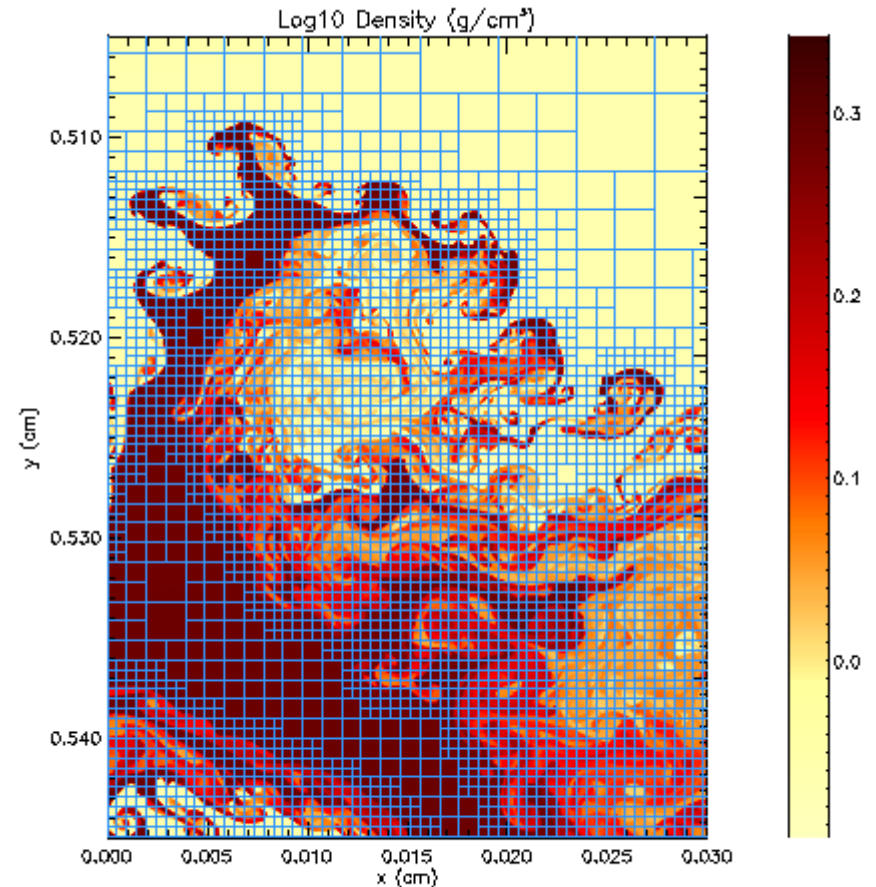


Lagrangian methods and Smoothed Particle Hydrodynamics (SPH)

Eulerian Grid Methods

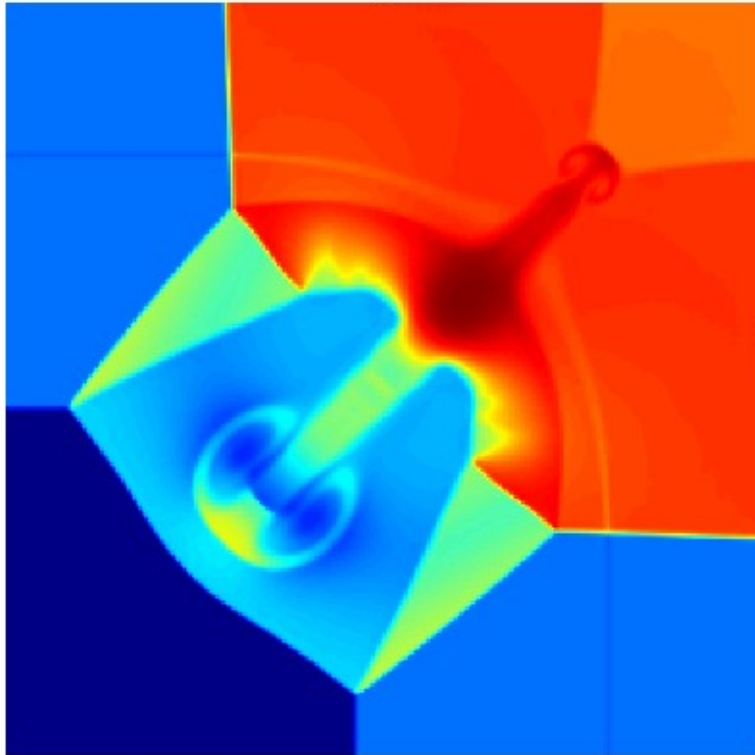
The methods covered so far in this course use an Eulerian grid:

- Prescribed coordinates
- In 'lab frame'
- Fluid elements flow through grid zones



time = 1.950 s
number of blocks = 224756
AMR levels = 10

Eulerian Grid Methods



This is probably the standard approach to solving the equations of fluid motions in most disciplines.

Many decades of research into solution techniques

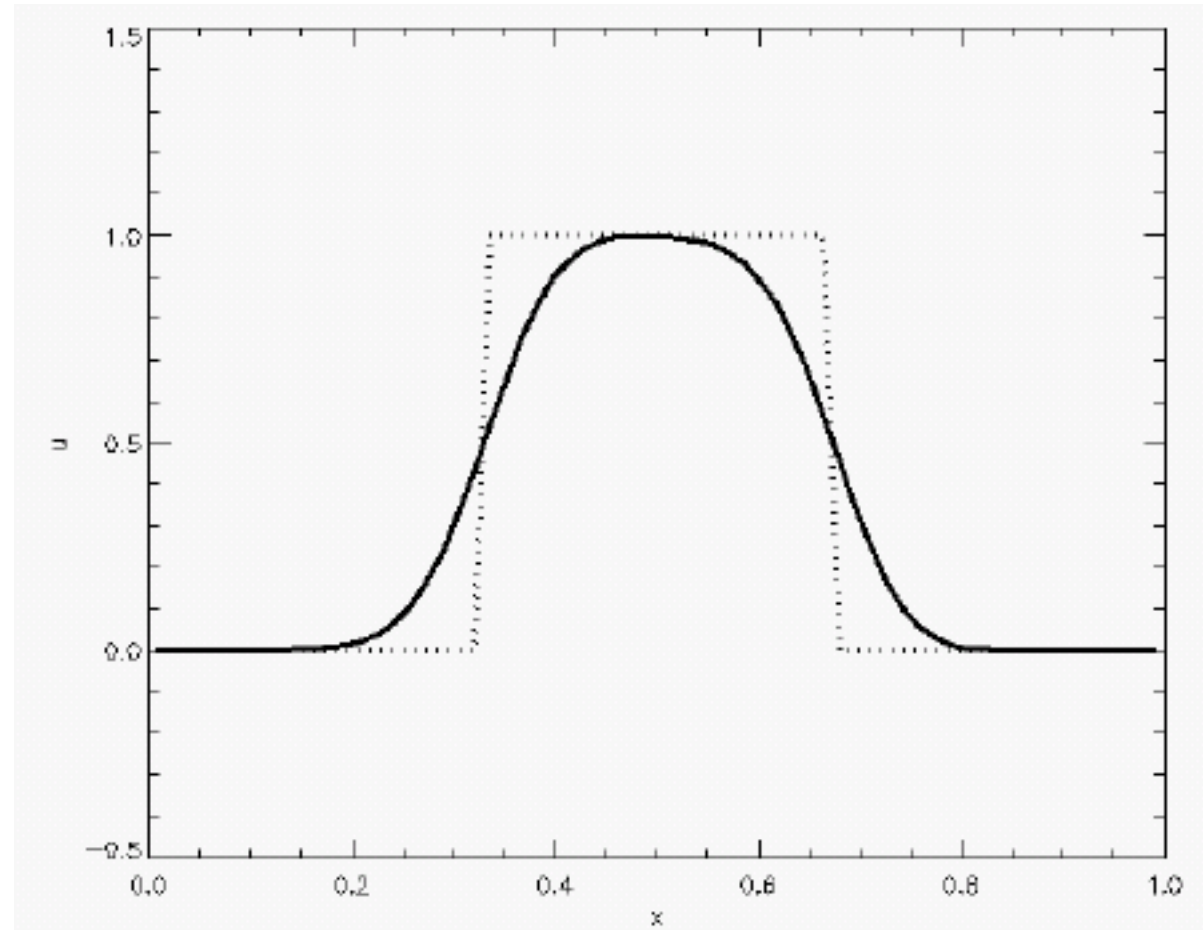
Extremely sophisticated, high-order accuracy methods

Can accurately describe very complex phenomena.

Eulerian Grid Methods

However, simplest possible dynamics fares surprisingly poorly

Even high-order methods are quite diffusive when advection over a large number of grid cells is necessary



Eulerian Grid Methods

This is fundamental to how Eulerian grid codes work.

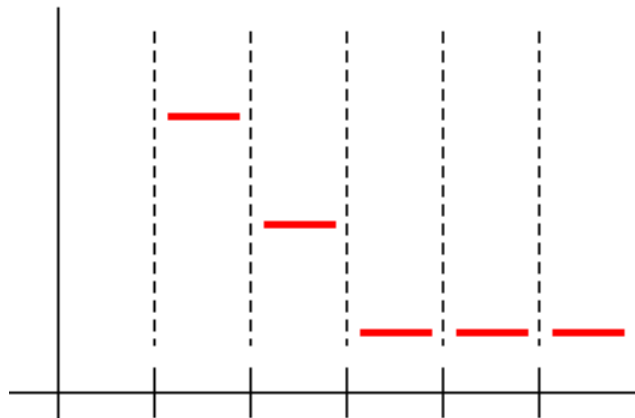
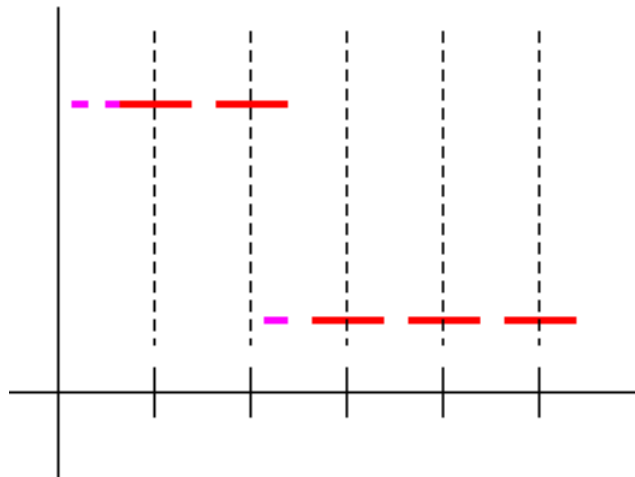
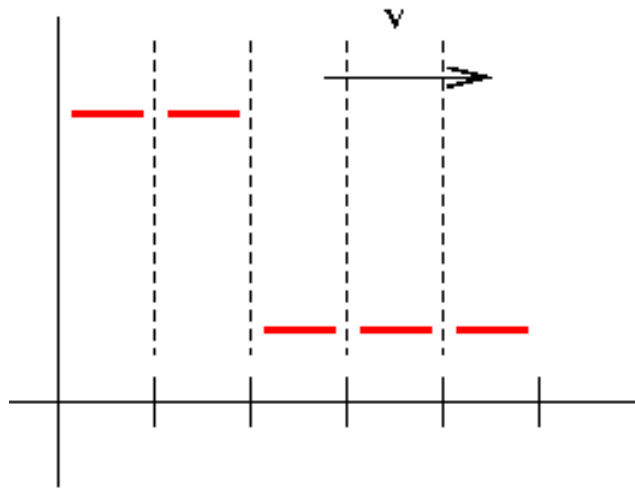
Can be ameliorated but not fixed.

Once some of a quantity enters a grid cell its contribution is spread throughout domain through some averaging procedure.

Higher order methods do this to a lesser degree than lower-order methods, but the effect remains.

Occurs for any evolved quantity.

``Numerical diffusion''

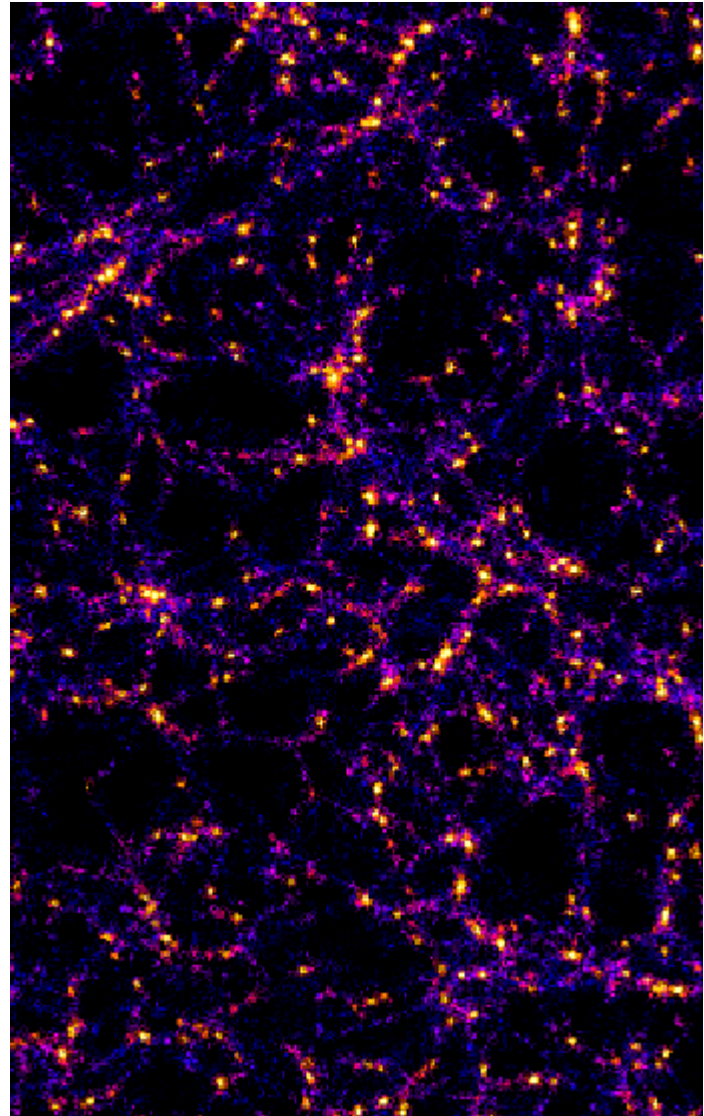


Advection-Dominated Flows

There are many systems in astrophysics which are dominated by large-scale advection of fluid

Eulerian grid is not necessarily the most natural approach in these systems

Cosmology: evolution is dominated by large scale falling of material onto local density enhancements

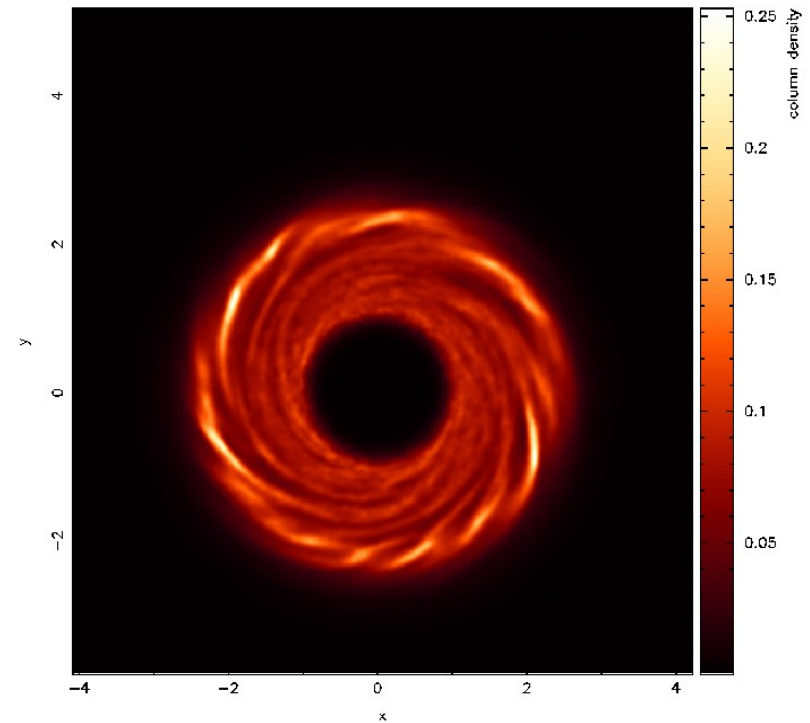


Advection-Dominated Flows

There are many systems in astrophysics which are dominated by large-scale advection of fluid

Eulerian grid is not necessarily the most natural approach in these systems

Accretion disks: Flow is dominated by differential Keplerian rotation around central object

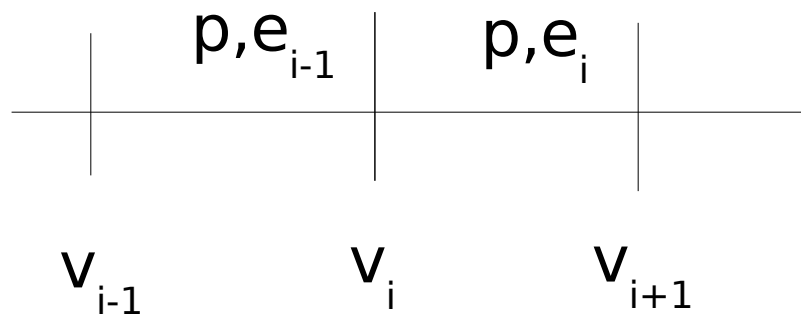


1D Lagrangian Formulation

Lagrangian formulation
given in lecture 4

No fluxes 'through' fluid
element interfaces, as
no transport through
interfaces*

Typically implemented on
a staggered grid:



$$\frac{D\rho}{Dt} = -\rho \frac{\partial u}{\partial x}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x}$$

$$\rho \frac{De}{Dt} = -p \frac{\partial u}{\partial x}$$

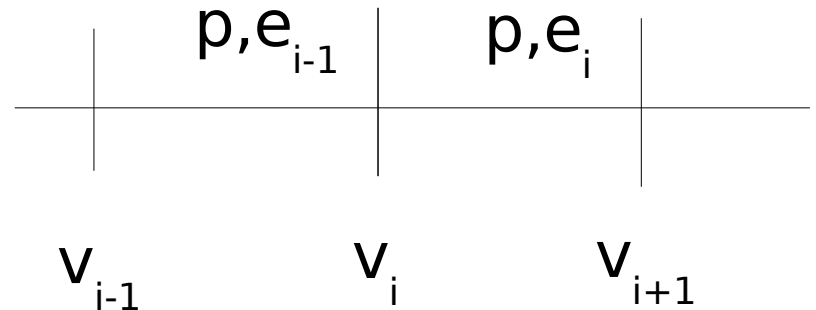
No purely advective fluxes!

*absent other physics like dissipative transport

1D Lagrangian Formulation

Each lagrangian cell has a fixed mass; density is just mass over (evolved) volume element

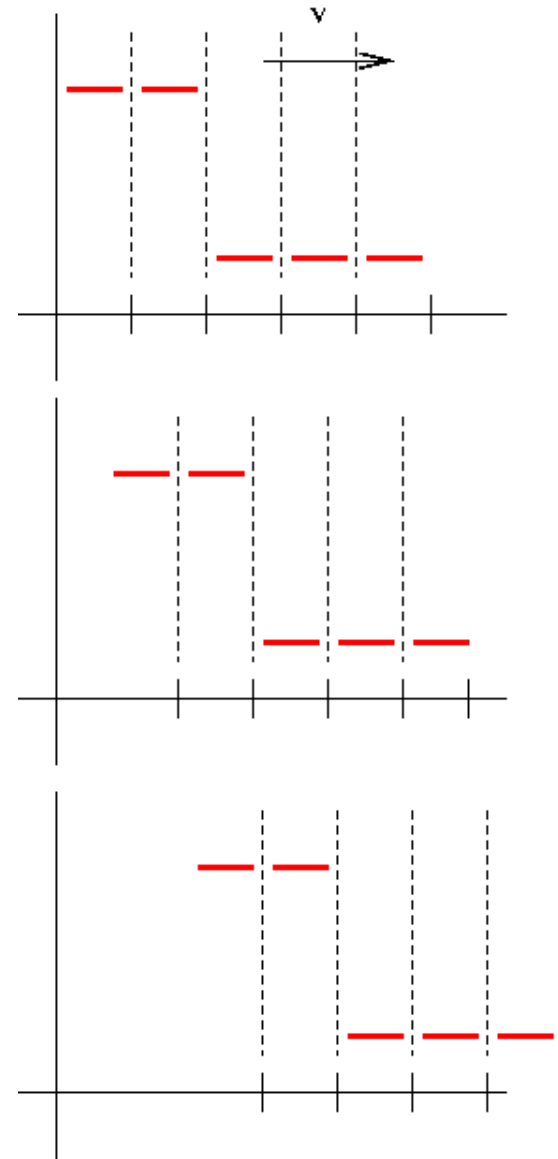
Guaranteed conservation of mass; shared cell edges guarantee conservation of volume, energy



1D Lagrangian Formulation

Huge benefit: open boundaries – mesh can expand as necessary

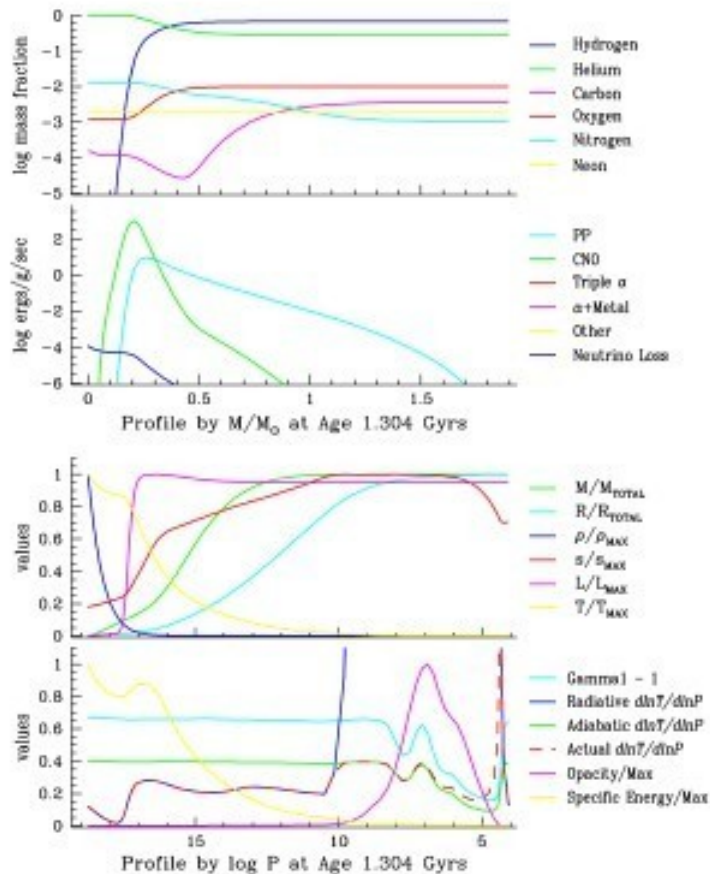
And, of course, no numerical diffusion from purely moving fluid around



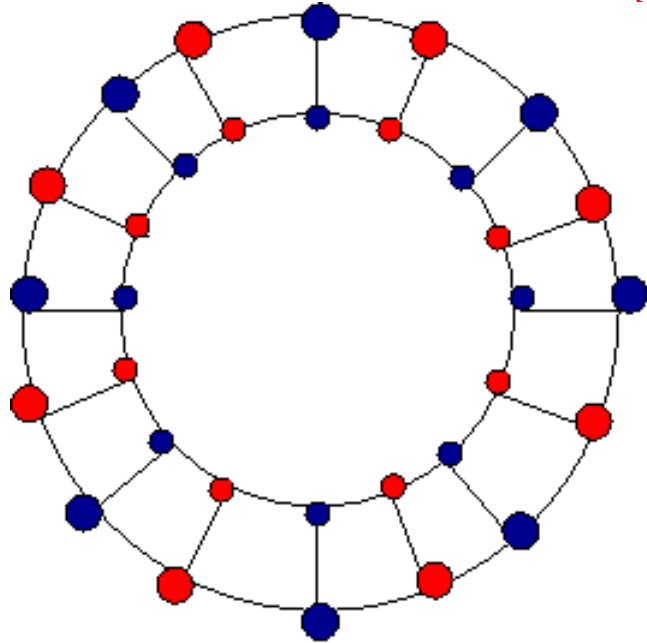
1D Lagrangian Formulation

These advantages make 1d lagrangian grid methods a natural choice for applications such as stellar evolution

Typically use 'mass coordinates' even outside of numerical context



Multi-D Lagrangian Gridding

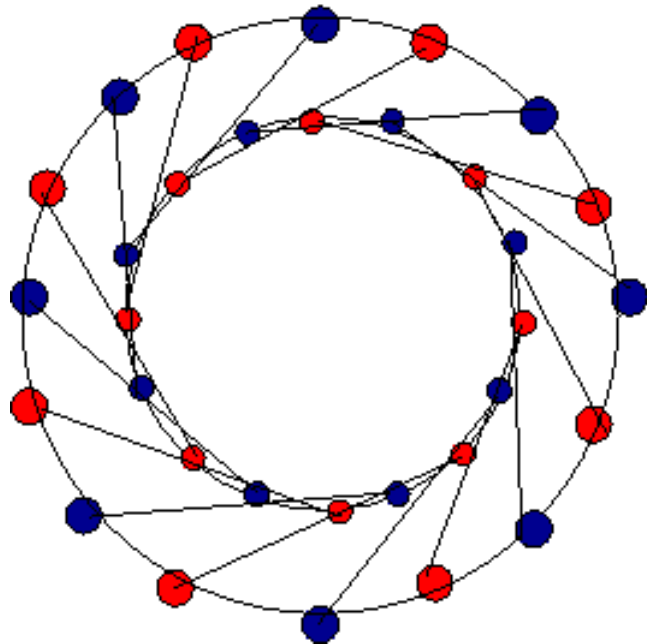


Works extremely well in 1d.

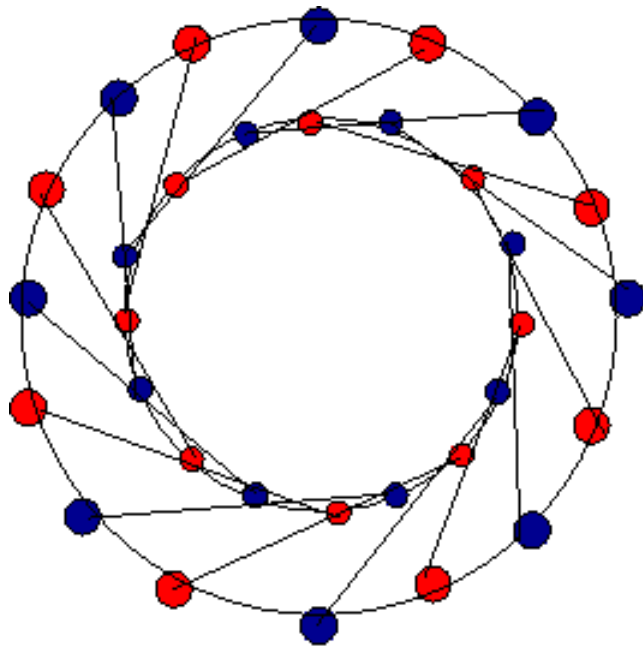
In multidimensions, more complexity possible in geometry

Even differential rotation / shear can lead to disastrously tangled meshes.

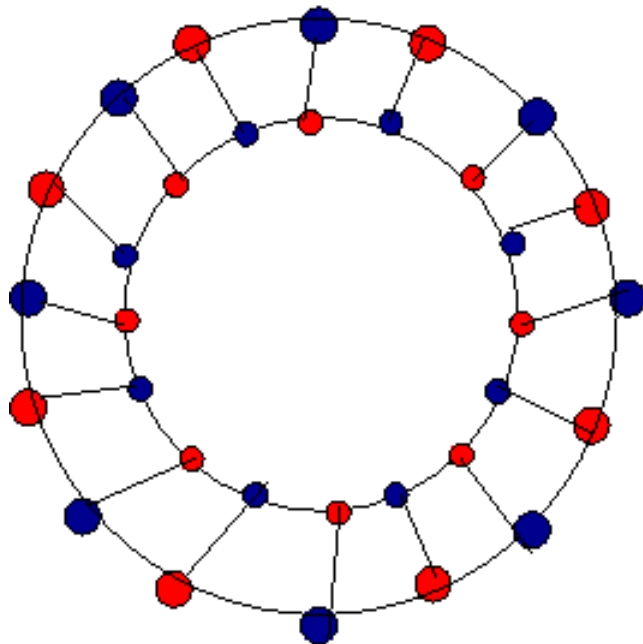
More complex motions almost hopeless



Multi-D Lagrangian Gridding



Remesh



Can deal with this problem by remeshing every so often

Remeshing can be a very expensive step (choosing an optimal new mesh for a set of points is difficult)

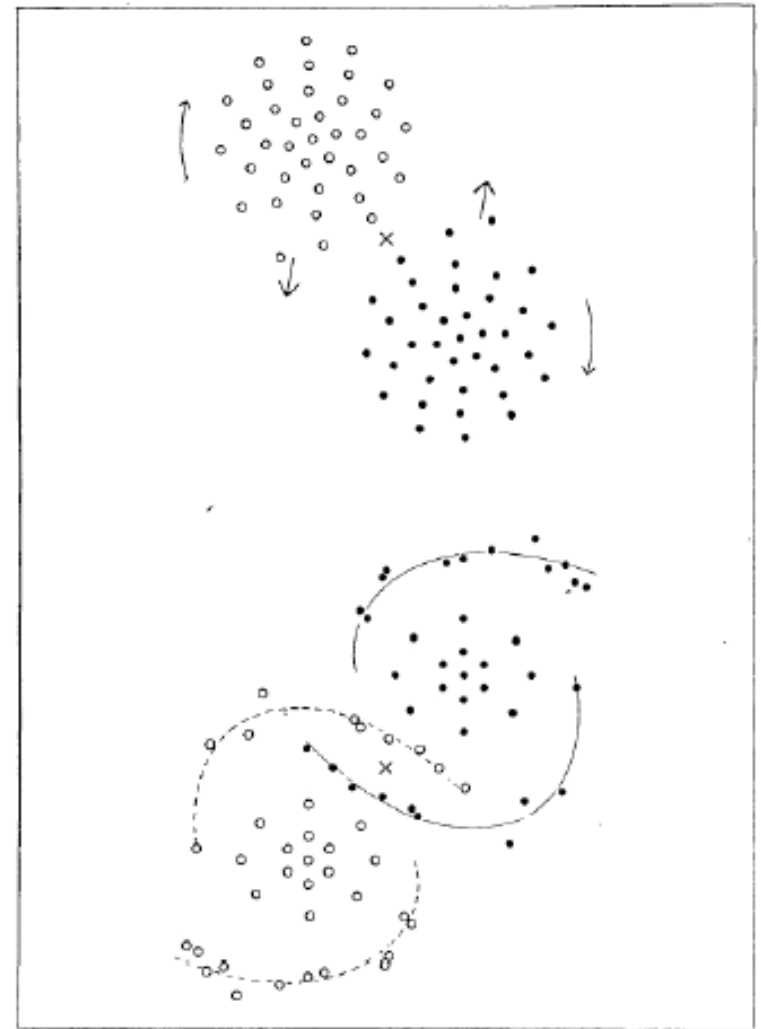
Loss of main benefit of Lagrangian method – diffusive (as fluid 'moves through' remeshing)

Gridless methods

Grid is a way of assigning neighbors to structure local interactions.

If can determine local neighbors without discretizing on a grid can avoid the issues with tangling/remapping

Astrophysics has a long (>60 yr) history with one gridless method – gravitational N-body calculations



Holmberg (1941)

FIG. 4a

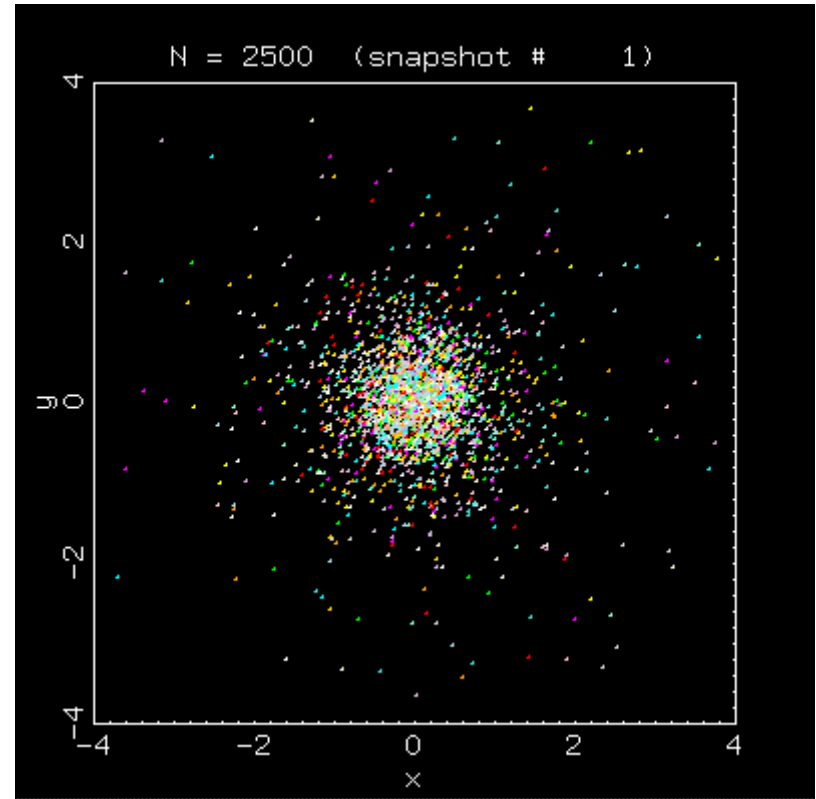
N-body calculation

It is clearly true that one can write the density field in a domain as integral over infinite number of point particles:

$$\rho(\vec{r}) = \frac{1}{V} \int_V \delta(\vec{r}' - \vec{r}) dm$$

N-body calculation approximates this by using a finite number of particles

$$\rho(\vec{r}) \approx \frac{1}{V} \sum_{i: \vec{r}_i \in V} m_i$$



<http://www.physics.drexel.edu/~steve/n-body.html>

Smoothed Particle Hydrodynamics

For hydrodynamics, interactions need to be local

Quantities stored at N 'free' particles

If infinite number of particles, any hydrodynamic quantity A could be defined as

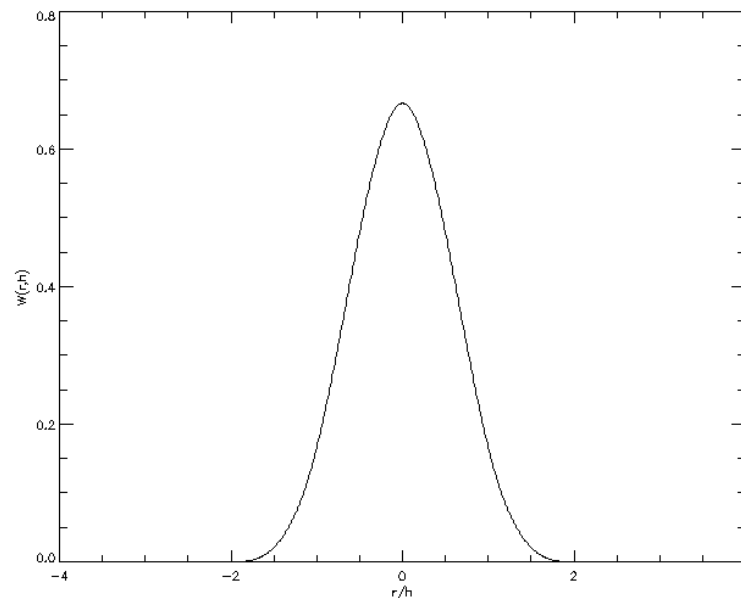
$$A(\vec{r}) = \int A(\vec{r}') \delta(\vec{r} - \vec{r}') dV$$

Finite number of particles – quantities must be smoothed over some finite smoothing length h

$$A(\vec{r}) = \int A(\vec{r}') W(|\vec{r} - \vec{r}'|, h) dV$$

Properties of smoothing function $W(r,h)$

- In small h limit, goes to delta function
- (usually) symmetric about $r=0$
- Compact support – W is exactly zero outside of some finite radius around the particle
- Cubic spline is typical choice



SPH `Discretization' Error

$$A(\vec{r}) = \int A(\vec{r}') W(|\vec{r} - \vec{r}'|, h) dV$$

Even in this limit, smoothed quantity has error $O(h^2)$ [Why?]

Very difficult (likely impossible) to have robust, stable smoothing with higher order accuracy.

With finite number of particles,

$$A(\vec{r}) \approx \sum_b A_b \frac{m_b}{\rho_b} W(|\vec{r} - \vec{r}_b|, h)$$

SPH 'Discretization' Error

$$A(\vec{r}) \approx \sum_b A_b \frac{m_b}{\rho_b} W(|\vec{r} - \vec{r}_b|, h)$$

Taylor expand this around particle 'a's position, and define $W_{ab} = W(r_a - r_b, h)$

$$A(r) = A_a \sum_b \frac{m_b}{\rho_b} W_{ab} + \nabla A_a \sum_b \frac{m_b}{\rho_b} \vec{r}_{ab} W_{ab} + \dots$$

Even for constant function (say, $A = 1$), not guaranteed exact; must divide by first term, eg do SPH interpolation of A / SPH interpolation of 1.

SPH `Discretization' Error

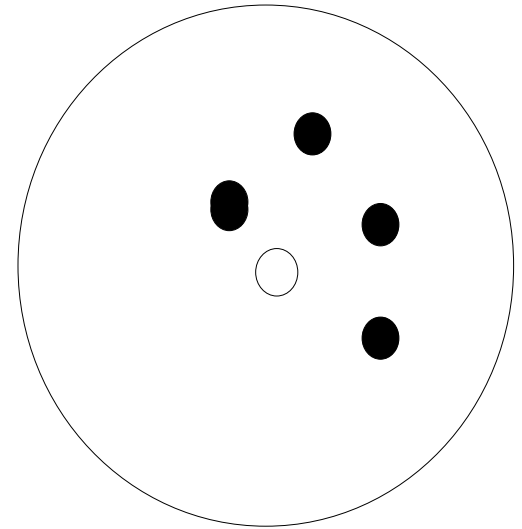
Error comes from discretization –
finite number of particles

No guarantee that there will be
enough particles well enough
distributed so that

$$\sum_b \frac{m_b}{\rho_b} W_{ab} = 1$$

although corrections can be made

This problem is much worse for
derivatives; numerical
derivatives of `noisy' data known
hard problem.



Equation of Continuity

Mass conservation is already guaranteed;
each particle has its own mass, which
doesn't change

Density has particularly simple form in SPH
summation interpolant

$$\rho_a = \sum_b m_b W_{ab}$$

Taking lagrangian time derivative results in

$$\frac{d\rho_a}{dt} = \sum_b m_b (\vec{v}_a - \vec{v}_b) \cdot \nabla_a W_{ab}$$

Lagrangian Formulation

Can now express other equations in terms of Lagrangian, Hamiltonian dynamics

Lagrangian for Hydrodynamics is

$$L = \int \left(\frac{1}{2} \rho v^2 - \rho u \right) dV$$

$$L = \sum_b m_b \left(\frac{1}{2} v_b^2 - u_b(\rho_b, s_b) \right) dV$$

from Euler-Lagrange equations, get eqn of motion

$$\frac{d\vec{v}_a}{dt} = - \sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} \right) \nabla_a W_{ab}$$

Momentum Equation

$$\frac{d\vec{v}_a}{dt} = - \sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} \right) \nabla_a W_{ab}$$

Note symmetry; contribution to momentum of particle a from b equal and opposite to b from a

Conserves momentum exactly

This form of gradient of pressure has some discreteness inaccuracies, but the symmetry is much more important

Energy Equation

Similarly, Hamiltonian can be written

$$H = \sum_b m_b \left(\frac{1}{2} v_a^2 + u_a \right)$$

implying a specific energy equation

$$\frac{de_a}{dt} = \sum_b m_b \left(\frac{P_a}{\rho_a^2} \vec{v}_b + \frac{P_b}{\rho_b^2} \vec{v}_a \right) \cdot \nabla_a W_{ab}$$

again, note symmetry.

Because of troubles with internal energy evolution in high-speed flows, some SPH practitioners evolve entropy rather than energy; applies to grid codes too.

Setting Initial Conditions in an SPH code

Unlike grid code (ICs are set everywhere in domain), have to sample (typically randomly) the density profile and put particles there

If under-resolving the density profile is a problem, some caution is necessary; fluctuations caused by particle placement can be a problem. May have to relax initial conditions.

Cosmology: extreme care needed --- any initial fluctuations in density become large scale structure! Specialized techniques.

Time Evolution

Many approaches possible, typically some variant on 'leapfrog' predictor-corrector step taken (2nd order in time, very efficient):

- At old timestep, calculate forces, rate of change of energy
- Half timestep: update velocities, energies; update positions with half-timestep velocities.
- Re-calculate forces, rate of change of energy with new values
- Full timestep: update with new forces, energies, velocities.

Time Evolution

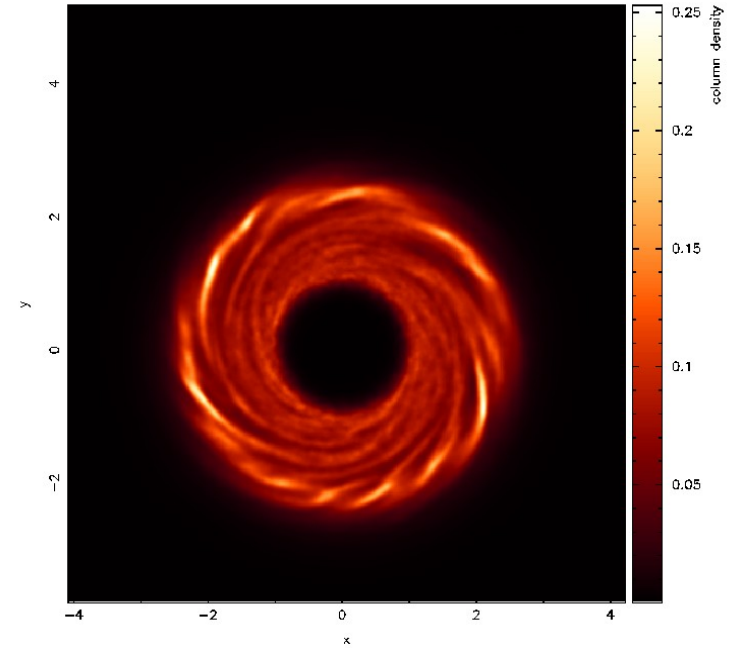
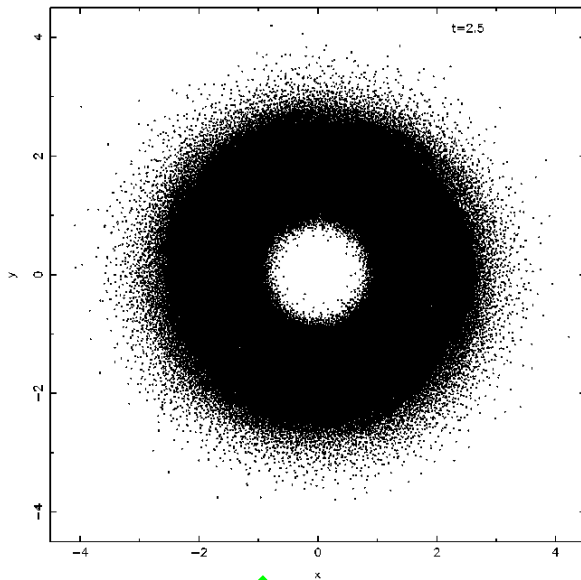
Visualization/Analysis

Grid code: output is a complete description of hydrodynamic states throughout domain.

Easy to visualize, analyze many instantaneous quantities – much harder to examine histories of fluid elements.

SPH: Opposite problem; can see what happens to any one fluid parcel immediately, but need some analysis tools even just to make a picture of what the domain looks like

Visualization/Analysis



Create a grid, use SPH interpolation to find quantities at each point on grid

Complications

(Things I really should talk about but don't have time)

- Need artificial viscosity to handle shocks
- Real world necessities – variable timesteps and smoothing lengths – complicate or break some of SPHs nice qualities
- There are a variety of techniques for finding neighbors within some distance h ; one common technique, using tree searches, integrates very nicely with using the same tree for gravitational N-body solving, making SPH + treecode gravity a very natural match.

SPH vs Grid Codes

- Handles open, free boundaries much better
- Much less diffusion for bulk motion
- Automatically resolves high density region
- No need to waste computation on empty space
- Couples naturally to N-body gravity
- Very robust
- Poor at dealing with shocks
- Low-order spatial accuracy
- Derivatives harder, making some physics (MHD) harder
- More caution required with initial conditions
- Hard to follow interesting dynamics in low-density regions
- Too robust?

SPH resources

References:

- Price (2005) astro-ph/0507472: Thesis chapter, very good review

Codes

- GADGET-2: Robust, widely-used SPH code
 - <http://www.mpa-garching.mpg.de/gadget/>
- StarCrash:
 - <http://www.astro.northwestern.edu/StarCrash/>
- SuperSPHplot: visualization tool
 - <http://www.astro.ex.ac.uk/people/dprice/supersphplot/>