Large-scale vortex formation in protoplanetary disks

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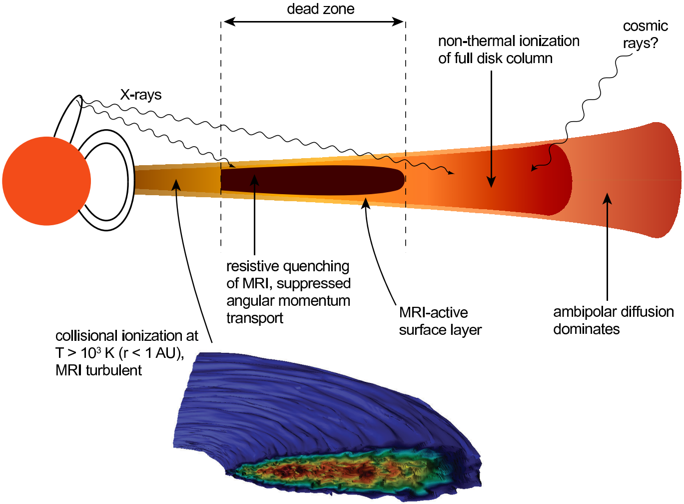
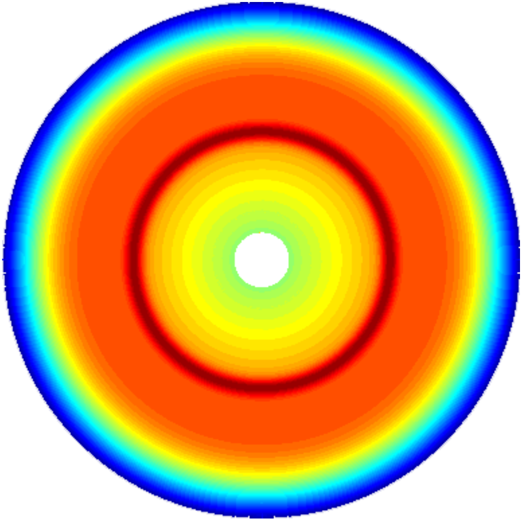
Complex structures in protoplanetary disks



(Armitage, 2011)

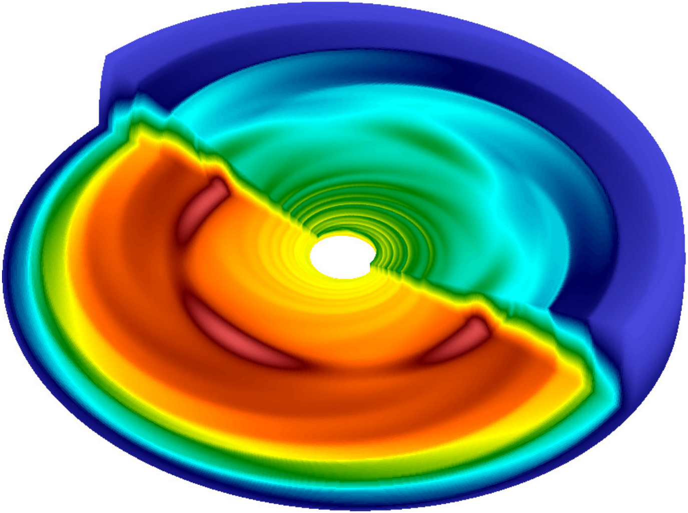
- Localized radial density gradients at boundaries of *dead zones*

Non-axisymmetric instability of structured disks



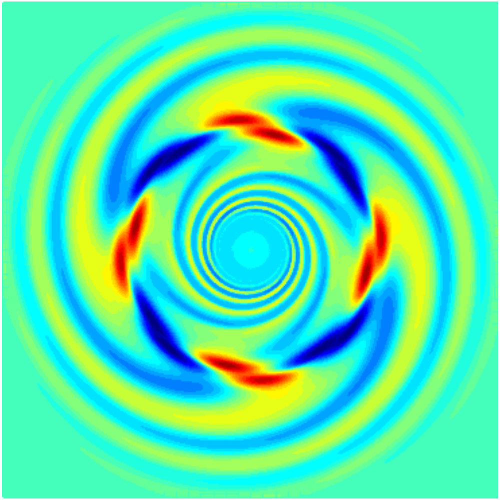
Toy model: axisymmetric over-dense ring

Non-axisymmetric instability of structured disks



ZEUS code: 3D self-gravitating adiabatic disk

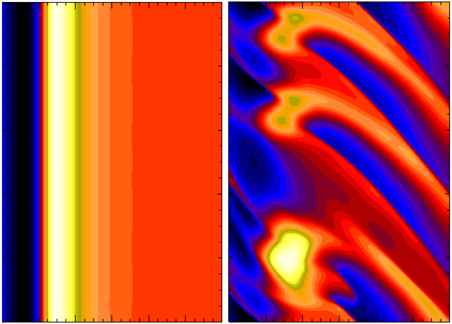
Non-axisymmetric instability of structured disks



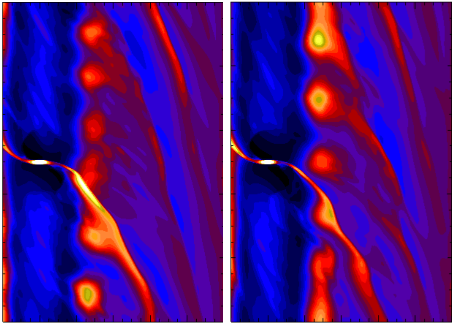
ATHENA code: 3D disk in a Cartesian box

Non-axisymmetric instability of structured disks

PLUTO code



3D disk with viscosity jump in radius



3D self-gravitating disk-planet simulation

Implications

- Hydrodynamic angular momentum transport (Li et al., 2001)
- Interaction with solids and planets (Inaba & Barge, 2006; Li et al., 2009; Meheut et al., 2012)

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Starting point → linear stability calculation for structured disks:

- Galactic disks: 'negative mass instability' (Lovelace & Hohlfeld, 1978), 'groove modes' (Sellwood & Kahn, 1991)
- Pressure-supported tori: 'Papaloizou-Pringle instability' (Papaloizou & Pringle, 1985; Narayan et al., 1987)
- Thin accretion disks: 'Rossby wave instability' (Lovelace et al., 1999; Li et al., 2000)

The linear problem

Original problem by Lovelace et al. (1999):

adiabatic non-self-gravitating 2D disk

Recent generalizations:

- Self-gravity 2D (Lin & Papaloizou, 2011a,b; Lovelace & Hohlfeld, 2012)
- Magnetic fields 2D (Yu & Li, 2009; Yu & Lai, 2013)
- Isothermal 3D (Meheut et al., 2012)

This talk:

- Polytropic 3D (Lin, 2012a, 2013a)
- Adiabatic 3D (Lin, 2013b)

The linear problem

After some manipulation, we have the basic equation for $\chi(=\delta p/\rho)$ as

$$\left[\frac{\partial}{\partial r}\left(a_{rr}\frac{\partial}{\partial r}+a_{rz}\frac{\partial}{\partial z}+b_r\right)+\frac{\partial}{\partial z}\left(a_{zz}\frac{\partial}{\partial z}+a_{rz}\frac{\partial}{\partial r}+b_z\right)+d_r\frac{\partial}{\partial r}+d_z\frac{\partial}{\partial z}+f\right]\chi=0,$$

with

$$a_{rr}=\frac{\rho\sigma r}{D}\left(1+\frac{\mu g_r^2}{DH}\right),\qquad a_{zz}=\frac{\rho r}{\sigma}\left(1+\frac{\mu g_z^2}{\sigma^2 H}\right),\qquad a_{rz}=\frac{\mu\rho g_r g_z r}{DH\sigma},$$

$$b_r=\frac{\mu\rho g_r}{DH}\left(\sigma r-\frac{2m\Omega g_r}{D}\right)-\frac{2m\Omega\rho}{D},\qquad b_z=\frac{\mu\rho g_z r}{\sigma H}\left(1-\frac{2m\Omega g_r}{\sigma D r}\right),$$

$$d_r=\frac{m\kappa^2\rho}{2\Omega D}-\left(\sigma r-\frac{m\kappa^2 g_r}{2\Omega D}\right)\frac{\mu\rho g_r}{DH},\qquad d_z=-\left(\sigma r-\frac{m\kappa^2 g_r}{2\Omega D}\right)\frac{\mu\rho g_z}{\sigma^2 H},$$

$$f=-\frac{m^2\sigma\rho}{Dr}-\left(\sigma r-\frac{m\kappa^2 g_r}{2\Omega D}\right)\left(1-\frac{2m\Omega g_r}{D\sigma r}\right)\frac{\mu\rho}{H}+\frac{(\mu+1)\sigma r\rho}{c^2},$$

(Kojima et al., 1989)

Motivations

Why bother with linear calculation when we can just download a well-tested astrophysical fluids code and directly simulate (and generalize) the problem?

- Necessary by definition
- A reason to believe AFD codes
- Fun mathematics

Linear problem for 3D polytropic disks ($p \propto \rho^{1+1/n}$)

1. Steady, axisymmetric, vertically hydrostatic density bump at $r = r_0$
2. Perturb fluid equations, e.g. $\rho \to \rho + \delta\rho(r, z) \exp i(m\phi + \sigma t)$
3. Combine linear equations to get equation for $W \equiv \delta p/\rho$:

$$\boxed{L(r, z; \sigma)W = 0.}$$

- $W \to$ eigenfunction ; $\sigma \to$ eigenvalue
- Note: σ appears through $\bar{\sigma} = \sigma + m\Omega(r)$
- RWI: $\mathrm{Re}[\bar{\sigma}(r_0)] \simeq 0$ *and* $\left.\frac{d\eta}{dr}\right|_{r_0} \simeq 0$ ($\eta = \kappa^2/2\Omega\Sigma$ is the vortensity)

Very complicated PDE even for numerical work!

Application of orthogonal polynomials

$L(r, z; \sigma)$ only depends on z through $\rho(r, z)$. For thin polytropic disks:

$$\rho(r, z) = \rho_0(r) \left[1 - \frac{z^2}{H^2(r)}\right]^n .$$

In new co-ordinates $(R, Z) = (r, z/H)$,

$$\rho(r, z) = \rho_0(R) w(Z; n),$$

$$\boxed{w(Z; n) \equiv \left(1 - Z^2\right)^n}$$

Notice

$$\int_{-1}^{1} \mathcal{C}_k^\lambda(x) \mathcal{C}_l^\lambda(x) \left(1 - x^2\right)^{\lambda - 1/2} dx \propto \delta_{kl}.$$

$\mathcal{C}_l^\lambda(x)$ are Gegenbauer polynomials (generalization of Legendre and Chebyshev polynomials)
[Separate treatment for $n \to \infty$ (isothermal): Hermite polynomials]

PDE to ODEs

L has vertical dependence only through $w(Z;n)$.

Assume

$$W(R,Z)=\sum_{l=0}^{\infty}W_l(R)\mathcal{C}_l^{\lambda}(Z)$$

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L has vertical dependence only through $w(Z;n)$.

Assume

$$W(R,Z) = \sum_{l=0}^{\infty} W_l(R)\mathcal{C}_l^{\lambda}(Z)$$

Then

$$\int_{-1}^{1} L(W)\mathcal{C}_k^{\lambda}(Z)dZ = \int_{-1}^{1} L(W_l\mathcal{C}_l^{\lambda})\mathcal{C}_k^{\lambda}dZ = 0$$

involve terms like $\int_{-1}^{1}\mathcal{C}_l^{\lambda}\mathcal{C}_k^{\lambda}(1-Z^2)^{\lambda-1/2}dZ$, so apply orthogonality relation $\longrightarrow$

vertical dependence removed $\rightarrow$ ODEs

Solving ODEs

- Coupled set of ODEs

$$A_l(W_l) + B_l(W_{l-2}) + C_l(W_{l+2}) = 0,$$

  for $l = 0,\, 2,\, \ldots l_{\mathrm{max}}$.
- Matrix representations of operators, e.g. $A_l \to \mathbf{A}_l$
- Vector representations of solutions, $W_l \to \mathbf{W}_l$
- Matrix equation, e.g for $l_{\mathrm{max}} = 2$ is

$$\begin{bmatrix} \mathbf{A}_0 & \mathbf{C}_0 \\ \mathbf{B}_2 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{W}_0 \\ \mathbf{W}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

  or

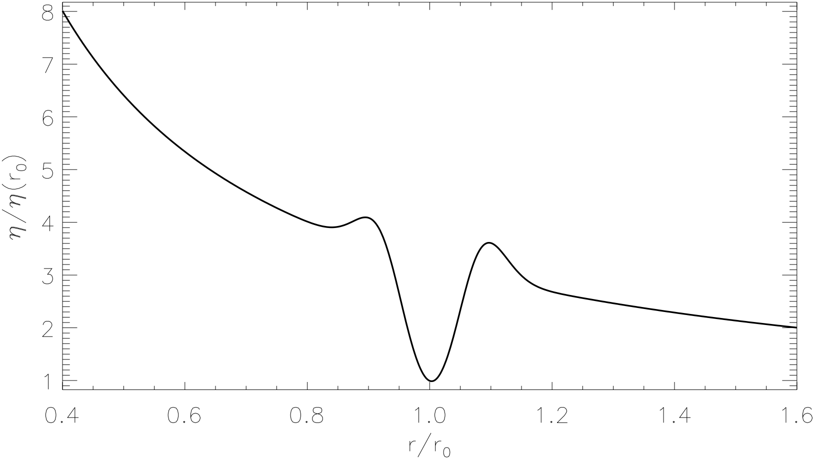
$$\mathbf{M}(\sigma)\mathbf{W} = \mathbf{0}$$

Summary:

PDE $\to$ ODEs $\to$ matrices $\to$ matrix solver $\to$ answer

Example problem

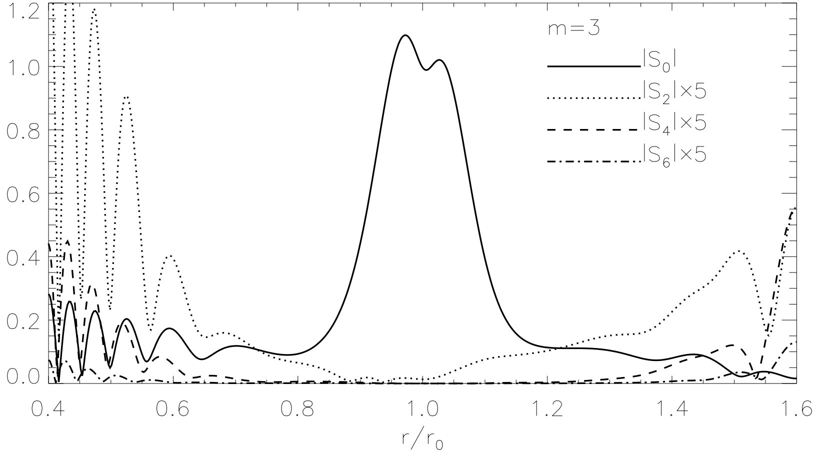
$n = 1.5$ polytrope with a surface density bump



Recall $\eta = \frac{1}{r\Sigma}\frac{d}{dr}\left(r^2\Omega\right)$ is the potential vorticity

Example solution

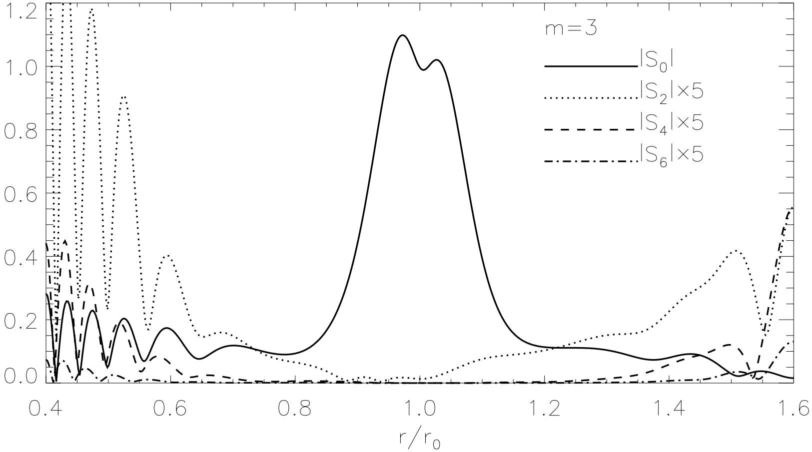
$$W(r,z) = W_0(r) + W_2(r)\mathcal{C}_2^{\lambda}(z/H) + \cdots$$



Growth rate $\sim 0.1\Omega$, same as 2D ($l_{\max} \equiv 0$). Instability is 2D.

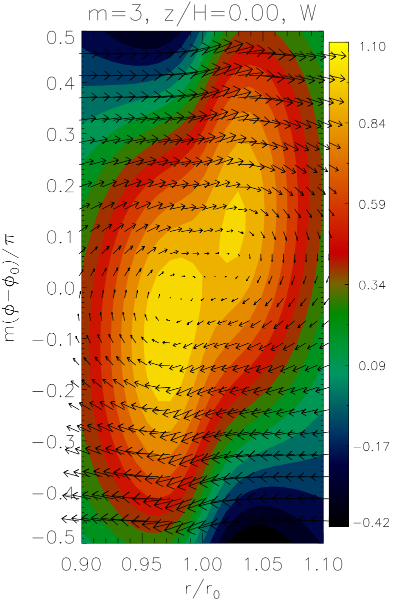
Example solution

$$W(r,z) = W_0(r) + W_2(r)\mathcal{C}_2^{\lambda}(z/H) + \cdots$$



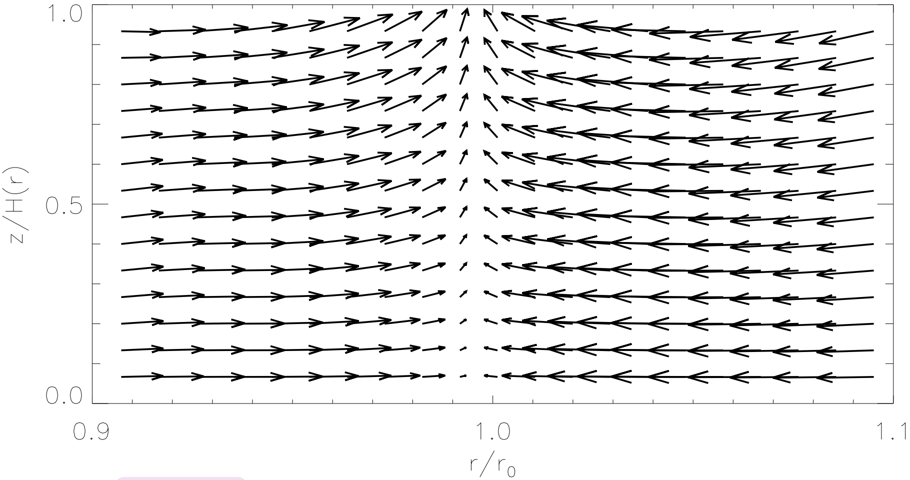
Note $\delta v_z = \mathrm{i}\left(\partial W/\partial z\right)/\bar{\sigma}$ but $|\bar{\sigma}| \sim 0$ at $r \sim r_0$

Horizontal flow



Anti-cyclonic motion associated with over-density

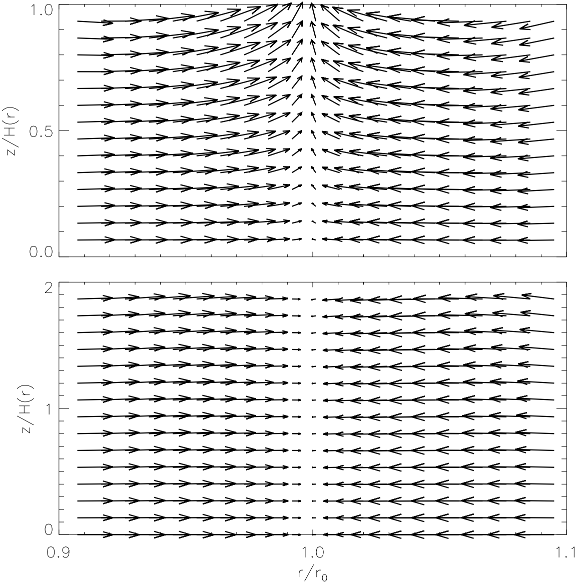
Vertical motion



Motion is upwards at (r_0, ϕ_0, z). Also seen in hydrodynamic simulations by Meheut et al. (2012).

Influence of equation of state

Magnitude of vertical motion decreases with increasing n (more compressible)



$\leftarrow$ $n = 1.0$ polytrope

$\leftarrow$ vertically isothermal disk ($n = \infty$)

The good, the bad, and the next step

GOOD

- Mathematical elegance (PDE$\rightarrow$ODEs done exactly)
- Solution in spectral space

BAD

- A lot of work
- No flexibility in vertical boundary conditions

Want a simpler method with freedom to impose vertical boundary conditions

Extension to adiabatic 3D disks

- $p \propto \rho^{\Gamma}$ in basic state only
- Energy equation $Ds/Dt = 0$, $s \equiv p/\rho^{\gamma} \propto \rho^{\Gamma-\gamma}$
- $\gamma \geq \Gamma \geq 1$, density bump $\rightarrow$ entropy dip

$$V_1 W + V_2 Q = 0$$
$$\bar{V}_1 W + \bar{V}_2 Q = 0$$

- $W = \delta p/\rho \rightarrow$ pressure perturbation
- $Q = c_s^2 \delta\rho/\rho \rightarrow$ density perturbation
- $S \equiv W - Q \rightarrow$ entropy perturbation

What should we look for?

$$\bar{S} \equiv Q - \frac{\gamma}{\Gamma} W = \left(1 - \frac{\gamma}{\Gamma}\right) W - S$$

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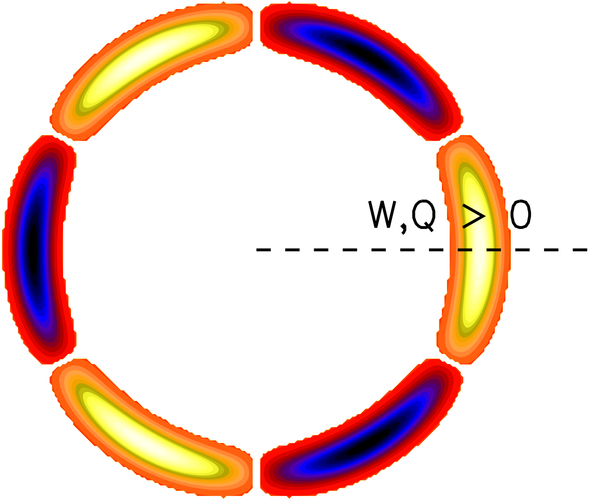

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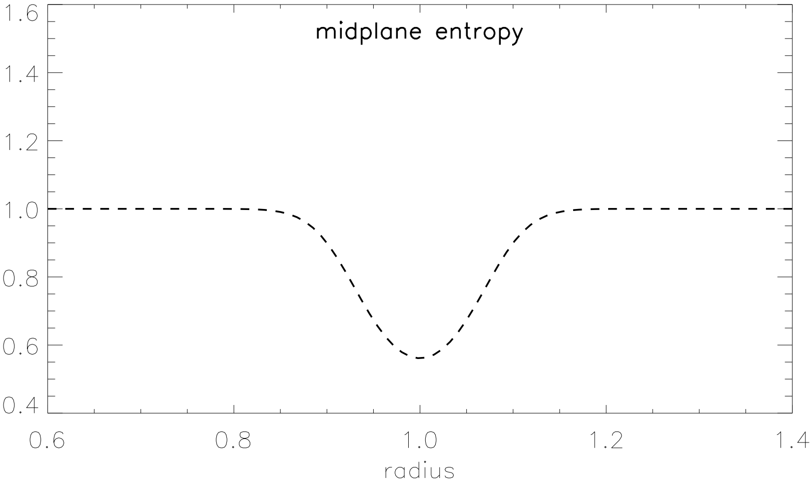

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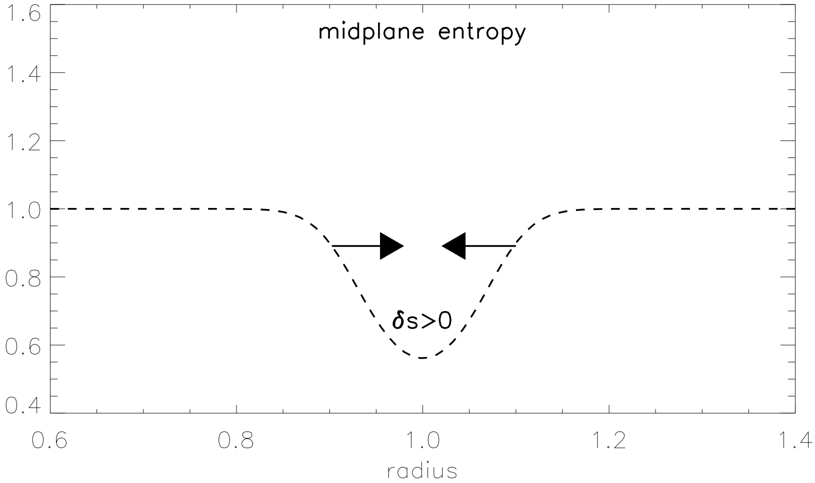

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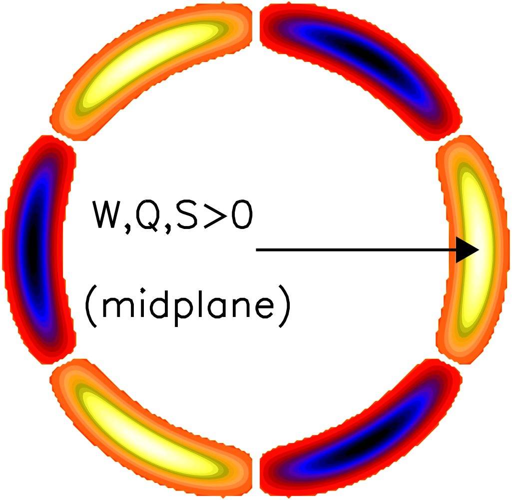

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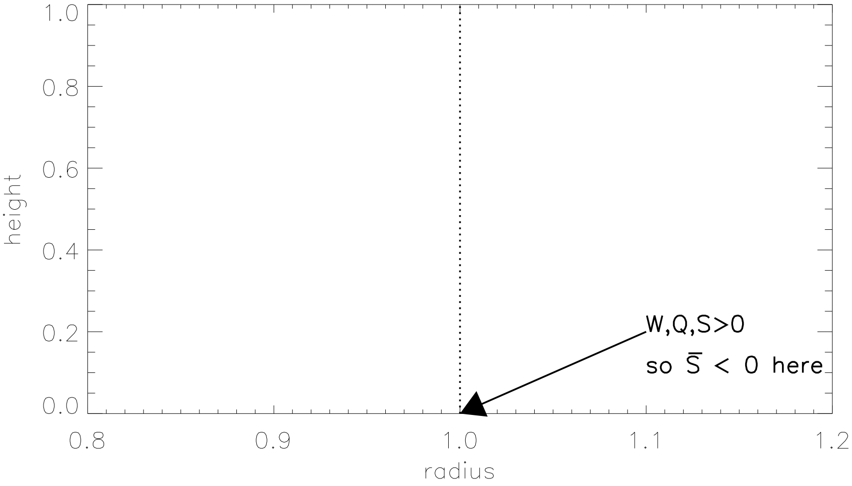

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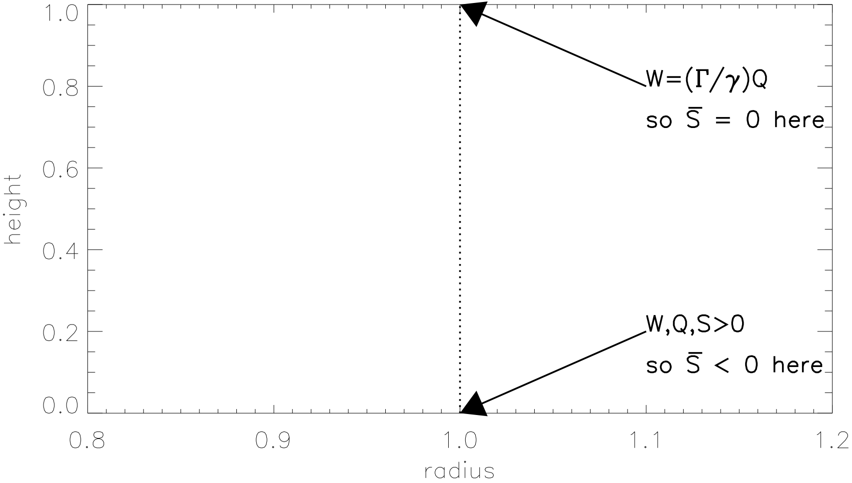

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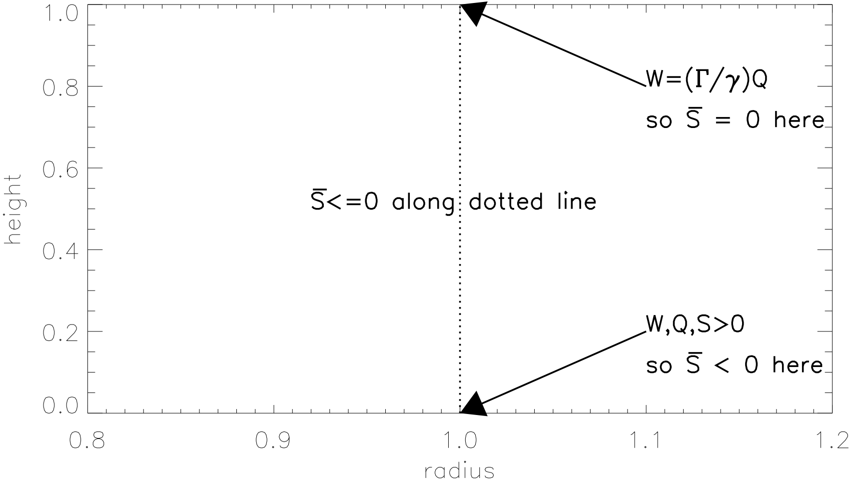

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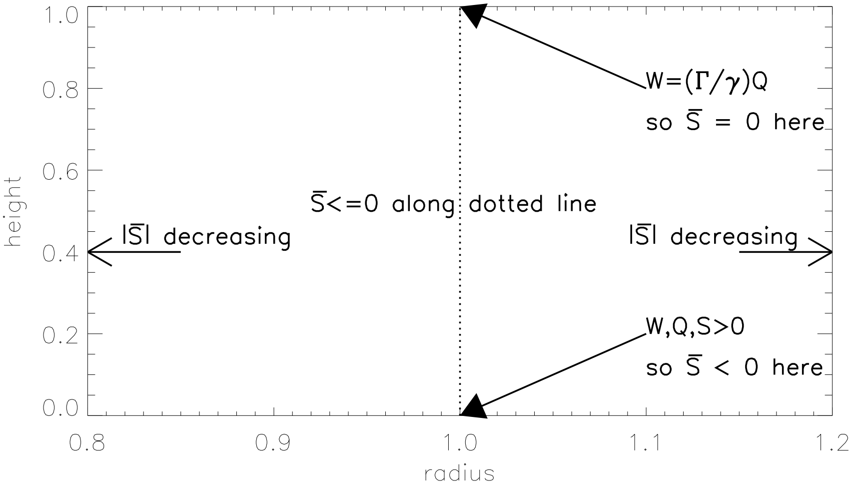
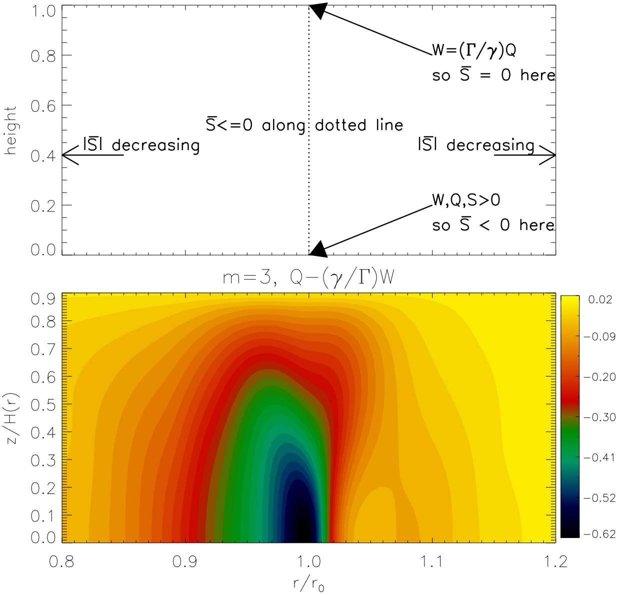


Expectation and reality



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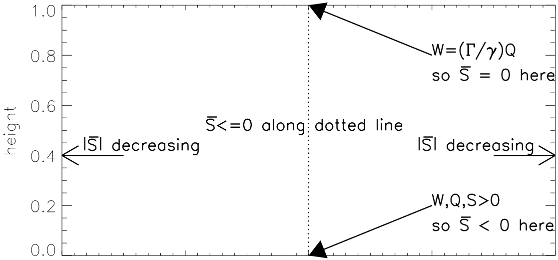
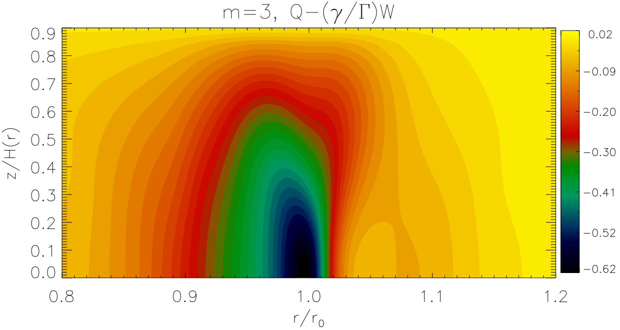
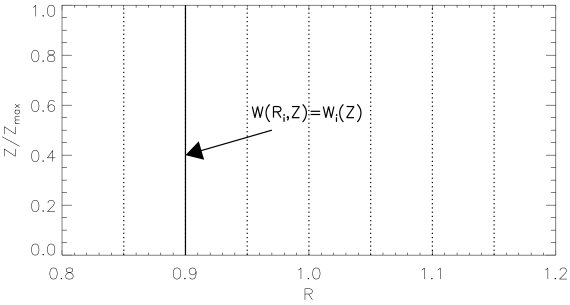




- $\bar{S} \Rightarrow \delta v_z$
- $\nabla \bar{S} \Rightarrow (\nabla \times \boldsymbol{\delta v})_\phi$

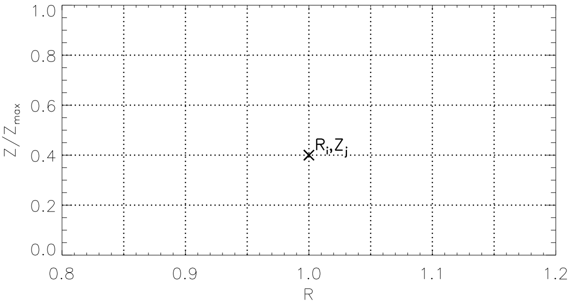
Pseudo-spectral method



$$W_i(Z) = \sum_{k=1}^{N_z} w_{ki}\psi_k(Z/Z_{\max})$$

- $\partial_R W \rightarrow$ finite diff.
- $\partial_Z W \rightarrow$ exact

Pseudo-spectral method

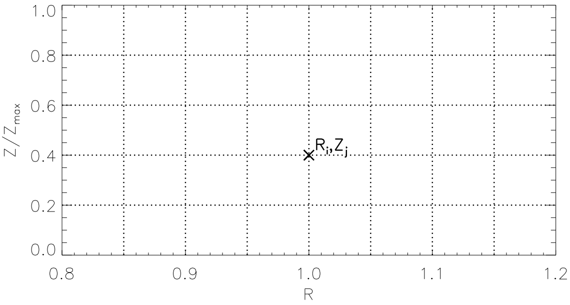


$$W_i(Z) = \sum_{k=1}^{N_z} w_{ki}\psi_k(Z/Z_{\max})$$

- $\partial_R W \longrightarrow$ finite diff.
- $\partial_Z W \longrightarrow$ exact
- evaluate PDE at (R_i, Z_j)

$$\left(V_1 W + V_2 Q\right)_{i,j} = 0$$
$$\left(\bar{V}_1 W + \bar{V}_2 Q\right)_{i,j} = 0$$

Pseudo-spectral method



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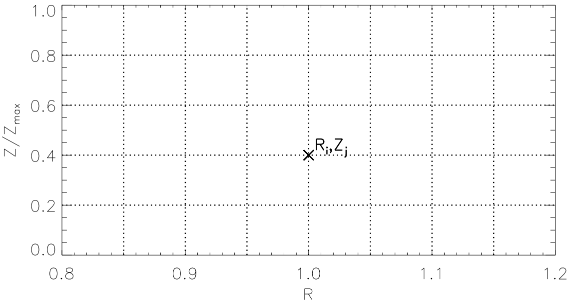
- evaluate PDE at (R_i, Z_j)

$$\mathbf{V}_1\mathbf{w} + \mathbf{V}_2\mathbf{q} = \mathbf{0}$$
$$\bar{\mathbf{V}}_1\mathbf{w} + \bar{\mathbf{V}}_2\mathbf{q} = \mathbf{0}$$

Pseudo-spectral method



$$W_i(Z) = \sum_{k=1}^{N_z} w_{ki}\psi_k(Z/Z_{\max})$$

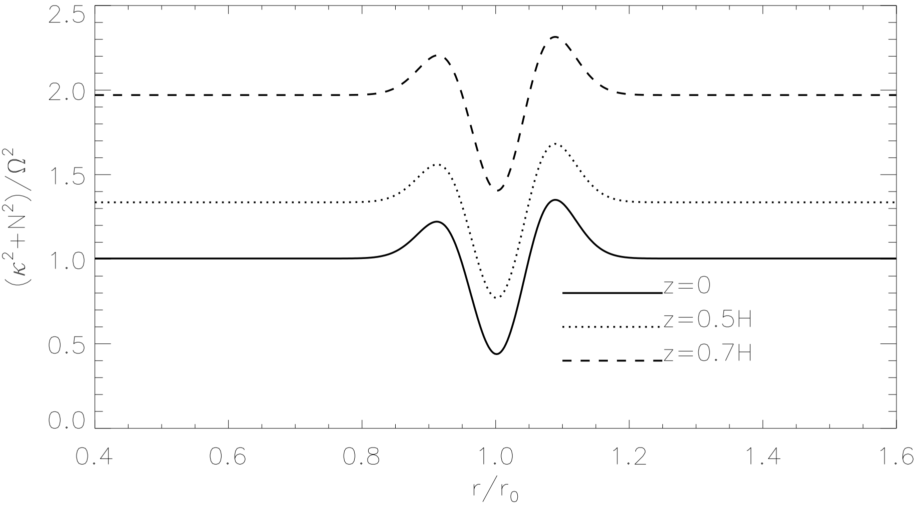
- $\partial_R W \rightarrow$ finite diff.
- $\partial_Z W \rightarrow$ exact
- evaluate PDE at (R_i, Z_j)

$$\mathbf{U}(\sigma)\mathbf{w} = \mathbf{0}$$

- $\mathbf{U} \rightarrow$ matrix representation of PDE operator
- $\mathbf{w} \rightarrow$ vector to store the w_{ki}
- Vertical boundary condition: $\Delta P = 0$, $\delta v_z = 0$ or $\delta v_\perp = 0$ at $Z = Z_{\max}$

Non-homentropic example

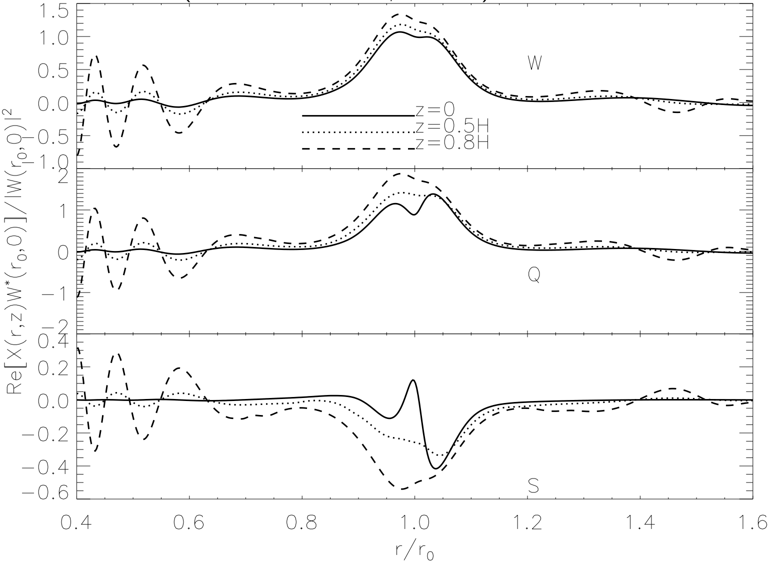
$\Gamma = 1.67$, $\gamma = 2.5$



N is the buoyancy frequency

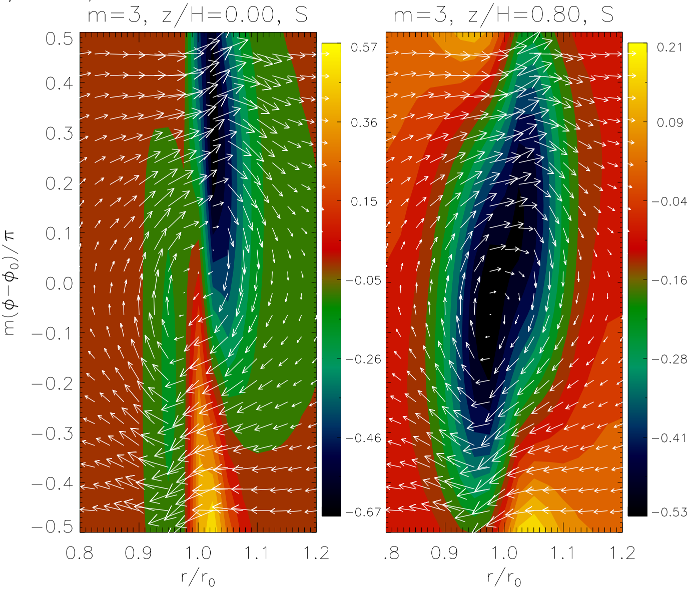
Non-homentropic example

$\Gamma = 1.67$, $\gamma = 2.5$, $m = 3$ along $\phi = \phi_0$.
Growth rate $0.1099\Omega_0$ (cf. $0.1074\Omega_0$ for $\gamma = 1.67$)



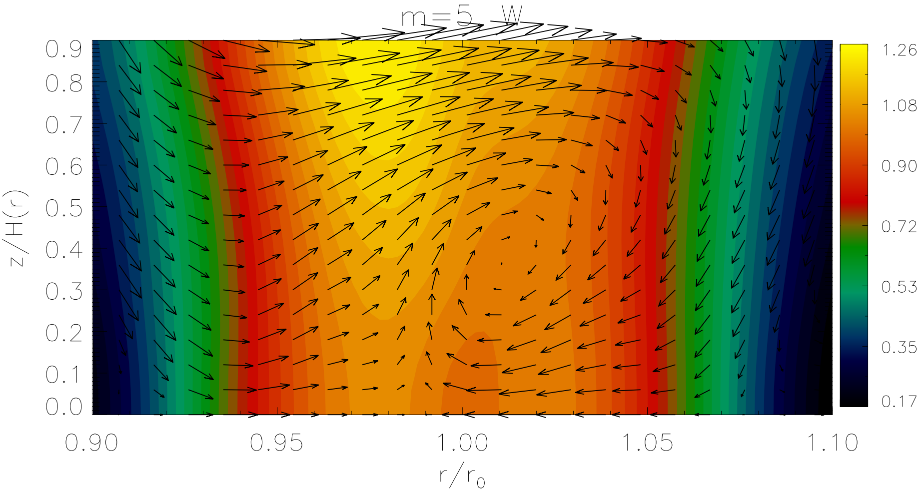
Entropy perturbation

$\Gamma = 1.67$, $\gamma = 2.5$, $m = 3$



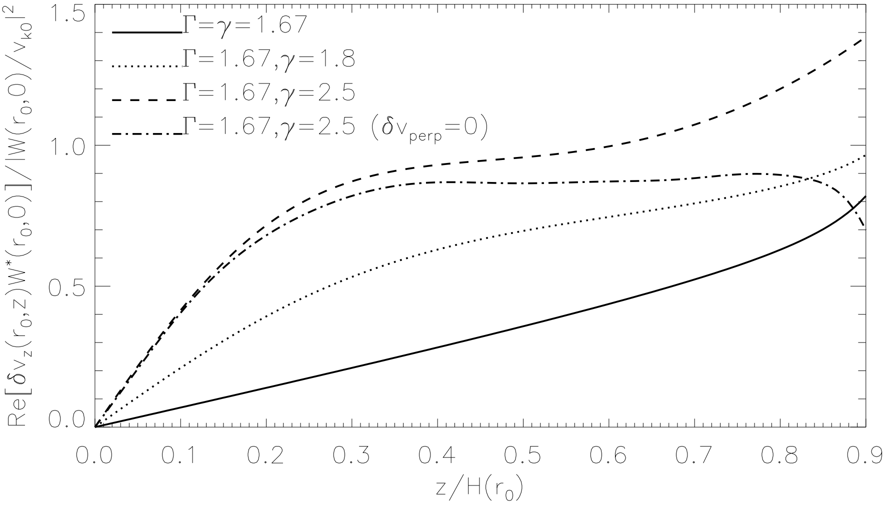
Meridional vortical motion

$\Gamma = 1.67$, $\gamma = 2.5$, $m = 5$ along $\phi = \phi_0$



Vertical motion

Fix $\Gamma = 1.67$, vary γ, plot δv_z along (r_0, ϕ_0, z).



Vertical motion

Kato (2001):

$$\delta v_z \sim -\frac{\nu}{N_z^2}\frac{\partial W}{\partial z} - \nu\rho\left(\frac{\partial p}{\partial z}\right)^{-1} W, \quad N_z^2 \neq 0$$

at co-rotation radius, and ν here is the growth rate. Compared to

$$\delta v_z \sim -\frac{1}{\nu}\frac{\partial W}{\partial z}, \quad N_z^2 \equiv 0.$$

Notice for $N_z^2 \neq 0$

$$\frac{\text{pressure}}{\text{buoyancy}} \sim \frac{\Omega^2}{N_z^2}\frac{\partial \ln W}{\partial \ln z},$$

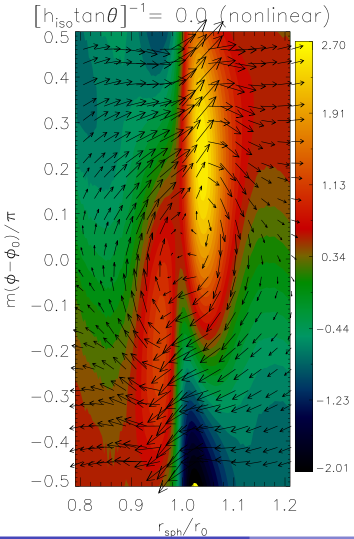
i.e. buoyancy dominates at large z as N_z^2 increases with height.

Origin of δv_z is different between homentropic and non-homentropic flow

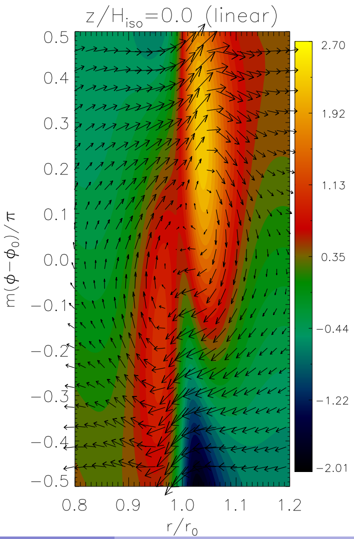
Comparison with hydrodynamic simulations

- Isothermal disk, adiabatic evolution ($\Gamma \equiv 1$, $\gamma = 1.4$)

ZEUS simulation



Linear code



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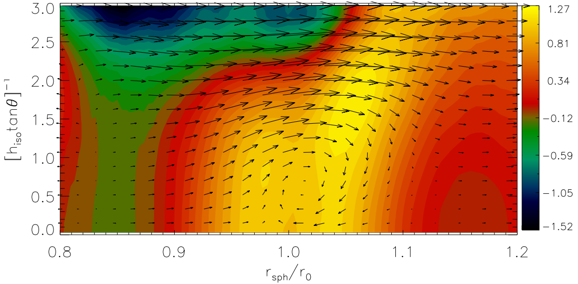
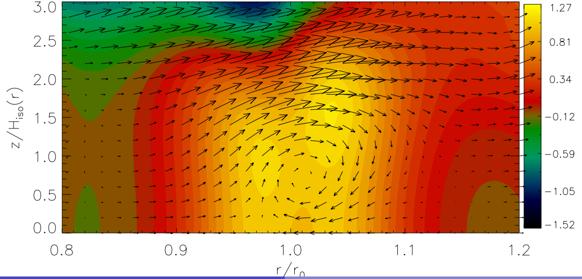


← ZEUS simulation

$\mathrm{Re}(\sigma) = -0.99m\Omega_0$

$\mathrm{Im}(\sigma) = -0.194\Omega_0$



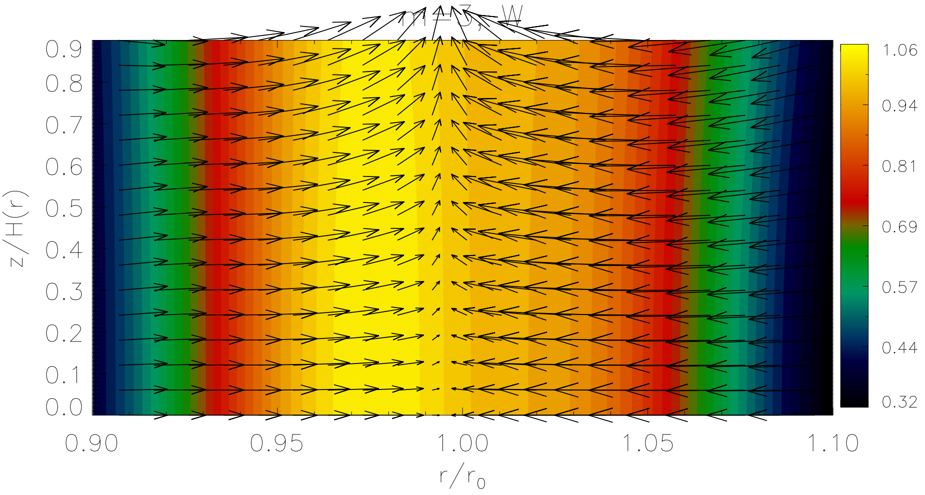
← linear code

$\mathrm{Re}(\sigma) = -0.9896m\Omega_0$

$\mathrm{Im}(\sigma) = -0.1937\Omega_0$

Upcoming

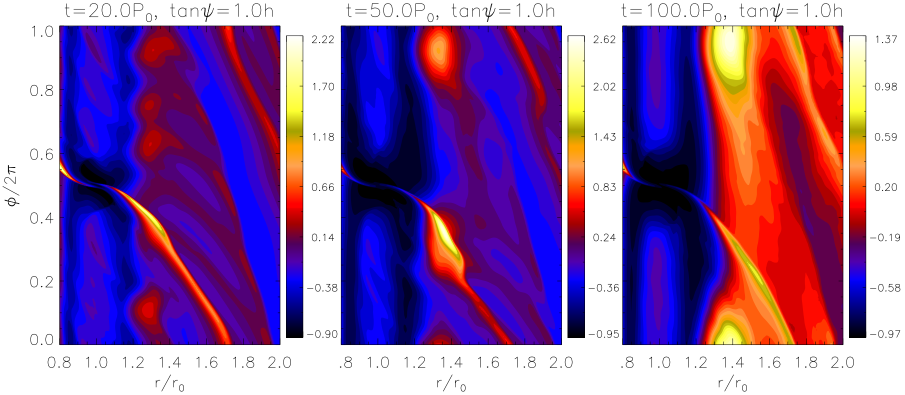
- Vortex-formation in layered-accretion disks?



Upcoming

- Vortex-formation in layered-accretion disks?

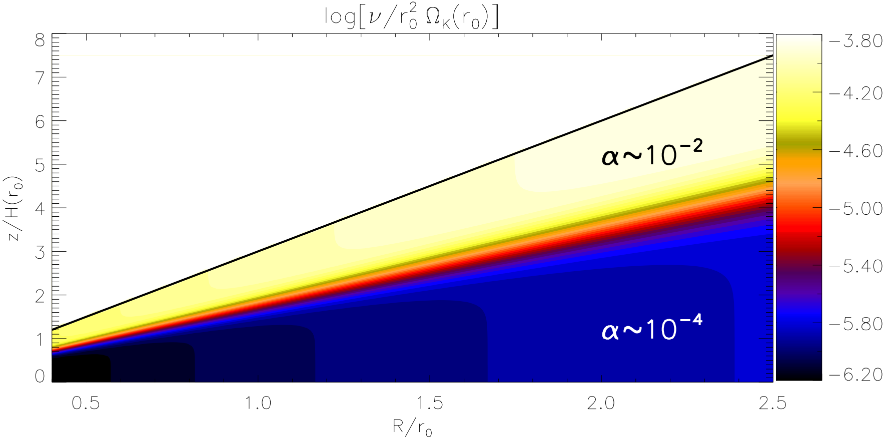
↓ PLUTO experiments ↓



Imposed viscosity $\alpha \sim 10^{-4}$ everywhere

Upcoming

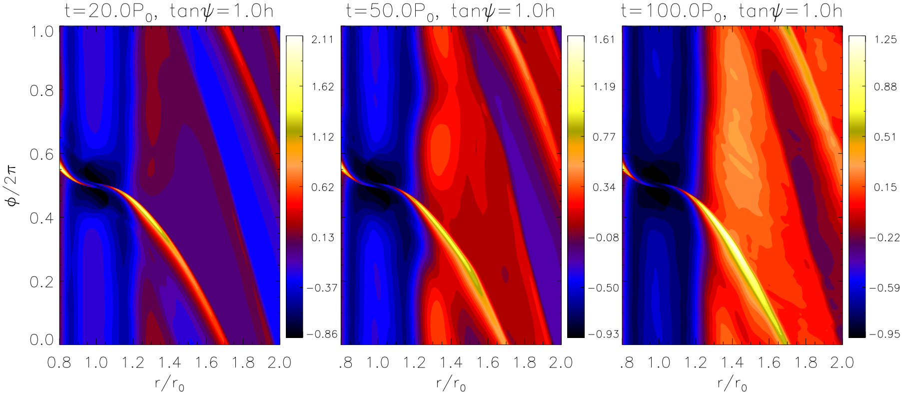
- Vortex-formation in layered-accretion disks?



Upcoming

- Vortex-formation in layered-accretion disks?

↓ PLUTO experiments ↓



$\alpha \sim 10^{-4}$ in bulk of the disk, $\alpha \sim 10^{-2}$ in atmosphere

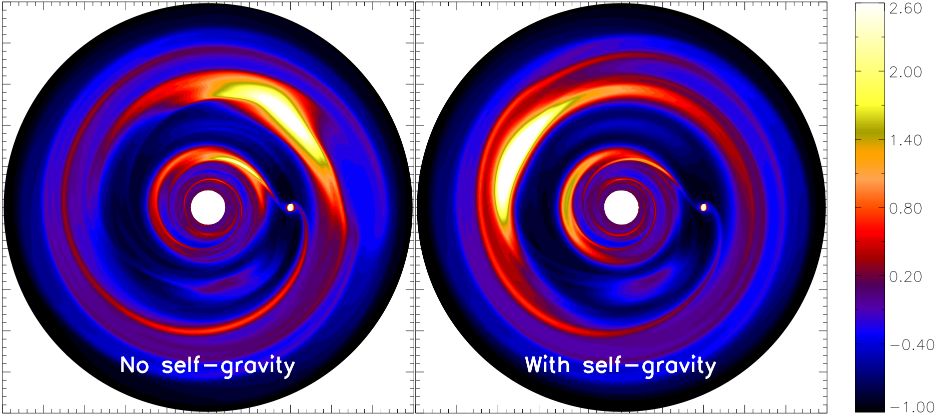
Self-gravity

- Vortensity and Toomre parameter are related:

$$\eta \equiv \kappa^2/2\Omega\Sigma, \quad \boxed{Q_T = \kappa c_s/\pi G\Sigma = (c_s/\pi G)\sqrt{2\Omega\eta/\Sigma}}$$

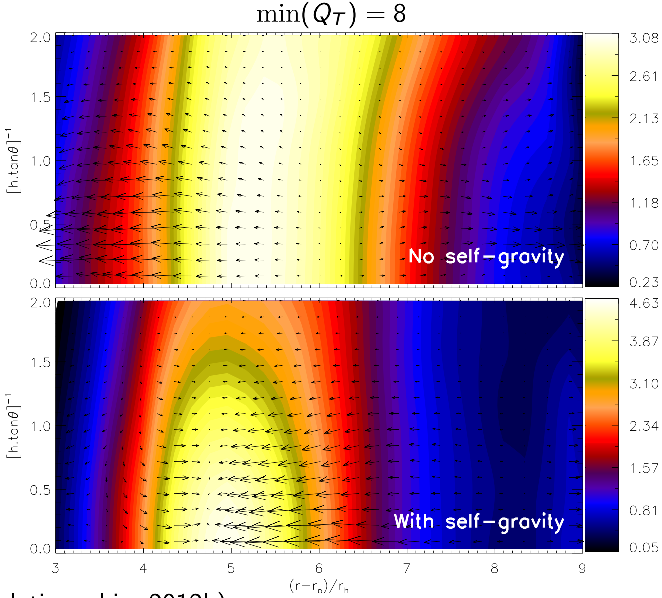
- SG *stabilizes* RWI (Lin & Papaloizou, 2011a)

$\downarrow \min(Q_T) = 8 \downarrow$



(ZEUS simulations, Lin, 2012b)

Self-gravity



(ZEUS simulations, Lin, 2012b)

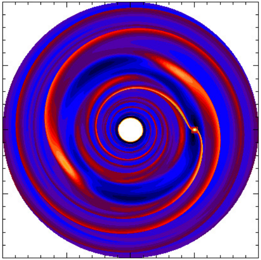
Future

Linear problem:

- Baroclinic equilibria, $\partial_z \Omega \neq 0$
- Vertical self-gravity

Numerical simulations:

- Vortex evolution in 3D self-gravitating disks
- Gravitational instabilities associated with disk structure
- ZEUS / PLUTO are MHD codes
  $\rightarrow$ magneto-gravitational instabilities



(FARGO simulation, Lin & Papaloizou, 2011b)

Application to other origins of disk structures

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