AN INTRODUCTION TO ASTROPHYSICAL MAGNETOHYDRODYNAMICS

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1 Basic equations

In this section the MHD equations are derived, starting with the non-magnetic fluid equations and then using Maxwell's equations to add the magnetohydrodynamic terms.

1.1 The hydrodynamic equations

First, we can write down the hydrodynamic equations for a compressible fluid, in which there are three variables: velocity $\mathbf{u}$, density ρ and pressure P. All are functions of position $\mathbf{r}$ and time t. The conservation of momentum, mass and energy give us partial differential equations containing the time derivatives of these three quantities. First of all, the application of Newton's second law to a fluid element gives us:

$$\rho\frac{D\mathbf{u}}{Dt} = \mathbf{F}_{\text{body}} + \mathbf{F}_{\text{surface}} \quad (1)$$

$$= -\rho\mathbf{g} - \boldsymbol{\nabla}P + \mathbf{F}_{\text{visc}} \quad (2)$$

where the terms on the right hand side represent various forces (per unit volume). These forces fall into two classes. First there are body forces such as gravity ($\mathbf{g}$ is the local gravitational force per unit mass). In the next section, we look at the electromagnetic body force. Secondly the surface forces, where the force on a fluid element comes from its immediate neighbours: the pressure gradient force, present in all fluids, and the viscous force. D/Dt is the Lagrangian (co-moving) derivative, which is related to the Eulerian (stationary) derivative $\partial/\partial t$ by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}; \quad (3)$$

in other words, the co-moving rate of change of a quantity in a particular fluid element momentarily located at $\mathbf{r}$ is equal to the rate of change at that location $\mathbf{r}$ plus the spatial derivative in the direction of the fluid velocity multiplied by the magnitude of the fluid velocity.

Imagining a volume V with boundary S, the rate of change of mass in the volume is equal to the mass flux $\rho\mathbf{u}$ into the volume through the boundaries, giving

$$\frac{\partial}{\partial t}\int\rho\ \mathrm{d}V = -\oint\rho\mathbf{u}\cdot\mathbf{dS}. \quad (4)$$

This leads, via Gauss' theorem, to the usual form of the continuity equation

$$\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho\mathbf{u}). \quad (5)$$

So far, we have two equations (2) and (5) and three unknowns $\mathbf{u}$, ρ and P. To close this set, one option is to find some way of directly relating ρ to P without involving any new variables. This is known as a 'barotropic' equation of state where $P = P(\rho)$. A special case is to assume a constant density ρ = const, so that ρ can be replaced by ρ_0 in (2) and (5) reduces to $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$. However, in the general case the equation of state of the fluid must be treated as a more complex $P = P(\rho, X)$ where X is some other thermodynamic variable. For instance, the ideal gas equation of state is $P = \rho RT/\mu$ where R is the gas constant, μ is mean molecular weight (in units of the hydrogen atom mass) and T is temperature. In most astrophysical contexts, fluid can be treated as ideal gas. Having introduced this new variable, we now need one extra equation to close the set – this equation comes from conservation of energy. Consider the internal energy per unit volume e, which from thermodynamics we know is related to pressure by $P = (\gamma - 1)e$, where the ratio of specific heats is $\gamma = c_p/c_v$, equal to 5/3 in a monatomic gas. The rate of change of e at a fixed location contains three contributions: rate of heat added Q, advected internal energy – note the similarity between advected energy and advected mass in (5) – and work done by compression (cf. $dW = -p\,dV$). Putting the three together gives us the heat equation

$$\begin{aligned} \frac{\partial e}{\partial t} &= Q - \boldsymbol{\nabla}\cdot(e\mathbf{u}) - P\,\boldsymbol{\nabla}\cdot\mathbf{u}, \\ \frac{DP}{Dt} &= (\gamma-1)Q - \gamma P\,\boldsymbol{\nabla}\cdot\mathbf{u}. \end{aligned} \tag{6}$$

where the second form is obtained with some manipulation. Q could contain contributions from thermal conduction, viscous heating, dissipation of electric currents, nuclear energy generation, release of latent heat, radiative cooling, and so on. Note that there are various ways of writing down this equation. Being sloppy with units, we can also write it down for instance as $D(\ln P\rho^{-\gamma})/Dt = Q/e$, which makes the relation to the thermodynamic equation $T\,dS = dQ$ more obvious; remember that $S = a + b\ln(P\rho^{-\gamma})$ where a and b are constants.

1.2 The MHD equations

I now expand this set of equations (2), (5) and (6) to include a magnetic field. This results in an extra term – the *Lorentz force*[1] – in the momentum equation (2) and a new partial differential equation called the induction equation.

The force on a single particle of charge q with velocity $\mathbf{v}$ in an electromagnetic field is given by:

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right), \tag{7}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields and c is the speed of light.[2] Note that this is valid only in the non-relativistic limit where terms in v^2/c^2 can be ignored; we will use the non-relativistic limit again below. Now, in MHD we are interested in the force on the fluid as a whole rather than on individual particles – the total force per unit volume is therefore

$$\mathbf{F}_{\rm Lor} = \mathbf{F}_{\rm i} + \mathbf{F}_{\rm e} = (n_{\rm i}q_{\rm i} + n_{\rm e}q_{\rm e})\mathbf{E} + \left(n_{\rm i}q_{\rm i}\frac{\overline{\mathbf{v}}_{\rm i}}{c} + n_{\rm e}q_{\rm e}\frac{\overline{\mathbf{v}}_{\rm e}}{c}\right)\times\mathbf{B}, \tag{8}$$

where $\mathbf{F}_{\rm i}$ is the total force on the ions, $n_{\rm i}$, $q_{\rm i}$ and $\overline{\mathbf{v}}_{\rm i}$ are the number density, charge and mean velocity of ions and the quantities with subscript e refer to electrons. We have assumed here that all ions have the same charge, but it is trivial to show that a generalisation to various species of ion does not affect the end result. The ratio of the two terms is $\sim (E/B)(v/c)^{-1}\epsilon$ where $\epsilon \equiv (n_{\rm i}q_{\rm i} + n_{\rm e}q_{\rm e})/n_{\rm e}q_{\rm e}$ is the fractional charge imbalance which must be extremely small in any realistic astrophysical object. For instance, for the electric field not to overcome the gravity of the Earth and cause it to explode, we need the charge imbalance ϵ to be less than 10^{-36}. Since the ratios E/B and v/c will have less extreme values (more on this below), it is safe to ignore the first term. Moreover, because of this charge balance between electrons and ions we can simplify the second term to $n_{\rm e}q_{\rm e}(\mathbf{v}_{\rm drift}/c)\times\mathbf{B}$, where $n_{\rm e}$ and $q_{\rm e}$ are the number density and charge of the electrons; their mean velocity relative to the ions (and therefore the bulk motion of the fluid, since the ions carry almost all of the momentum, i.e. $\mathbf{u} \approx \mathbf{v}_{\rm i}$) is $\mathbf{v}_{\rm drift} \equiv \overline{\mathbf{v}}_{\rm e} - \mathbf{u}$. It is now convenient to introduce the concept of electric current density $\mathbf{J} = n_{\rm e}q_{\rm e}\mathbf{v}_{\rm drift}$. Note that in astrophysical contexts, $\mathbf{v}_{\rm drift} \ll \mathbf{u}$. We can now write

$$\mathbf{F}_{\rm Lor} = n_{\rm e}q_{\rm e}\frac{\mathbf{v}_{\rm drift}}{c}\times\mathbf{B} = \frac{1}{c}\mathbf{J}\times\mathbf{B}. \tag{9}$$

[1]There is some disagreement in the literature concerning terminology, some authors preferring "Laplace force" for the magnetic force in MHD. Here, I stay with the majority and use "Lorentz force".

[2]In some sense we can use this as the *definition* of $\mathbf{E}$ and $\mathbf{B}$, by saying that the force on a particle is some function of its charge and velocity which can be characterised by two vector fields in the form of (7).

We still need to answer the question of the origin of this relative velocity of the electrons to the ions; this comes essentially from the difference in force on the two species. The electrons experience a force relative to the fluid given by

$$\mathbf{F}_{\mathrm{e}} - \mathbf{F}_{\mathrm{Lor}} = n_{\mathrm{e}} q_{\mathrm{e}} \left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} \right). \tag{10}$$

This force will accelerate the electrons relative to the ions, and there will quickly be a balance established between this acceleration and losses through collisions between electrons and ions; the drift velocity (and therefore current) established will be proportional to this acceleration. This gives us Ohm's law:

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} \right), \tag{11}$$

where σ is the conductivity of the fluid, which will depend on density, mean free path, etc.

So far, we have added a term (9) to the momentum equation containing two new variables $\mathbf{B}$ and $\mathbf{J}$. Ohm's law (11) introduces yet another new variable $\mathbf{E}$, so that we need an additional two equations to close the set. Maxwell's equations are

$$\nabla \cdot \mathbf{E} = 4\pi\rho_{\mathrm{e}}, \tag{12}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{13}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{14}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}, \tag{15}$$

where ρ_{e} is the net charge density.[3] First of all, note that if (13) is satisfied at some point in time, (14) ensures that it is satisfied at all other times, since the divergence of the curl of any vector field is zero. In standard magnetohydrodynamics we now make the approximation that the charge density ρ_{e} is small; also that the displacement current in (15) can be neglected, i.e. that $4\pi\mathbf{J} \gg \partial\mathbf{E}/\partial t$. See section 1.2.1 for a justification.

From Ohm's law (11) we obtain an expression for the electric field $\mathbf{E} = (1/\sigma)\mathbf{J} - (\mathbf{u}/c) \times \mathbf{B}$, which we can use in conjunction with (14) to obtain

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} = \nabla \times \left(\mathbf{u} \times \mathbf{B} - \frac{c}{\sigma}\mathbf{J} \right), \tag{16}$$

and dropping the displacement current from (15) and substituting for $\mathbf{J}$ gives the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{u} \times \mathbf{B} - \frac{c^2}{4\pi\sigma}\nabla \times \mathbf{B} \right). \tag{17}$$

1.2.1 The validity of the MHD approximation

In addition to the standard conditions under which the fluid approximation is valid, e.g. collision frequency, etc. the 'MHD approximation' employed above makes two further assumptions: that the flow is non-relativistic and that the conductivity of the material is high enough so that the charge density is low and (12) can be ignored.

In the rest frame of a test particle (in which quantities are denoted with a prime), the force experienced is $\mathbf{F}' = q\mathbf{E}'$. In transforming to an inertial frame, we ignore terms in v^2/c^2 (so that the Lorentz factor $\Gamma \approx 1$) to obtain the lab frame relation (7). Returning to the fluid picture, a high conductivity σ ensures that current can flow in order to almost neutralise the rest frame electric field $\mathbf{E}' = \mathbf{E} + (\mathbf{u}/c) \times \mathbf{B}$ so that $\mathbf{E}' \ll \mathbf{E}$ and $E \sim (u/c)B$. We can now justify neglecting the displacement current in (15), because it is of order $E(u/c)/L \sim B(u/c)^2/L$ where L is a typical length scale of the flow. This is smaller by a factor v^2/c^2 than the curl of the magnetic field; meaning that $J \sim cB/L$ (dropping factors of 4π).

[3] Note the similarity between (12) and the equation relating the gravitational field to density $\nabla \cdot \mathbf{g} = -4\pi G\rho$. No constant is required in the electromagnetic equivalent because it is built into the unit of charge.

Looking at the first of Maxwell's equations (12), also known as Gauss' law, we see that $\rho_e \sim E/L \sim (u/c)B/L \sim (u/c)J/c$. Since this law holds in every reference frame it follows from $E' \ll E$ that $\rho_e' \ll \rho_e$. Current density J can be considered equal in lab and co-moving frames since assuming that $\Gamma \approx 1$ gives $\mathbf{J}' = \mathbf{J} - \rho_e \mathbf{u}$; the ratio of the two terms is $(u/c)^2$ so that $\mathbf{J}' = \mathbf{J}$. Magnetic field can also be considered frame-independent, since a transformation assuming $\Gamma \approx 1$ gives $\mathbf{B}' = \mathbf{B} - (\mathbf{u}/c) \times \mathbf{E}$, and the electric field $\mathbf{E}$ is itself smaller than the magnetic field by a factor u/c so we are left with a ratio u^2/c^2 between the two terms so that $\mathbf{B} = \mathbf{B}'$.

There are obviously astrophysical contexts in which this MHD approximation does not hold, for instance relativistic flows such as GRB jets; these are outside the scope of this mini-course.

1.2.2 A brief note concerning units

At this juncture it is worth commenting on the difference between the c.g.s. units employed here and the S.I. units often taught in undergraduate courses. The reader will notice that the equations above contain only one constant of nature: the speed of light c. In contrast, a glance at some of the text books reveals that the S.I. system is burdened not only with c but also with the rather nineteenth century concepts of the permittivity and permeability of the ether, ϵ_0 and μ_0. Another advantage of c.g.s. is that the electric and magnetic fields have the same units. However, a word of caution: there exist variations of c.g.s. units, for instance 'Gaussian c.g.s.' in which the current density J is equal to that defined above divided by c.

1.3 Diffusivities

We noted above that a fluid will in general experience some viscous force, which appears in the momentum equation (2), some contribution to the evolution of the magnetic field from a finite conductivity σ, and some contribution to the pressure from thermal conductivity. We are now ready to explore this more formally. First we look at viscosity (kinetic diffusion) before going onto heat conduction (thermal diffusion) and finite electrical conductivity (magnetic diffusion), all three of which make a contribution to the heating rate Q in the heat equation (6). We shall see below that these three diffusions are parametrised by three diffusivities and that it is convenient to have the same units for the three, namely cm^2 s^{-1}. If these diffusivities are set to zero, we speak of *ideal MHD*.

1.3.1 Kinetic diffusion

We shall assume here both that the fluid is *Newtonian*, meaning that the viscous stress is proportional to the deformation, as this is the case in almost all astrophysical fluids.[4] We also assume that the bulk viscosity is negligible, which is a very good approximation for monatomic gases. I state here without proof that the viscous force per unit volume is

$$F_{\text{visc},i} = \frac{\partial}{\partial x_j}\left(2\mu e_{ij} - \frac{2}{3}\mu\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{u}\right). \tag{18}$$

where I have used the Einstein summation notation; δ_{ij} is the Kronecker delta whose value is 1 if $i = j$ and 0 if $i \neq j$. μ is the viscosity of the fluid and e_{ij} is the rate of strain tensor defined as $e_{ij} \equiv (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$. It turns out that μ is normally proportional to density, so it makes sense to define a kinetic diffusivity $\nu \equiv \mu/\rho$ which depends only on temperature and has units cm^{2}s^{-1}. Furthermore, we can argue that in a subsonic flow where $\boldsymbol{\nabla}\cdot\mathbf{u}$ is small, it is often a good approximation to take the viscosity outside of the brackets and to assume that the second term in the viscous stress is negligible, giving the much simpler form

$$\mathbf{F}_{\text{visc}} = \rho\nu\boldsymbol{\nabla}^2\mathbf{u}. \tag{19}$$

[4] Blood and yoghurt are good examples of non-Newtonian fluids, as well as the "nuclear pasta" layer of a neutron star crust.

In addition, deformation of the fluid converts energy from kinetic to thermal form at a rate

$$Q_{\rm visc} = 2\mu\left(e_{ij} - \frac{1}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{u}\right)^2 \tag{20}$$

per unit volume.

1.3.2 Thermal diffusion

In many astrophysical contexts we can treat heat transfer as thermal conduction, where the heat flux obeys the Fourier law $\mathbf{q} = -K\boldsymbol{\nabla}T$ where the thermal conductivity K is some property of the fluid and T is temperature. This treatment is valid not only in cases where the heat transport mechanism is actually conduction, but also where heat transfer is by radiation, as long as we have local thermodynamic equilibrium (LTE), such as in the interior of a star. Now, the contribution to the heating rate per unit volume from heat conduction is

$$Q_{\rm th} = -\boldsymbol{\nabla}\cdot\mathbf{q} = \boldsymbol{\nabla}\cdot(K\boldsymbol{\nabla}T). \tag{21}$$

Under similar conditions to those applied in making the simplified equation (19), we can simplify this to

$$Q_{\rm th} = \rho\kappa c_p\boldsymbol{\nabla}^2 T, \tag{22}$$

where we have defined a thermal diffusivity $\kappa \equiv K/\rho c_p$ which has units $\mathrm{cm^2 s^{-1}}$.

1.3.3 Magnetic diffusion

If we assume that the electrical conductivity σ is uniform, we can rearrange the induction equation (17) using the constraint $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and the vector identity $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A}$, to

$$\frac{\partial\mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) + \eta\boldsymbol{\nabla}^2\mathbf{B}, \tag{23}$$

introducing the magnetic diffusivity $\eta \equiv c^2/4\pi\sigma$, which, like the other diffusivities ν and κ, has units $\mathrm{cm^2 s^{-1}}$. Note the similarity in the three diffusive terms – all contain a diffusive coefficient multiplied by $\boldsymbol{\nabla}^2$ of the relevant variable. Since the other terms in the MHD equations contain at most first spatial derivatives, we can see that all kinds of diffusion will be more important on smaller length scales present in the flow and will dominate on the smallest scale. From analysis of units, we see that there are three characteristic diffusive timescales, equal to $\mathcal{L}^2/(\nu,\ \kappa,\ \eta)$ where $\mathcal{L}$ is the characteristic length scale of the system. We return to this later.

Finally, it is worth mentioning the heating from Ohmic dissipation: this is equal to $\mathbf{J}\cdot\mathbf{E}'$ where $\mathbf{E}'$ is the electric field in the comoving frame. Expressed differently, this means that

$$Q_{\rm mag} = \frac{1}{\sigma}J^2 = \frac{\eta}{4\pi}(\boldsymbol{\nabla}\times\mathbf{B})^2. \tag{24}$$

2 Basic concepts

In this section we look at some useful concepts which follow from the MHD equations presented in the previous section.

2.1 Making the equations dimensionless

First let us summarise the set of equations we have so far, including the approximations made in the previous section and assuming an ideal gas equation of state $P = \rho RT/\mu$.

$$\rho\frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla}P + \frac{1}{4\pi}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} + \rho\nu\boldsymbol{\nabla}^2\mathbf{u} + \rho\mathbf{g}, \quad (25)$$

$$\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho\mathbf{u}), \quad (26)$$

$$\frac{DP}{Dt} = (\gamma-1)Q - \gamma P\,\boldsymbol{\nabla}\cdot\mathbf{u}, \quad (27)$$

$$\frac{\partial\mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) + \eta\boldsymbol{\nabla}^2\mathbf{B}. \quad (28)$$

The Lorentz force has been rearranged with the aid of (15). The heating rate is given by

$$Q = 2\rho\nu e_{ij}e_{ij} + \rho\kappa c_p\boldsymbol{\nabla}^2T + \frac{\eta}{4\pi}(\boldsymbol{\nabla}\times\mathbf{B})^2 + Q_{\mathrm{other}}. \quad (29)$$

It is now informative to assign typical flow parameters $\mathcal{L}$, $\mathcal{U}$ and $\mathcal{T} = \mathcal{L}/\mathcal{U}$ – typical length scale, velocity and timescale, and to compare the size of various terms in the equations. We can rewrite equations (25) and (28), dropping gravity as well as factors of order unity such as γ, in terms of non-dimensional variables and gradients, such as $\mathbf{u}' = \mathcal{U}\mathbf{u}$ (where we drop the prime below), as:

$$\begin{aligned}\frac{D\mathbf{u}}{Dt} &= -\frac{c_{\mathrm{s}}^2}{\mathcal{U}^2}\boldsymbol{\nabla}P + \frac{v_{\mathrm{A}}^2}{\mathcal{U}^2}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} + \frac{1}{\mathrm{Re}}\boldsymbol{\nabla}^2\mathbf{u},\\ &= \frac{1}{M^2}\left[-\boldsymbol{\nabla}P + \frac{1}{\beta}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}\right] + \frac{1}{\mathrm{Re}}\boldsymbol{\nabla}^2\mathbf{u}, \quad (30)\end{aligned}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) + \frac{1}{\mathrm{Re_m}}\boldsymbol{\nabla}^2\mathbf{B}, \quad (31)$$

where $c_{\mathrm{s}}^2 = \mathrm{d}P/\mathrm{d}\rho = \gamma P/\rho$ and $v_{\mathrm{A}}^2 = B^2/4\pi\rho$ are the sound and Alfvén speeds and $\mathrm{Re} \equiv \mathcal{U}\mathcal{L}/\nu$ and $\mathrm{Re_m} \equiv \mathcal{U}\mathcal{L}/\eta$ are the Reynolds number and magnetic Reynolds number respectively, measures of the ratio of inertia to diffusivity of two kinds. We can also define a plasma $\beta \equiv 8\pi P/B^2 \approx c_{\mathrm{s}}^2/v_{\mathrm{A}}^2$, a (likely) ratio of the first and second terms on the right hand side of the momentum equation. The Mach number is the ratio of flow speed to sound speed $M \equiv \mathcal{U}/c_{\mathrm{s}}$. We also give names to the ratios of diffusivities: the Prandtl number and magnetic Prandtl number are defined as $\mathrm{Pr} \equiv \nu/\kappa$ and $\mathrm{Pr_m} \equiv \nu/\eta$ respectively. Clearly, $\mathrm{Re_m}/\mathrm{Re} = \mathrm{Pr_m}$.

2.2 Different regimes in MHD

In an unmagnetised fluid, we have two dimensionless parameters which describe the flow: the Mach number M and the Reynolds number Re. It can be shown that if $M \ll 1$ the flow is roughly incompressible, i.e. that $D\rho/Dt \approx 0$, which gives $\boldsymbol{\nabla}\cdot\mathbf{u} \approx 0$ via some rearrangement of the continuity equation (26). This comes from the fact that in the $M^2 \approx E/e$ where E and e are the kinetic and thermal energies per unit volume; at low Mach number there is not enough kinetic energy available to produce any significant compression or expansion, since an energy of order e would be required. This is the regime we assumed in simplifying the viscous force in the momentum equation. The other parameter of the flow is the Reynolds number, which describes the importance of viscosity. At low Reynolds number, the flow is viscous: waves are damped, and the flow is laminar; at high Reynolds number turbulence often appears. Note that the viscous term in (30) contains second spatial derivatives of the velocity, whereas the other two terms on the right hand side contain only first spatial derivatives. This means that viscosity will be more important on the smaller length scales of the flow.

In a magnetised medium, the same applies to the magnetic diffusion which appears in the induction equation (31) with a second spatial derivative. We now have two extra parameters $\mathrm{Re_m}$ and β; at low magnetic Reynolds number the magnetic field will be smooth and at high $\mathrm{Re_m}$ turbulence may appear, although this also depends on Re. The value of the 'plasma β' is very important in MHD. If $\beta \ll 1$, a glance at the momentum equation reveals that the Lorentz force will be much larger than the pressure gradient term. In astrophysical contexts, the gravitational term is small in most low-β plasmas, so that velocities of order the Alfvén speed (which can be very high) will result unless the current $\mathbf{J} = (c/4\pi)\boldsymbol{\nabla}\times\mathbf{B}$ is almost parallel to $\mathbf{B}$, the so-called 'force-free' regime (since the Lorentz force goes to zero). Conversely, if $\beta \gg 1$ then the current need not be parallel to the magnetic field, and the magnetic field will only have much affect on the flow in directions where the Lorentz force is not opposed by the pressure gradient or other stronger forces such as gravity.

2.3 Field lines, flux conservation and flux freezing

There are several useful concepts in MHD which can help develop an intuitive understanding of the subject. In this section we derive some results for the case of a fluid with infinite conductivity ($\eta = 0$) which can be considered approximately true in the more realistic case of finite conductivity. Now, if $\eta = 0$ then

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}). \tag{32}$$

Let us define a magnetic flux ϕ as a integral of the normal component of $\mathbf{B}$ on a surface S

$$\phi = \int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} \tag{33}$$

and let us calculate the change of flux through that surface it moves with the flow of the fluid:

$$\frac{D\phi}{Dt} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot \mathrm{d}\mathbf{S} + \oint_l (\mathbf{u}\times \mathrm{d}\mathbf{l})\cdot \mathbf{B}, \tag{34}$$

where the first term comes from the rate of change of flux through the surface if it were fixed in space and the second comes from the movement of the surface from the fluid velocity $\mathbf{u}$. The surface is bounded by a line l. Substituting (32) into this equation and using Stokes' theorem, the first term becomes $\oint_l \mathbf{u}\times\mathbf{B}\cdot \mathrm{d}\mathbf{l}$. From the triple vector product rule we now see that the two terms cancel and that the flux through the co-moving surface is constant in time.

If we now imagine the fluid being composed of small co-moving fluid elements, each threaded by a constant flux, it becomes clear that the concept of field lines and of their being 'frozen' into the fluid are useful tools in understanding MHD. This will be discussed below at greater length.

2.4 Magnetic pressure, tension and energy density

The Lorentz force can be written in an alternative form, making use of a vector identity and the solenoidal constraint $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$

$$\mathbf{F}_{\mathrm{Lor}} = \frac{1}{4\pi}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} = \frac{1}{4\pi}(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B} - \boldsymbol{\nabla}\left(\frac{B^2}{8\pi}\right). \tag{35}$$

The second term looks like the pressure gradient term $-\boldsymbol{\nabla}P$, so the quantity $B^2/8\pi$ is called the *magnetic pressure*. The first term on the right can be thought of as a magnetic tension. However, we need to remind ourselves that the total Lorentz force is always perpendicular to the magnetic field, so that the components of these two terms parallel to the field must cancel. After some manipulation we can rewrite the Lorentz force without these cancelling components as

$$\mathbf{F}_{\mathrm{Lor}} = \boldsymbol{\kappa}\frac{B^2}{4\pi} - \boldsymbol{\nabla}_{\perp}\frac{B^2}{8\pi}, \tag{36}$$

where $\boldsymbol{\kappa}$ is the curvature vector (directed along the radius of curvature of the field and equal in magnitude to the reciprocal of that radius) and $\boldsymbol{\nabla}_{\perp}$ is the part of the gradient perpendicular to the field. The first term can now be called the 'curvature force' and resembles tension in a string in that it will tend to restore a perturbed straight field line to its original shape.

It can be shown that the magnetic pressure $B^2/8\pi$ is also the energy density of the magnetic field; to demonstrate this in a non-rigorous way is reasonably straightforward. Imagine that we have a slab, infinite in the y and z directions but bounded at $x = 0$ and $x = l$, containing a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{y}}$ where $\hat{\mathbf{y}}$ is the unit vector in the y direction. Outside the slab $B = 0$. The Lorentz force clearly has no y or z component but its x component has delta functions at the two boundaries which are balanced by delta functions in the thermal pressure gradient if we have an external thermal pressure equal to $B^2/8\pi$. Now imagine we change l adiabatically so that flux is conserved, meaning that $Bl = k =$ const. If the magnetic field has energy e per unit volume, then the energy per unit area of the slab is el. From the $\mathrm{d}U = -p\mathrm{d}V$ relation in thermodynamics we therefore have $\mathrm{d}(el) = -(B^2/8\pi)\mathrm{d}l = -(k^2/8\pi)l^{-2}\mathrm{d}l$ and so assuming that at the total energy el goes to zero at $l = \infty$ we can integrate from $l' = \infty$ to l to give

$$\int_0^{el} \mathrm{d}(el)' = -\frac{k^2}{8\pi}\int_\infty^l l^{-2}\mathrm{d}l' \qquad \Longrightarrow \qquad el = \frac{k^2}{8\pi l} \qquad \Longrightarrow \qquad e = \frac{B^2}{8\pi}. \tag{37}$$

2.5 Magnetic helicity

In section 2.3 we saw that the magnetic flux through a co-moving fluid surface is constant in the limit of high conductivity. We can now look at an additional quantity which is also conserved in this limit. Magnetic helicity is a global quantity defined as

$$H \equiv \int_V \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}V \tag{38}$$

where $\mathbf{A}$ is the vector potential defined from $\mathbf{B} = \nabla\times\mathbf{A}$. Now, since the curl of the divergence of a scalar is zero, we can add any gradient of a scalar $\nabla\phi$ to the vector potential without changing the magnetic field; however this will in general affect the magnetic helicity. It can be shown in the following way though that magnetic helicity is gauge invariant provided that no magnetic field lines pass through the surface of the volume of integration. Consider some new vector potential $\mathbf{A}' = \mathbf{A} + \nabla\phi$. The helicity is now

$$H' = \int_V [\mathbf{A}\cdot\mathbf{B} + (\nabla\phi)\cdot\mathbf{B}]\,\mathrm{d}V \tag{39}$$

$$= H + \int_V [\nabla\cdot(\phi\mathbf{B}) - \phi(\nabla\cdot\mathbf{B})]\,\mathrm{d}V \tag{40}$$

$$= H + \oint_S \phi\,\mathbf{B}\cdot\mathrm{d}\mathbf{S}, \tag{41}$$

since the first term on the first line is simply equal to the original helicity; the second term was expanded with a standard vector identity to give the expression on the second line. Since the magnetic field is solenoidal, the second of the new terms vanishes, and the first can be rewritten with the aid of Gauss' theorem to give a surface integral. Therefore if $\mathbf{B}\cdot\mathrm{d}\mathbf{S} = 0$ everywhere on the boundary, the helicity is unchanged and therefore gauge invariant.

Helicity is a useful concept because of its conservation properties. It can be shown that it is perfectly conserved in the limit of infinite conductivity: let us first note that in this limit, (28) becomes $\mathbf{B}_t = \nabla\times(\mathbf{u}\times\mathbf{B})$ and so $\mathbf{A}_t = \mathbf{u}\times\mathbf{B}$. Now

$$\begin{aligned}\frac{\partial H}{\partial t} &= \int_V \mathrm{d}V\,[\mathbf{A}_t\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{B}_t] = \int_V \mathrm{d}V\,[\mathbf{u}\times\mathbf{B}\cdot\mathbf{B} + \mathbf{A}\cdot\nabla\times(\mathbf{u}\times\mathbf{B})] \\ &= \int_V \mathrm{d}V\,[(\mathbf{u}\times\mathbf{B})\cdot\nabla\times\mathbf{A} - \nabla\cdot(\mathbf{A}\times(\mathbf{u}\times\mathbf{B}))] \\ &= -\oint_S \mathrm{d}\mathbf{S}\cdot\mathbf{A}\times(\mathbf{u}\times\mathbf{B}).\end{aligned} \tag{42}$$

We simply require then that the velocity goes to zero on the boundary of the domain.

In a fluid with finite conductivity it is still approximately conserved – we can see this from an argument with units. Now, the diffusive timescale on which the magnetic field decays due to finite conductivity, as we saw above, is $\tau \sim \mathcal{L}^2/\eta$. This is shorter on shorter length scales so that when magnetic energy is converted to heat via Ohmic dissipation, it is mainly the small-scale structure where the energy is converted. Helicity however has units of length times energy and is therefore present more in the large scale components of the magnetic field than the magnetic energy, and less is therefore lost due to diffusive processes on small scales. Often then during MHD processes where the flow contains a range of length scales, energy is lost at the smallest scales while helicity is roughly conserved.

Helicity also has units of flux squared, and can in fact be thought of in some sense of the product of two fluxes of different components of the magnetic field. It is often said that helicity is a measure of the 'twist' of the magnetic field, because a twisted field must contain at least two components. As we have seen, that twist is conserved even as magnetic energy is lost.

2.6 MHD equilibria

In many astrophysical contexts, we are interested in equilibrium situations where the forces are balanced. However, before we proceed, it is important to clarify what we mean by equilibrium. If we simply set the velocity to zero, the momentum equation (25) gives us a relation between P, $\mathbf{B}$ and $\rho\mathbf{g}$, so that any combination of the three which satisfy that relation will be an equilibrium. Now, both sides of the continuity equation (26) go to zero, but the heat equation (27) contains Q, which does generally not go to zero when $\mathbf{u} = \mathbf{0}$ due to its terms with κ and η. Likewise, the term with η in the induction equation (28) will not in general vanish. The result of this is that the magnetic field and pressure field will evolve, giving rise to a non-zero velocity field – a truly stationary state is in general achievable only where $\kappa = \eta = 0$. However, provided that these diffusivities are small, we can still find a *dynamic equilibrium* by setting $\mathbf{u} = \mathbf{0}$ and balancing forces. This equilibrium will not chance appreciably on a dynamic timescale, i.e. the time taken for a sound or Alfvén wave to travel across the domain; rather, it will evolve over a longer timescale due to the diffusive terms.

So, finding a (dynamic) equilibrium is simply a matter of finding a solution to the following equation:

$$-\nabla P + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} + \rho\mathbf{g} = \mathbf{0}, \tag{43}$$

together with the constraint $\nabla \cdot \mathbf{B} = 0$. It is interesting to explore some of the properties of this equation. First of all, note that in the non-magnetised case it reduces to the equation of hydrostatic equilibrium which one often sees in the form $\partial P/\partial z = -\rho g$ where gravity is directed downwards along the z-axis. In the magnetised case, taking the dot product with $\hat{\mathbf{B}}$, the unit vector in the direction of the magnetic field, gives

$$(\hat{\mathbf{B}} \cdot \nabla)P = \rho(\mathbf{g} \cdot \hat{\mathbf{B}}) \qquad \text{or} \qquad \frac{dP}{ds} = \rho g_s, \tag{44}$$

where dP/ds is the derivative along a field line and g_s is the component of gravity along the field line. In other words, in an MHD equilibrium there is hydrostatic balance *along field lines*.

In a situation where $\beta \gg 1$ and the Lorentz term in (43) is much smaller than the pressure and gravity terms, we can imagine first constructing a non-magnetic equilibrium where $\nabla P = \rho\mathbf{g}$, adding a weak magnetic field and then making small adjustments to the pressure and density fields to balance the Lorentz force. In principle this should be possible, because an arbitrary magnetic field and its associated Lorentz force have two degrees of freedom – three dimensions minus the one constraint ($\nabla \cdot \mathbf{B} = 0$) – and we also have two degrees of freedom in balancing the Lorentz force since we can adjust both the pressure and density fields independently of each other.[5] Note that in a fluid with a barotropic equation of state $P = P(\rho)$ we only have one degree of freedom

[5]Strictly speaking, adjusting the density field will affect the gravitational field $\mathbf{g}$, but if only small adjustments to the non-magnetic equilibrium are needed it is hard to imagine that changes in $\mathbf{g}$ will prevent the existence of an equilibrium.

in adjusting the pressure and density fields, so that depending on the context it may be either more difficult or impossible to construct an equilibrium.

In the special case without gravity we see from (44) that pressure is constant along field lines. Also, the gradients in thermal pressure P and magnetic pressure $B^2/8\pi$ must be comparable unless we can construct an approximately force-free field where current and magnetic field are almost parallel. Also note that density is no longer relevant for the structure of the equilibrium, meaning that we now only have one scalar field P to balance the Lorentz force with its two degrees of freedom, just as in the case above with gravity and a barotropic E.O.S.

3 Astrophysical contexts

We now illustrate these principles and introduce further principles in the astrophysical contexts of the solar corona, jets and accretion discs.

3.1 The solar corona

On the photosphere of the Sun, we observe magnetic fields in the quiescent regions which are structured on the granulation scale ($\sim$ 1000km) and hundreds of gauss in strength, and in addition we see active regions with sunspots of sizes 10 to 100 times the granulation scale in which the magnetic field is in the range $1-3$kG. The thermal energy density at the photosphere is about the same as a magnetic field of about $3kG$, which we call the 'equipartition field strength'. Since the flow speeds at and just below the photosphere are roughly sonic, the kinetic energy density is about the same. This explains why a field of around 3kG is required to have much effect on the appearance of the photosphere. The thermal energy density increases rapidly below the photosphere and decreases rapidly above it – more rapidly than the magnetic energy density (pressure falls exponentially whereas the magnetic field tends to fall geometrically) so we can think of a $\beta = 1$ surface which lies at or just above the photosphere. Below this surface, the magnetic field has an either negligible or subtle effect on the flow of gas; above this surface the magnetic field dominates. This region above the photosphere is called the corona, although strictly speaking the two are separated by the chromosphere and the transition region. The corona has a temperature of around $1-2$ million Kelvin, which contrasts to the photospheric temperature of 5800K – the origin of this high temperature, the 'coronal heating problem', is one of the best-known unsolved problems in astrophysics. It is generally agreed that the magnetic field transports energy through the photosphere and that it is converted from magnetic to thermal form in the corona; what is not understood is how the magnetic energy is dissipated. Theories normally invoke either reconnection or excitation and dissipation of magnetic waves.

Now let us look at some of the parameters in the corona: $T \sim 10^6$ K, $\rho \sim 10^{-15}$ g cm^{-3}, $P = \rho TR/\mu \sim 10^{-1}$ erg cm^{-3}, $|\mathbf{g}| \sim 3\times10^4$ cm s^{-2}, $B \sim 10$ G. This means that the scale height $H = P/\rho g \sim 3\times10^9$ cm, plasma $\beta = 8\pi P/B^2 \sim 3\times10^{-2}$, sound speed $c_\mathrm{s} = \sqrt{\gamma P/\rho} \sim 10^7$ cm s^{-1}, Alfvén speed $v_\mathrm{A} = B/\sqrt{4\pi\rho} \sim 10^8$ cm s^{-1}.

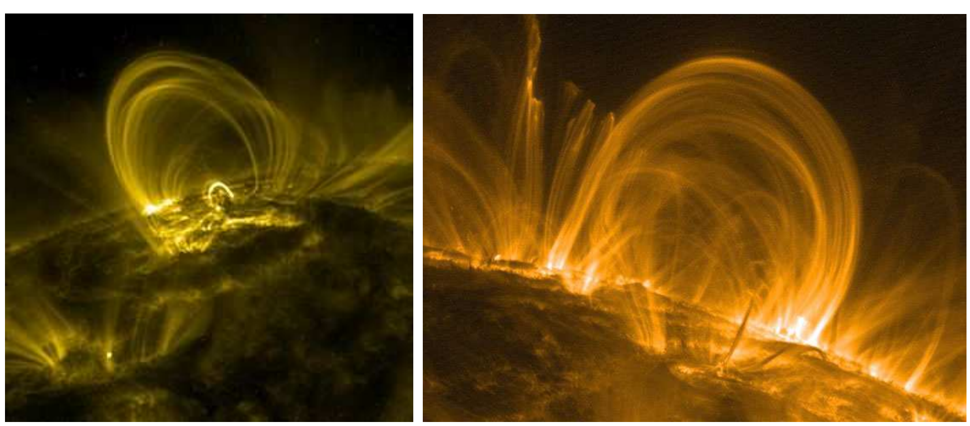

Figure 1: Images of coronal loops taken in the Fe IX line at 171Å by TRACE.

Looking at (30) it appears that since the Lorentz force is much greater than the pressure gradient force, it must be balanced by the inertia term on the left hand side, meaning that flow speeds are comparable to the Alfvén speed. However, structures such as those in fig. 1 are observed to last for anything up to weeks, much greater than the Alfvén timescale $\tau_A = H/v_A \sim 30$ s. The only way out of this is for the Lorentz to be reduced by having the current and magnetic field almost parallel to each other – we call this a 'force-free' field. The properties of these fields are explored in the next section. Often however we observe loop structures in the corona which, having apparently been in such a force-free equilibrium for some time, suddenly depart from equilibrium and convert much of their magnetic energy into heat on a timescale comparable to the Alfvén timescale. This process is explored in section 3.1.2.

3.1.1 Force-free fields

In any low β plasma we see from (30) that the Lorentz force cannot apparently be balanced by the pressure gradient and it is difficult to imagine a gravitational field with the necessary geometry; balancing the Lorentz force with inertia, the term on the left-hand side of the momentum equation (25), would mean Alfvénic flow speeds and nothing approaching an equilibrium. We therefore speak of a 'force-free' field, where the current and magnetic field are almost parallel so that $|(\nabla\times\mathbf{B})\times\mathbf{B}| \ll B^2/\mathcal{L}$. This gives:

$$\nabla\times\mathbf{B} = \alpha\mathbf{B}, \tag{45}$$

$$\mathbf{B}\cdot\nabla\alpha = 0, \tag{46}$$

where the second equality comes from taking the divergence of the first and using the solenoidal condition $\nabla\cdot\mathbf{B} = 0$; it means that α is constant along field lines. In other words, as we follow a field line we see that the neighbouring lines curve around it in the same sense all the way along the line – force-free fields are 'twisted' in some sense. If α is a constant everywhere, we can write the Helmholtz equation $(\alpha^2 + \nabla^2)\mathbf{B} = 0$ by taking the curl of (45).

A special case is where $\alpha = 0$, which we call a *potential* or *curl-free* field, where the current vanishes. This is obviously the case in a vacuum, and is a good approximation in some other astrophysical contexts such as above the surface of some magnetic main-sequence stars. We call it a potential field because being curl-free we can express it as the gradient of a scalar potential $\mathbf{B} = \nabla\phi$. Since the divergence of the field is zero, we have the Laplace equation $\nabla^2\phi = 0$. This we can solve if we know the normal component $\mathbf{B}\cdot\mathbf{n}$ everywhere on the boundary of the domain.

There is a theorem which states that no equilibrium can be force-free everywhere. Imagine a force-free equilibrium in a region of volume V surrounded by an unmagnetised region, and imagine an isotropic expansion or contraction of the region. Under such a change, the position vector of any fluid element changes from $\mathbf{r}$ to $\mathbf{r}'$ and the field from $\mathbf{B}$ to $\mathbf{B}'$. In a uniform expansion by a factor a we have $\mathbf{r}' = a\mathbf{r}$ and from flux conservation ($\phi = Br^2$ is constant) we see that $r'^2\mathbf{B}'(\mathbf{r}') = r^2\mathbf{B}(\mathbf{r})$. The energy of the field after the expansion is

$$E' = \int_{V'} \frac{B'^2}{8\pi}\mathrm{d}V' = \frac{1}{a}\int_V \frac{B^2}{8\pi}\mathrm{d}V \tag{47}$$

since $\mathrm{d}V' = a^3\mathrm{d}V$ and $\mathbf{B}' = a^{-2}\mathbf{B}$. The region will therefore expand until either some force opposes it, at which point it is no longer force-free, or it reaches infinite extent and $E \to 0$. [More generally, anything with positive energy will tend to expand.] A force-free region must be subject to forces on its boundary.

3.1.2 Reconnection

We saw in above in section 2.1 and from equation (28) that the timescale over which the magnetic diffusivity acts is $\tau_{\rm diff} \sim \mathcal{L}^2/\eta$. However, we see in many astrophysical contexts such as the solar corona that changes in global magnetic topology, i.e. deviations from flux-freezing, and the associated dissipation of magnetic energy can occur on much shorter timescales. For instance, energy is released during solar flares over timescales of seconds and minutes although $\tau_{\rm diff} \gtrsim 10^6$yr, assuming a standard Spitzer conductivity. There are two possible reasons for this. The first is that somehow the diffusion is locally brought to work on shorter length scales than the global length scale $\mathcal{L}$, the second is that there is some 'anomalous resistivity' – higher than the standard resistivity – perhaps when the current density is particularly high; plasma instabilities may also be involved. It seems likely that in some situations both mechanisms must be invoked.

Producing structure on small length scales from a global configuration initially lacking such small scales is a common phenomenon in physics. For instance, to mix two paints together so that they combine on the microscopic scale, it is sufficient to stir with a large spoon. In a turbulent medium, large-scale driving leads to a cascade of energy to smaller and smaller length scales until a scale is reached on which the viscous dissipation timescale is equal to the flow timescale. In the example of solar flares, we do not see turbulence; rather, the field is thought to dissipate in thin current sheets which separate regions with magnetic field in different directions. This process is known as 'reconnection'.

The first model of reconnection, the *Sweet-Parker* mechanism, is illustrated in fig. 2. Material flows perpendicular to the field lines at speed v_0 towards the current sheet. This speed is equated to the 'diffusion speed' within the current sheet, which from the induction equation is equal to $\eta/\delta \approx v_0$. Assuming incompressibility, conservation of mass gives $v_0 L \approx v_* \delta$. Now, if we consider the force balance in the y direction it is clear that there is an excess thermal pressure at the centre of the current sheet, where the magnetic field vanishes, equal to the magnetic pressure at the boundaries of the sheet, i.e. $B^2/8\pi$. The material is accelerated in the x direction by this thermal pressure and escapes at the ends of the sheet; we have from the force balance that $\rho v_*^2/2 \approx B^2/8\pi$ which can be rearranged to $v_* \approx v_A = B/\sqrt{4\pi\rho}$. Now we can solve for the reconnection velocity v_0:

$$v_0 \approx \frac{\eta}{\delta} \approx \frac{v_A \eta}{v_0 L} \qquad \Longrightarrow \qquad \frac{v_0}{v_A} \approx \sqrt{\frac{\eta}{v_A L}} = \mathrm{Re_A}^{-1/2}, \tag{48}$$

where $\mathrm{Re_A}$ is the Alfvénic Reynolds number. This model produces reconnection speed ratios of $v_0/v_A \sim 10^{-6}$ in the solar corona and other astrophysical plasmas, which is unfortunately rather less than the generally observed value of ~ 0.1. One way out of this is the Petschek reconnection model in which most of the energy is dissipated in standing shocks attached to a small central Sweet-Parker-like diffusion region. Some kind of anomalous resistivity probably also plays a role.

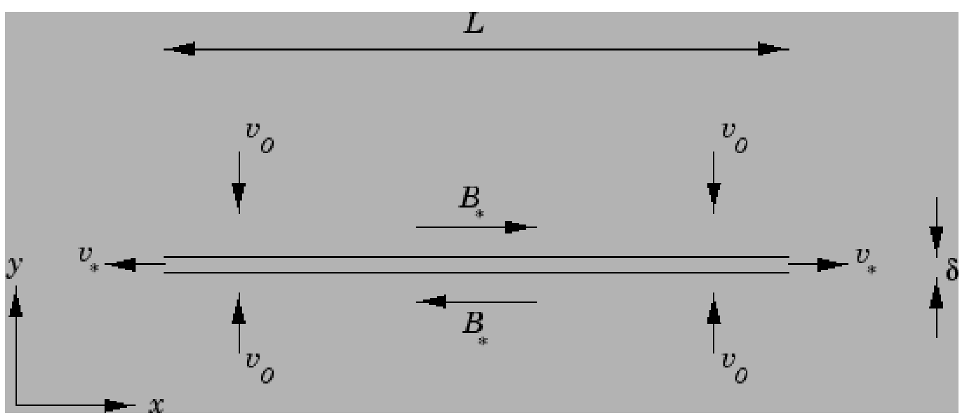

Figure 2: The Sweet-Parker reconnection mechanism. Regions of opposing magnetic field B_* are brought together, separated by a thin sheet of thickness δ.

3.2 Jets: launching, collimation and instabilities

Jets are found in many astrophysical accretion settings, for example protostars, neutron stars, AGN. In this section we examine the magneto-centrifugal model of jet launching and collimation.

3.2.1 Launching

Imagine a Keplerian disc around a central object threaded by a magnetic field. The field which emerges from the disc is 'ordered' in some sense. If, for simplicity, we assume that the field component emerging normal to the disc is of uniform sign and its strength varies with cylindrical radius as $B_z = B_0(\varpi^2/\varpi_0^2 + 1)^{-1/2}$, and then assume that the field above the disc is curl-free (i.e. zero-current, force-free with $\alpha = 0$) we have the field illustrated in fig. 3. Near the disc the lines are inclined away from the centre because of the greater field strength at the centre, but further from the disc they tend towards the vertical because of the fact that flux per unit cylindrical radius increases outwards – in other words, most of the flux is threaded through the outer disc and so from a distance the inner part is of lesser importance. We expect that $B_z \propto \Sigma$ where Σ is the column density of the disc, so this picture of flux increasing outwards is realistic as long as $\varpi\Sigma$ increases outwards, i.e. $\partial \ln \Sigma / \partial \ln \varpi > -1$.

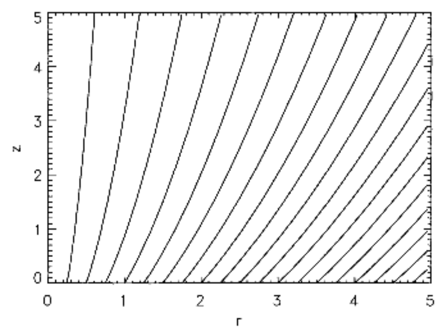

Figure 3: The field above a disc from which a vertical field component $B_z = B_0(\varpi^2/\varpi_0^2 + 1)^{-1/2}$ emerges, assuming the field is curl-free. Note that the field lines curve towards the vertical. [From Spruit et al. 1997.]

We now allow material to evaporate from the disc. Just above the disc this material will have a low density in the sense that the magnetic energy density $B^2/8\pi$ is much greater than the thermal (i.e. $\beta \ll 1$) and also than the kinetic $\rho u^2/2$. This means that the field must be force-free and that it is little affected by the material. Flux freezing requires that the material flows along the field lines, and since the field lines point away from the central object, the material is centrifugally accelerated away from the disc; it can be shown that if the angle between the field and the vertical exceeds some threshold then this centrifugal acceleration exceeds the downwards gravitational acceleration. The material is forced to co-rotate with the magnetic field.

As the material is accelerated its kinetic energy density eventually exceeds the magnetic, i.e. $\rho u^2/2 > B^2/8\pi$ or alternatively $u > v_A$; we say that this transition happens at the 'Alfvén surface' which is analogous to the sonic point in non-magnetised flows such as in the nozzle of a rocket engine. Flux freezing holds on both sides, but whereas inside the Alfvén surface the flow follows the field lines, outside the Alfvén surface the field lines follow the flow. This means that the material ceases to co-rotate with the foot points of the field lines; rather, the field lines are 'wound up' so that a significant toroidal component B_t is produced.

3.2.2 Collimation

We saw above that the poloidal field may have a tendency to collimate the flow (fig. 3). In this section we examine the collimation after the initial acceleration phase, when the energy in the z component of the motion dominates and therefore $u_z \approx$ const.

Imagine a jet with circular cross-section of radius $a(z)$ at a distance z from the central object (outside the Alfvén surface). It contains a spiral magnetic field with toroidal and poloidal components B_t and B_p. If the toroidal and poloidal fluxes are both conserved as the material moves away from the source, then as the radius of the jet changes we have $B_t \propto a^{-1}$ and $B_p \propto a^{-2}$. After the initial acceleration phase the jet will expand ballistically with $a \propto z$ in the absence of significant pressure or magnetic forces. Obviously in the non-magnetic case, if the thermal pressure in the jet is greater than that in the surroundings, the jet will expand faster than $a \propto z$ – the jet will be 'flared'. The poloidal component of the field will also tend to make the jet flare as it exerts a pressure $B_p^2/8\pi$ on the jet's surroundings; the toroidal field exerts no pressure because its pressure and tension forces are equal and opposite. Another way of thinking about this is in terms of the energy per unit length of the jet in the toroidal and poloidal components of the field: $E_t = \pi a^2 B_t^2 = \pi \Phi_t^2$ and $E_p = \pi a^2 B_p^2 = \Phi_p^2/(\pi a^2)$. It is the drop in E_p which drives the expansion of the jet. It is therefore necessary to have some external pressure for collimation to occur – one can imagine that with some constant pressure in the ambient medium the jet would be flared near the source and then further away would settle at constant a when its internal pressure $P_{\rm jet} + B_{\rm p}^2/8\pi$ is equal to the external.

It is worth looking in more detail at the collimating effect of the toroidal field. A jet with $B_{\rm t} = B_{\rm t}(\varpi)$ carries a current $\mathbf{J} = \hat{\mathbf{z}}(c/4\pi\varpi)\partial(\varpi B_{\rm t})\partial\varpi$ and so the Lorentz force is

$$\mathbf{F}_{\rm Lor} = -\hat{\boldsymbol{\varpi}} \frac{B_{\rm t}}{4\pi\varpi} \frac{\partial(\varpi B_{\rm t})}{\partial\varpi} \tag{49}$$

where $\hat{\boldsymbol{\varpi}}$ is the unit vector in the ϖ direction. At least near the axis this force must be directed inwards: this effect is often referred to as 'hoop stress'. This has lead to the misleading concept of 'self collimation', according to which a jet can be collimated by its own toroidal magnetic field. The problem with this is that to avoid having the energy diverge towards infinity, the partial derivative must change sign at some radius outside which the Lorentz force is directed outwards, requiring in effect some external pressure support.

3.2.3 Instability

We saw above that as a jet expands the poloidal field falls faster than the toroidal, which leads eventually to an instability driven by the free energy in the toroidal field. It can be shown that the dominant modes have azimuthal wavenumbers $m = 0$ (sausage mode) and $m = 1$ (kink mode); here we look at a simple derivation of the instability criterion for the $m = 0$ mode.

A purely toroidal field is some function of cylindrical radius, $B = B(\varpi)$. Imagine two thin annuli at radii ϖ and $\varpi + \delta\varpi$ with magnetic fields B and $B + \delta B$, each of area A and therefore of thicknesses $A/(2\pi\varpi)$ and $A/(2\pi(\varpi + \delta\varpi))$. The energy per unit length jet of the magnetic field in the annuli is

$$E = \frac{A}{8\pi}\left[B^2 + (B + \delta B)^2\right]. \tag{50}$$

We now exchange adiabatically the positions of the two annuli, keeping the volume of each constant. In general, the most unstable modes of any instability will be incompressible (density unchanged), since compressing the gas will require work to be done by the magnetic field. Since the volumes of the annuli remain the same, the total thermal energy is unchanged. Since flux is conserved, the new fields at locations ϖ and $\varpi + \delta\varpi$ are $(B + \delta B)\varpi/(\varpi + \delta\varpi)$ and $B(\varpi + \delta\varpi)/\varpi$, so that the new energy is

$$E + \delta E = \frac{A}{8\pi}\left[\left(\frac{(B + \delta B)\varpi}{\varpi + \delta\varpi}\right)^2 + \left(\frac{B(\varpi + \delta\varpi)}{\varpi}\right)^2\right]. \tag{51}$$

For stability we need the exchange to have increased the energy, i.e. $\delta E > 0$. Subtracting (50) from (51) and dividing by $A/8\pi$ we have

$$\left(\frac{(B + \delta B)\varpi}{\varpi + \delta\varpi}\right)^2 + \left(\frac{B(\varpi + \delta\varpi)}{\varpi}\right)^2 - B^2 - (B + \delta B)^2 > 0 \tag{52}$$

$$\left(1 + 2\frac{\delta B}{B} + \frac{\delta B^2}{B^2}\right) + \left(1 + 2\frac{\delta\varpi}{\varpi} + \frac{\delta\varpi^2}{\varpi^2}\right)\left[1 + 2\frac{\delta\varpi}{\varpi} + \frac{\delta\varpi^2}{\varpi^2} - 1 - \left(1 + 2\frac{\delta B}{B} + \frac{\delta B^2}{B^2}\right)\right] > 0, \tag{53}$$

where the zeroth and first order terms cancel; keeping only the second order terms we have

$$\delta\varpi - \delta B > 0 \qquad \text{or} \qquad \frac{\partial \ln B}{\partial \ln \varpi} < 1. \tag{54}$$

This corresponds to the result of Tayler (1957). For obvious reasons, this is known as an 'interchange' mode. A more general treatment including the non-axisymmetric modes reveals that $m \geq 1$ modes are stable if $\partial \ln B/\partial \ln \varpi < m^2/2 - 1$, meaning that the $m = 1$ mode (the 'kink mode') sets in first. Note that to avoid a current singularity we need $\partial \ln B / \partial \ln \varpi \geq 1$ on the axis, so it is impossible in practice to construct a toroidal field which is stable everywhere. The growth timescale of all modes is comparable to the dynamical timescale, i.e. the Alfvén timescale ϖ / v_A.

A jet contains not only a toroidal component, of course, but also an axial component B_z which can help to stabilise the jet against this instability. We can see approximately how strong this component needs to be by means of the following energy argument. It is clear that as the instability grows, work needs to be done against the axial component of the field and for stability this must be greater than the energy released from the toroidal part of the field via the instability. Now, since $\ddot{\boldsymbol{\xi}} = \sigma^2 \boldsymbol{\xi}$ where the growth rate $\sigma = v_A/\varpi$, the $m = 1$ mode releases an energy per unit volume equal to $\frac{1}{2}\rho\ddot{\xi}\xi = \frac{1}{2}\rho\sigma^2\xi^2$. As the axial field lines are stretched, they exert a restoring force $\frac{1}{4\pi}B_z^2\xi/l_z^2$ where l_z is the length scale of the instability in the z direction, which is equal to $1/k_z = \lambda_z/2\pi$. The work to be done therefore is $\frac{1}{8\pi}B_z^2\xi^2/l_z^2$, which for stability needs to be greater than the energy released so that

$$\frac{1}{8\pi} B_z^2 \xi^2 / l_z^2 > \frac{1}{2} \rho \sigma^2 \xi^2 \tag{55}$$

which can be rewritten as

$$\frac{B_z^2}{4\pi\rho} \frac{1}{l_z^2} > \sigma^2 \tag{56}$$

$$\frac{B_z^2}{4\pi\rho} \frac{\varpi^2}{l_z^2} > \frac{B_\varpi^2}{4\pi\rho} \tag{57}$$

$$\frac{B_z}{B_\varpi} > \frac{\lambda_z}{2\pi\varpi}. \tag{58}$$

The axial field therefore stabilises the shortest wavelengths, because it is the shortest wavelengths which have to bend the axial field lines to a greater degree for a given $|\boldsymbol{\xi}|$. Another way of expressing this result is that instability sets in for wavelengths greater than the distance over which a field line makes one full circle around the jet. This is often referred to in the literature as the *Kruskal-Shafranov* condition. Unfortunately, the ratio λ_z/ϖ could be very large in a narrow jet so this instability is expected to be present in essentially all collimated jets; work continues therefore on its non-linear development.

3.3 Angular momentum transport in discs

Accretion discs are found in many astrophysical contexts, such as during star formation, mass transfer in binary systems, and accretion of gas onto supermassive black holes. When matter is accreted, it falls deeper into a gravitational potential well and energy must be lost from the system, generally via radiation. However, it is not possible for a disc of material in isolation to accrete entirely onto the central object because of angular momentum conservation – angular momentum must be removed from the accreting material by transfer to other material which does not accrete. Therefore the lowest-energy end-state of a disc in isolation is to have an infinitesimally small amount of mass move outwards towards infinity and infinite specific angular momentum and the rest of the mass accreted onto the central object. This requires transport of angular momentum outwards.[6] Since all systems like to relax to energy minima, we should expect to find some mechanism operating in discs for the outward transport of angular momentum. According to the model of Shakura & Sunyaev (1973), the observational properties of accretion discs can be reproduced well by imagining there is some viscous stress equal in magnitude to some fraction α of the thermal pressure P. However, microscopic viscosity is far too small to account for values $\alpha \sim 0.1$ inferred from the observations; we turn our attention therefore to possible instabilities which could generate turbulence and the resultant 'turbulent viscosity'.

[6]Transport is outwards in the Langrangian sense of each fluid element transfering its angular momentum to its outside neighbour. Imagining the rate of change of angular momentum inside a fixed volume containing the central object (whose angular momentum is increasing as it is spun up and becomes more massive) and part of a steady-state disc, we see that net transport must be *inwards*; in other words, the inwards advection of angular momentum exceeds the transport though turbulent stress, albeit only by some small amount.

An example of a shear instability in a differentially rotating flow is the Rayleigh instability, an interchange instability. Imagine exchanging two annuli of equal volume and density between two radii ϖ and $\varpi+\delta\varpi$ which are initially moving with angular velocities Ω and $\Omega+\delta\Omega$. The kinetic energies before and after the exchange are E and $E+\delta E$:

$$
\begin{aligned}
E &= \frac{\rho V}{2}\left[\varpi^2\Omega^2 + (\varpi+\delta\varpi)^2(\Omega+\delta\Omega)^2\right], &(59)\\
E+\delta E &= \frac{\rho V}{2}\left[\varpi^2(\Omega+\delta\Omega)^2\left(\frac{\varpi+\delta\varpi}{\varpi}\right)^4 + (\varpi+\delta\varpi)^2\Omega^2\left(\frac{\varpi}{\varpi+\delta\varpi}\right)^4\right]. &(60)\\
\delta E &\approx 2\rho V\varpi^2\Omega^2\delta\ln\varpi^2\left[2+\frac{\delta\ln\Omega}{\delta\ln\varpi}\right] &(61)
\end{aligned}
$$

so that the stability condition is $q \equiv \partial\ln\Omega/\partial\ln\varpi > -2$, or in other words that the specific angular momentum $\varpi^2\Omega$ increases outwards. An accretion disc ($q=-3/2$) is therefore stable to this mechanism. However, another kind of shear instability, the *magneto-rotational instability* has the stability condition that $q>0$.

3.3.1 Magneto-rotational instability: physical mechanism and stability condition

The physical mechanism can be thought of in the following way. Consider two fluid elements, initially at the same radius, one above the other so that they are threaded by the same field line. They are given a perturbation in the radial direction, in opposite senses. Initially their angular momenta remain unchanged so that the element which has been perturbed inwards moves faster than the other. As it moves forwards with respect to the other, the field line connecting them is stretched so that the inner element pulls on the outer, transferring angular momentum to it. This causes the inner element to move further inwards and the outer element to move further outwards. In some sense the field line can be thought of as a spring which oscillates at a frequency v_A/λ where λ is the wavelength. Clearly the spring has to have a lower intrinsic frequency than that at which it is driven in order that it can be stretched instead of oscillating, meaning that $v_A/\lambda < \Omega/2\pi$. The field has therefore to be 'weak' in some sense for the instability to proceed.

Another way of imagining the instability is the following. Consider a fluid element at radius ϖ threaded by a vertical magnetic field B. In the rotating frame, in equilibrium the centrifugal force per unit mass at any radius $\Omega^2\varpi$ is balanced by some net inward-pointing force which in a disc is a combination of a pressure gradient and a gravitational force. The fluid element is now displaced a distance $\delta\varpi$ to a radius $\varpi+\delta\varpi$, while the magnetic field couples it to the other fluid elements on the same field line so that it retains its original angular velocity (angular momentum is transferred to it). The inward-pointing force at this new position is $(\Omega+\delta\Omega)^2(\varpi+\delta\varpi)$ but the centrifugal force is now $\Omega^2(\varpi+\delta\varpi)$. In addition there is a magnetic restoring force, giving the total (radial) force

$$
\begin{aligned}
F &= \Omega^2(\varpi+\delta\varpi) - (\Omega+\delta\Omega)^2(\varpi+\delta\varpi) - \frac{B^2}{4\pi\rho}\frac{\delta\varpi}{(\lambda_z/2\pi)^2},\\
&= \Omega^2\varpi\left[(1+\delta\ln\varpi) - (1+\delta\ln\Omega)^2(1+\delta\ln\varpi) - \frac{v_A^2}{\Omega^2}\frac{\delta\ln\varpi}{(\lambda_z/2\pi)^2}\right],\\
&= \Omega^2\varpi\left[-2\delta\ln\Omega - \frac{v_A^2}{\Omega^2}\frac{\delta\ln\varpi}{(\lambda_z/2\pi)^2}\right],\\
&= -\Omega^2\delta\varpi\left(2q+\kappa_z^2\right), &(62)
\end{aligned}
$$

where λ_z is the wavelength of the perturbation in the vertical direction, $v_A^2 = B^2/4\pi\rho$ is the Alfvén speed and $\kappa_z = k_z v_A/\Omega$ is a dimensionless wavenumber where $k_z = 2\pi/\lambda$ is the vertical wavenumber. For stability we need $F<0$ and therefore

$$2q+\kappa_z^2 > 0, \tag{63}$$

remembering that $v_A^2 = B^2/4\pi\rho$. For stability at all vertical wavenumbers, we clearly need $q > 0$, and in the unstable case the maximum unstable wavelength is given by $\kappa_z^2 = 2|q|$; as we shall see below, these estimates agree with the more rigorous treatment. Note that F is equal to the acceleration $\partial^2(\delta\varpi)/\partial t^2$; we can replace $\partial^2/\partial t^2$ with $-\omega^2$ where ω is the oscillation frequency (if real) or the growth rate (if imaginary). This gives, from (62),

$$-\omega^2 = -\Omega^2\left(2q + \kappa_z^2\right), \tag{64}$$

so that the growth rate will generally be comparable to the rotation frequency.

3.3.2 The dispersion relation

In our rotating fluid, locally we can look at a small corotating Cartesian volume at distance ϖ_0 from the centre, rotating at Ω_0, and change variables to $x = \varpi - \varpi_0$; the azimuthal direction is y and the vertical direction z. Since $\Omega(x) \approx \Omega_0 + qx\Omega_0/\varpi_0$, the bulk velocity in the rotating frame is $U \approx q\Omega_0 x$ (where $q = -3/2$ in the Keplerian case). In a rotating frame of reference, we must generally add to the momentum equation (25) both a Coriolis force $-2\rho\mathbf{\Omega}_0 \times \mathbf{v}$ (where $\mathbf{v}$ is the total velocity field) and a centrifugal force $\hat{\varpi}\rho\varpi\Omega_0^2$, but here we can drop the centrifugal term, the steady part of the Coriolis force coming from the basic flow U_y and the x component of the gravitational term because they cancel each other, leaving just the Coriolis force associated with any additional velocity field $\mathbf{u} = \mathbf{v} - U\hat{\mathbf{y}}$ on top of the basic flow, $-2\rho\mathbf{\Omega} \times \mathbf{u}$. In an accretion disc we also have the vertical part of the gravitational acceleration $-\rho GMz\varpi_0^{-3}\hat{\mathbf{z}} = -\rho z\mathbf{\Omega}_0^2\hat{\mathbf{z}}$, where the assumption is made that $z \ll \varpi_0$. This vertical stratification may be important in realistic discs, but we shall ignore it for the time being.

A general linear analysis of the MRI is quite involved, so we make two further simplifications: the initial magnetic field is of uniform strength and in the z direction so that $\mathbf{B} = B\hat{\mathbf{z}}$, and we consider only axisymmetric modes, meaning that $\partial/\partial y = 0$. The most unstable modes, at least for a weak field, will be incompressible and we assume that in the following. The perturbation to the magnetic field is $B\mathbf{b}$, so that $\mathbf{b}$ is dimensionless. We now linearise the MHD equations by subtracting equilibrium (zero-order) terms in the momentum equation, and keeping terms to first order in the perturbed quantities, ignoring the diffusive terms. Beginning with the momentum equation we have

$$\begin{aligned}
\partial_t\mathbf{u} + \mathbf{U}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{U} &= -\frac{1}{\rho}\nabla\delta P + \frac{1}{4\pi\rho}\left[(\nabla\times(B\mathbf{b}))\times\mathbf{B} + (\nabla\times\mathbf{B})\times(B\mathbf{b})\right] - 2\mathbf{\Omega}\times\mathbf{u},\\
\partial_t\mathbf{u} + \hat{\mathbf{y}}u_x\partial_x U &= -\frac{1}{\rho}\nabla\delta P + v_A^2(\nabla\times\mathbf{b})\times\hat{\mathbf{z}} - 2\Omega\hat{\mathbf{z}}\times\mathbf{u},\\
\partial_t\mathbf{u} &= -\frac{1}{\rho}\nabla\delta P + v_A^2(\nabla\times\mathbf{b})\times\hat{\mathbf{z}} - \Omega(2\hat{\mathbf{z}}\times\mathbf{u} + q\hat{\mathbf{y}}u_x),
\end{aligned} \tag{65}$$

noting that $\mathbf{U}\cdot\nabla\mathbf{u} = \mathbf{0}$ and $\nabla\times\mathbf{B} = \mathbf{0}$. We can now consider perturbations which vary in space and time as $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ so that $\partial_x = ik_x$ and so on:

$$-i\omega\mathbf{u} = -\frac{i\mathbf{k}}{\rho}\delta P + iv_A^2(\mathbf{k}\times\mathbf{b})\times\hat{\mathbf{z}} - \Omega(2\hat{\mathbf{z}}\times\mathbf{u} + q\hat{\mathbf{y}}u_x), \tag{66}$$

$$-i\psi\mathbf{u} = -i\boldsymbol{\kappa}\frac{\delta P}{\rho v_A} + iv_A(\boldsymbol{\kappa}\times\mathbf{b})\times\hat{\mathbf{z}} - 2\hat{\mathbf{z}}\times\mathbf{u} - q\hat{\mathbf{y}}u_x, \tag{67}$$

where we have introduced a dimensionless wavenumber $\boldsymbol{\kappa} \equiv \mathbf{k}v_A/\Omega$ and a dimensionless frequency $\psi \equiv \omega/\Omega$. The induction equation becomes (assuming $\nabla\cdot\mathbf{u} = 0$)

$$\partial_t(B\mathbf{b}) = \nabla\times(\mathbf{u}\times\mathbf{B} + \mathbf{U}\times(B\mathbf{b})). \tag{68}$$

$$\partial_t\mathbf{b} = \nabla\times(\mathbf{u}\times\hat{\mathbf{z}} + U\hat{\mathbf{y}}\times\mathbf{b}). \tag{69}$$

$$\partial_t\mathbf{b} = \partial_z\mathbf{u} + q\Omega\hat{\mathbf{y}}b_x, \tag{70}$$

$$-i\psi\mathbf{b} = i\kappa_z\mathbf{u}/v_A + q\hat{\mathbf{y}}b_x, \tag{71}$$

$$i\psi v_A\mathbf{b} = -\kappa_z\left(i\mathbf{u} - q\hat{\mathbf{y}}u_x/\psi\right), \tag{72}$$

where the last line was obtained by substituting for b_x back into the y component of the equation. This can be substituted into the momentum equation (multiplied by ψ) to give:

$$-i\psi^2\mathbf{u} = -i\boldsymbol{\kappa}\psi\frac{\delta P}{\rho v_{\rm A}} - \kappa_z\left[\boldsymbol{\kappa}\times(i\mathbf{u} - q\hat{\mathbf{y}}u_x/\psi)\right]\times\hat{\mathbf{z}} - \psi(2\hat{\mathbf{z}}\times\mathbf{u} + q\hat{\mathbf{y}}u_x). \tag{73}$$

Along with the incompressibility condition we have four equations and four quantities to be eliminated, δP and the three components of $\mathbf{u}$. Therefore:

$$\begin{aligned} -i\psi^2 u_x &= -i\kappa_x\psi\frac{\delta P}{\rho v_{\rm A}} - i\kappa_z(\kappa_z u_x - \kappa_x u_z) + 2\psi u_y, & (74)\\ -i\psi^2 u_y &= -\kappa_z^2(iu_y - qu_x/\psi) - (2+q)\psi u_x. & (75)\\ -i\psi^2 u_z &= -i\kappa_z\psi\frac{\delta P}{\rho v_{\rm A}}. & (76)\end{aligned}$$

From the third of these, we see that $-i\kappa_x\psi\delta P/\rho v_{\rm A} = -i\psi^2 u_z\kappa_x/\kappa_z$ which, using the incompressibility condition $\kappa_x u_x + \kappa_z u_z = 0$, replaces the first term on the right hand side of the x-component equation and reduces the set to two equations and two variables u_x and u_y:

$$\begin{aligned} -i\psi^2 u_x &= i\psi^2 u_x\frac{\kappa_x^2}{\kappa_z^2} - iu_x(\kappa_z^2+\kappa_x^2) + 2\psi u_y, & (77)\\ -i\psi^2 u_y &= -\kappa_z^2(iu_y - qu_x/\psi) - (2+q)\psi u_x. & (78)\end{aligned}$$

Collecting terms with u_x and u_y gives

$$\begin{aligned} \left(i\psi^2\frac{\kappa^2}{\kappa_z^2} - i\kappa^2\right)u_x + 2\psi u_y &= 0, & (79)\\ \left(\frac{q\kappa_z^2}{\psi} - (2+q)\psi\right)u_x + \left(i\psi^2 - i\kappa_z^2\right)u_y &= 0, & (80)\end{aligned}$$

where $\kappa^2 = \kappa_x^2 + \kappa_z^2$. Taking the determinant of $\mathbf{A}$ in $\mathbf{A}\cdot\mathbf{u} = \mathbf{0}$ to be zero we have the following quadratic in ψ^2:

$$\left(\psi^2\frac{\kappa^2}{\kappa_z^2} - \kappa^2\right)\left(\psi^2 - \kappa_z^2\right) + 2\psi\left(\frac{q\kappa_z^2}{\psi} - (2+q)\psi\right) = 0, \tag{81}$$

which we rearrange to:

$$\frac{\kappa^2}{\kappa_z^2}\psi^4 - 2\left(\kappa^2 + 2 + q\right)\psi^2 + \kappa_z^2\left(\kappa^2 + 2q\right) = 0. \tag{82}$$

Solving the quadratic in ψ^2 we have

$$\psi^2 = \frac{\kappa_z^2}{\kappa^2}\left[\kappa^2 + 2 + q \pm \sqrt{4\kappa^2 + (2+q)^2}\right]. \tag{83}$$

It is straightforward to show from this that the stability condition, i.e. the condition that both roots are postive, is $2q + \kappa^2 > 0$ which is identical to (63) derived above. For stability at all wavenumbers we require $q > 0$, i.e. an angular velocity increasing with radius. For $q < 0$, there is instability for a range of wavenumbers $0 < \kappa^2 < 2|q|$ and the maximum growth rate is $|\psi_{\rm max}| = |q|/2$ at a wavenumber $\kappa^2_{z,\,\rm max} = (1+q/4)|q|$ (and $\kappa_x = 0$).

3.3.3 MRI: remarks

We saw above that an accretion disc with a weak vertical magnetic field suffers an instability with some minimum wavelength. From vertical force balance, we see that $H \sim \varpi c_s / v_{\mathrm{Kep}}$ where H is the thickness of the disc, c_s is the sound speed and v_{Kep} is the Keplerian orbit speed. Evidently, if this instability is to be effective the minimum wavelength must be less than H, so that since $q = -3/2$ we see from (63) that

$$\frac{2\pi v_A}{\sqrt{3}\Omega} \lesssim \frac{\varpi c_s}{v_{\mathrm{Kep}}} \qquad \Longrightarrow \qquad \beta \gtrsim \frac{4\pi^2}{3\gamma}. \tag{84}$$

The magnetic field must not therefore become too strong relative to the thermal pressure. Also, a strong magnetic field would be buoyantly unstable.

We looked here at the axisymmetric modes where the unperturbed field is parallel to the rotation axis; in reality we would expect the field to be dominated by its azimuthal component since this is the direction of the shear. The instability does operate on a purely azimuthal field but the modes are non-axisymmetric and the dispersion relation is more complex. Numerical nonlinear analysis of this instability shows a steady-state dynamo effect as well as the desired angular momentum transport with a Shakura-Sunyaev α parameter of roughly the right magnitude, 0.01 to 0.1.

Finally, note that in the non-magnetic limit $v_A \to 0$ the dispersion relation becomes

$$\frac{\omega^2}{\Omega^2} = 2(2+q)\frac{k_z^2}{k^2}, \tag{85}$$

so that we recover the Rayleigh stability condition $q > -2$ derived at the beginning of this section.

4 Problems

4.1 Equilibria in non-convective stars

An upper main-sequence star is radiative apart from a small convective core (which can be ignored); the velocity field is therefore zero everywhere. The star contains a magnetic field in ‘dynamic equilibrium’, meaning that the only evolution is due to the diffusive terms in the MHD equations and that the equilibrium can be considered static on the dynamic (Alfvén or sound-crossing) timescale.

(a) The equilibrium is axisymmetric. Using cylindrical coordinates (ϖ, ϕ, z) show that the field can be expressed as the sum of poloidal and toroidal components as

$$\varpi \mathbf{B} = \boldsymbol{\nabla}\psi \times \hat{\boldsymbol{\phi}} + F\hat{\boldsymbol{\phi}} \tag{86}$$

where $\hat{\boldsymbol{\phi}}$ is the azimuthal unit vector and ψ and F are functions of ϖ and z, and that the poloidal and toroidal fields are associated with toroidal and poloidal currents, respectively. By considering azimuthal force balance, show that contours of F in the meridional plane are parallel to those of ψ, i.e. that $F = F(\psi)$. [Hint: show that $(\boldsymbol{\nabla}\psi) \times (\boldsymbol{\nabla}F) = \mathbf{0}$.]

(b) In contrast to many other astrophysical objects, the interiors of stars have $\beta \gg 1$. Show that an arbitrary axisymmetric magnetic field configuration can be added to a star and an equilibrium constructed by making small adjustments to the pressure and density fields. Ignore surface effects as well as any changes in the gravitational potential. Argue that this result can be generalised to non-axisymmetric fields.

(c) Assuming that the equilibrium is structured on length scales comparable to the size of the star, use the Spitzer conductivity to make an order-of-magnitude estimate of the diffusive timescale on which the equilibrium evolves, and compare this to the main-sequence lifetime of the star and to the dynamic (Alfvén) timescale for a magnetic field of 1kG.

4.2 Kink instability in solar coronal loops

A coronal loop of magnetic field links two sunspots of opposite polarity. According to one theory of solar flares, reconnection events are triggered when a loop crosses the kink instability threshold.

(a) In a straight flux tube, the growth rate of the kink $m = 1$ instability is comparable to the Alfvén frequency, defined as $\omega_{\rm A} \equiv v_{\rm A}^{\phi}/\varpi$ where $v_{\rm A}^{\phi}$ is the Alfvén speed associated with the azimuthal component B_ϕ of the magnetic field and ϖ is the cylindrical radius. An axial field component B_z can stabilise the field by providing an extra tension against which the instability must do work. In a tube where $B_\phi \propto \varpi$, by consideration of the force balance perpendicular to the tube axis show that the stability criterion is

$$k_z B_z > \frac{B_\phi}{\varpi}, \tag{87}$$

where $k_z = 2\pi/\lambda_z$ is the wavenumber of the instability. [Hint: first show what force is required to produce the growth rate $\omega_{\rm A}$ and then equate this to the restoring Lorentz force from the axial field.]

(b) We can make the approximation that the stability criterion for a curved flux tube does not differ enormously from that in a straight tube. The field between the two sunspots is initially untwisted, i.e. $B_\phi = 0$ and then one of the spots slowly rotates. Calculate the energy required to twist one sunspot up to the instability threshold by consideration of the Lorentz force in a shallow layer in the sunspot where B_ϕ changes from to its value in the coronal loop. As the tube is twisted, it passes quasi-statically through a series of force-free equilibria; show that the α parameter in the force-free equation $\boldsymbol{\nabla} \times \mathbf{B} = \alpha \mathbf{B}$ increases from 0 up to some value. When the instability threshold is passed, the field in the corona relaxes back to the lowest energy state, i.e. the curl-free field $\alpha = 0$; make an estimate of the energy released in the flare and equate this to the sunspot-rotating energy calculated earlier.