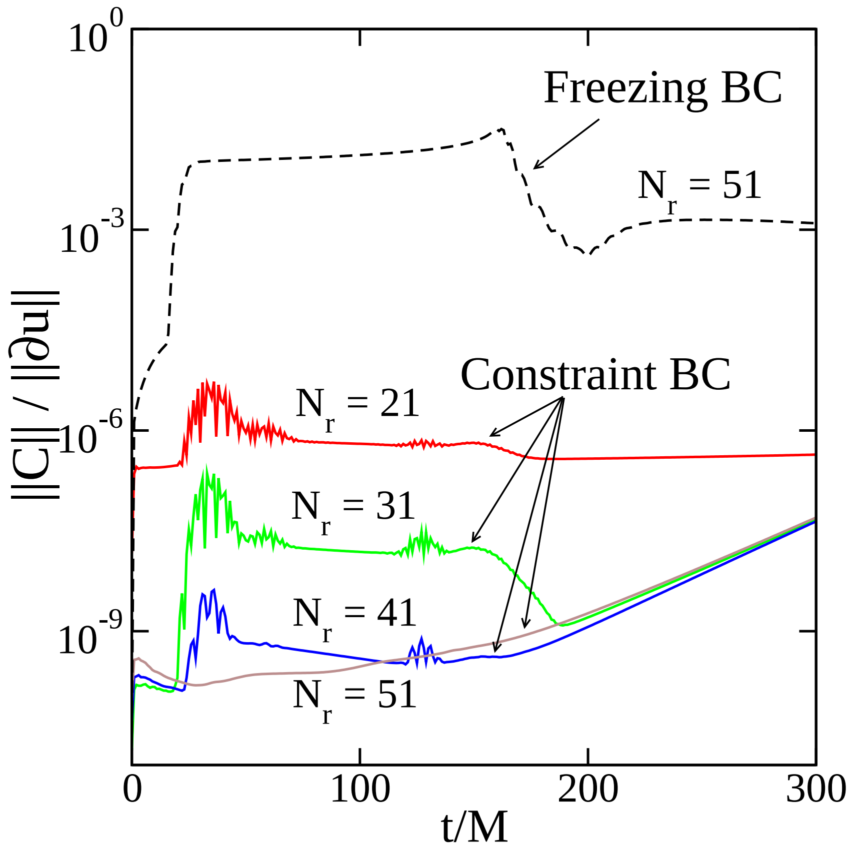
$$B_{\mu\nu} = \frac{1}{2} C_{\mu\omega\sigma\tau}\, \epsilon^{\sigma\tau}{}_{\nu\rho} n^\omega n^\rho, \tag{6}$$

- The **evolution system** for $E_{\mu\nu}$ and $B_{\mu\nu}$ is strongly hyperbolic – compute its **characteristic fields**
- One of its incoming characteristic fields, $U_{ij}^{8-}$, is proportional to $\Psi_0$ (Newman-Penrose component)
- **Prescribing** $\boldsymbol{\Psi_0}$ on the boundary determines $U_{ij}^{8-}$, which in turn **determines** $\boldsymbol{U_{ij}^{4-}}$.
- NB: $U_{ij}^{4+} \sim \Psi_4$

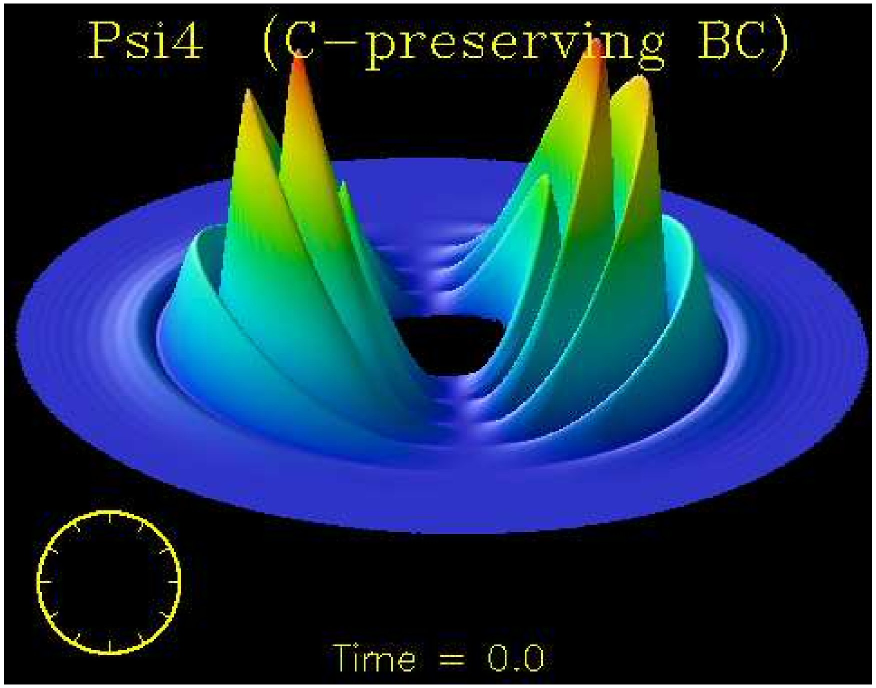
# Gauge fixing boundary conditions

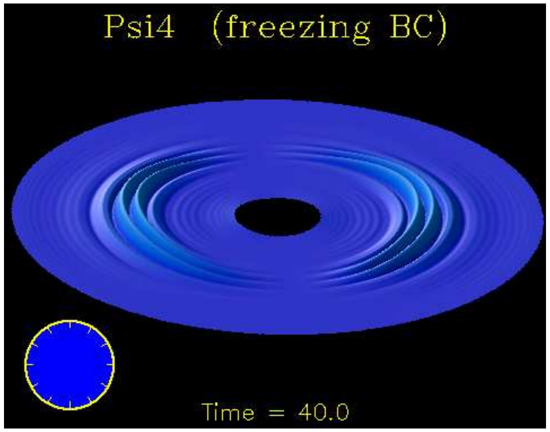
- **No boundary conditions yet** for the fields $U^{1-}$ and $Z_i^4$ $\longrightarrow$ Gauge modes
- $\partial_t K = 0$ (on bdry) results in condition for $U^{1-}$
- "Gamma-freezing" (on bdry) gives conditons on $Z_i^{4-}$
- In practice, we use $\partial_t K$ for the $l=0,1,2$–components of $U^{1-}$ and $\partial_t U^{1-} = \partial_t Z_i^4 = 0$ for the remaining components

# Success




outer bdry at $r_{\max} = 21.9M$

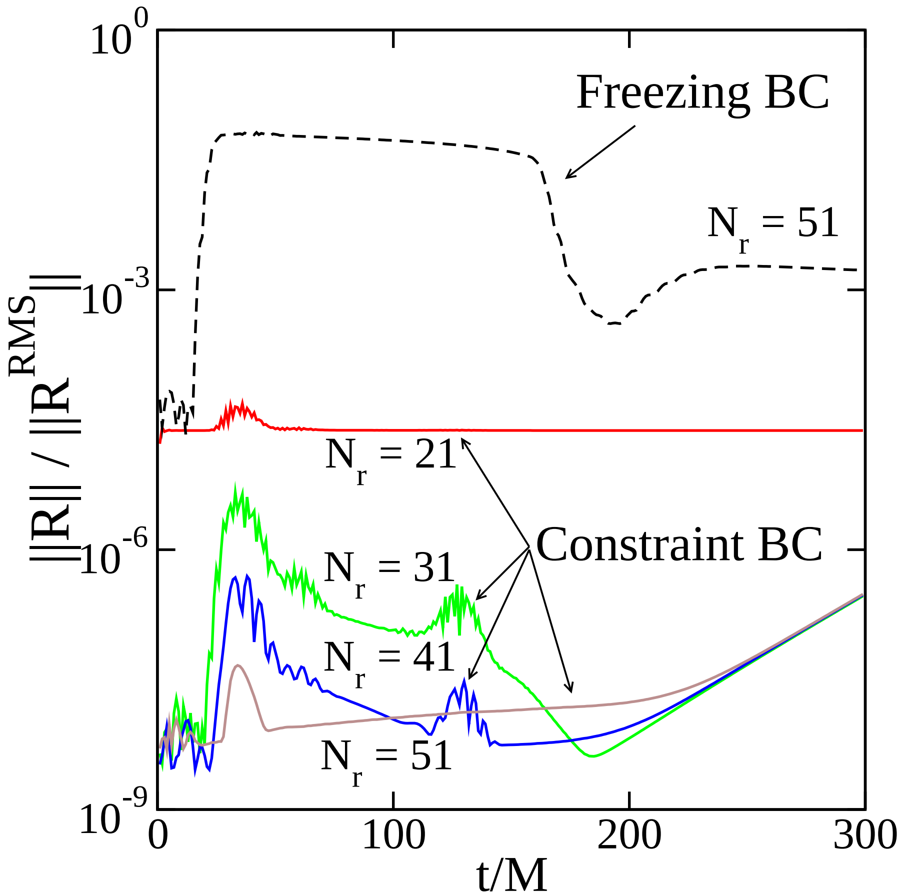


# Independent residual evaluator

Are we solving vacuum Einstein's equations?

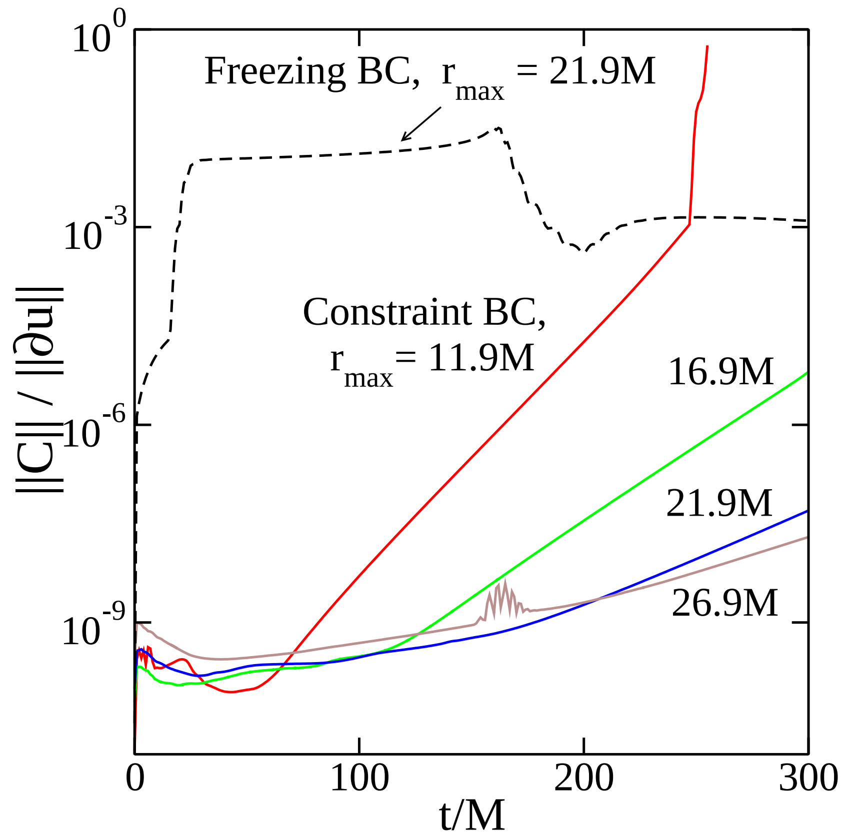
$$\boxed{{}^{(4)}\boldsymbol{R_{\mu\nu} = 0}}$$




- At each timestep, assemble ${}^{(4)}g_{\mu\nu}$
- Compute $\partial_t^{(4)}g_{\mu\nu}$ and $\partial_t\partial_t^{(4)}g_{\mu\nu}$ by finite differences
- Assemble ${}^{(4)}R_{\mu\nu}$ and compute suitable norm $||R||$

# Fineprint...

- **Exponentially growing** C-violation remains
- growth slower (but there) for larger $r_{\max}$
- **Similar growth seen** in evolutions of time-independent unperturbed Kerr-Schild solution **with freezing BCs**
- **Bulk constraint violation??**

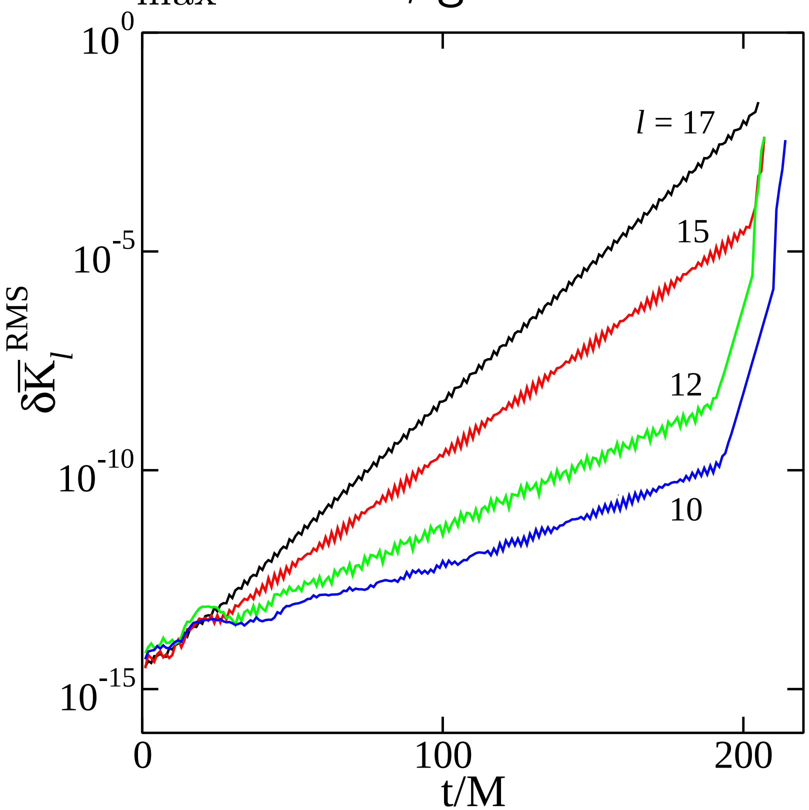


# More fineprint...

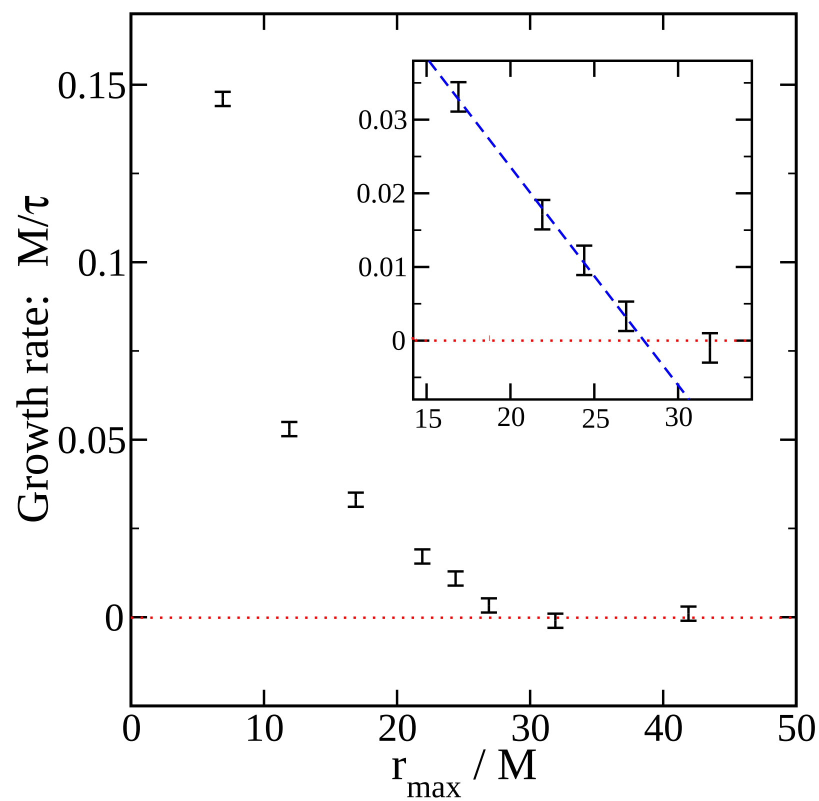
A **non-convergent angular instability** exists (if one looks hard enough)

(Unperturbed Kerr-Schild, Ylm-decomposition of $K - K_{\mathrm{analytic}}$)

$r_{\mathrm{max}} = 5.9M$, growth vs time




growth in $l = 17$ vs. $r_{\mathrm{max}}$




**Instability not present with freezing BC's**

# Summary

- For constrained evolution systems, C-violations can enter through time-like boundaries, or can grow in the bulk
- Experiments with the "evil" scalar wave system showed
  1. C-preserving BC's necessary, but don't cure bulk C-violations
  2. C-preserving BC's *and* C-projection control the system
- We developed BC's for the Einstein system...
  1. C-preserving & physical & gauge
- ...and presented tests in time-dependent situations
  1. Very effective against influx of C-violations
  2. Slowly growing (convergent) bulk C-violation remains
  3. Weak angular (nonconvergent) instability appears