Perturbations and critical behavior in the self-similar gravitational collapse of a massless scalar field

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This paper studies the perturbations of the continuously self-similar critical solution of the gravitational collapse of a massless scalar field (Roberts solution). The perturbation equations are derived and solved exactly. The perturbation spectrum is found to be not discrete, but occupying a continuous region of the complex plane. The renormalization group calculation gives the value of the mass-scaling exponent $\beta = 1$. [S0556-2821(97)02422-3]

I. INTRODUCTION

The numerical calculation of the spherically symmetric gravitational collapse of a massless scalar field by Choptuik [1] and subsequent results for different matter models and symmetries (see, for example, Refs. [2–5]) spectacularly demonstrate that critical phenomena occur in gravitational collapse, and that the near-critical behavior is universal in some important aspects.

There are two possible late-time outcomes of gravitational collapse, distinguished by whether or not a black hole is formed in the course of evolution. Which end state is realized depends on initial conditions, specified by a control parameter p that characterizes the gravitational interaction strength in the ensuing evolution. For $p < p^*$ (subcritical solution) the gravitational field is too weak to form a black hole, while for $p > p^*$ (supercritical solution) a black hole is produced. The critical solution, corresponding to a control parameter value p^* just at the threshold of black hole formation, acts as an intermediate attractor for nearby initial conditions, and has an additional symmetry: discrete or continuous self-similarity, also referred to as echoing. For supercritical evolution there is usually no mass gap, and the mass of the resultant black hole is described by the mass-scaling relation

$$M_{\mathrm{BH}}(p) \propto |p - p^*|^\beta, \tag{1}$$

where β is mass-scaling exponent.

General gravitational collapse is a very difficult problem, even if restricted to the case of spherical symmetry. As a result, most studies of critical behavior resort to numerical calculations at some stage, with few fully analytical solutions known. Finding critical solutions is made easier by the fact that they have additional symmetry; requiring the solution to be continuously self-similar can simplify the problem enough to make it solvable for simpler matter models. Both the stability of the critical solution and the exact mass-scaling exponent can then be determined by linear perturbation analysis, as suggested by Evans and Coleman [3] and carried out in Refs. [6–8].

In this paper we consider an analytical continuously self-similar solution of gravitational collapse of a massless scalar field constructed by Roberts [9]. Originally it was intended as a counterexample to cosmic censorship conjecture, but it was later rediscovered in context of the critical gravitational collapse by Brady [10] and Oshiro, Nakamura, and Tomimatsu [11]. Although this solution does exhibit critical behavior, there are two problems. First, as any self-similar solution, it is not asymptotically flat, and the mass of the black hole grows infinitely. This can be prevented if we cut the self-similar solution away from the black hole and glue it to some other solution to make the spacetime asymptotically flat and of finite mass, which was in fact done in Ref. [12]. A second, and bigger, problem is that growing non-self-similar modes will dominate the mass-scaling exponent if they exist, but they are not accounted for by the self-similar solution.

This paper addresses the second problem by doing linear perturbation analysis of the Roberts solution. We calculate the mass-scaling exponent β and analyze the stability of the critical solution. The remarkable feature of this model is that it allows exact analytical treatment.

II. SELF-SIMILAR SOLUTION

We begin by presenting a continuously self-similar spherically symmetric solution of the gravitational collapse of a massless scalar field (Roberts solution). It is most easily derived in null coordinates. Therefore we write the metric as

$$dS^2 = -2e^{2\Sigma}du\,dv + R^2 d\Omega^2, \tag{2}$$

where Σ and R are functions of both u and v. The coordinates u, v can be reparametrized as $u(\bar{u})$, $v(\bar{v})$, so coordinate choice freedom is given by two functions of one variable each; it can be fixed by setting $\Sigma = 0$ at two null hypersurfaces $v = \mathrm{const}$ and $u = \mathrm{const}$. With this choice of metric the Einstein scalar field equations become

$$(R^2)_{,uv} + e^{2\Sigma} = 0, \tag{3a}$$

$$2\Sigma_{,uv} - \frac{2R_{,u}R_{,v}}{R^2} - \frac{e^{2\Sigma}}{R^2} + 2\Phi_{,u}\Phi_{,v} = 0, \tag{3b}$$

$$R_{,vv} - 2\Sigma_{,v}R_{,v} + R(\Phi_{,v})^2 = 0, \tag{3c}$$

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$$R_{,uu} - 2\Sigma_{,u}R_{,u} + R(\Phi_{,u})^2 = 0, \qquad (3d)$$

$$\Phi_{,uv} + \frac{R_{,v}\Phi_{,u}}{R} + \frac{R_{,u}\Phi_{,v}}{R} = 0. \qquad (3e)$$

The first four equations (3a)–(3d) are the Einstein equations for the metric (2), and Eq. (3e) is the scalar wave equation $\Box\Phi = 0$. Equations (3c) and (3d) are constraints. Note that in the case of a massless scalar field the wave equation follows from the Einstein equations, so one of the equations (3a), (3b), and (3e) is redundant.

We turn on the influx of the scalar field at the advanced time $v = 0$, so that the spacetime is Minkowskian to the past of this surface. The initial conditions for system (3) are specified there by the continuity of the solution. We use coordinate freedom to set $\Sigma|_{v=0} = 0$.

An assumption that the collapse is continuously self-similar, i.e., that there exists a vector field ξ such that $\pounds_\xi g_{\mu\nu} = 2g_{\mu\nu}$, allows the Einstein-scalar equations (3) to be solved analytically. The critical solution is given by

$$\Sigma_0 = 0,$$

$$R_0 = \sqrt{u^2 - uv},$$

$$\Phi_0 = \frac{1}{2}\ln\left[1 - \frac{v}{u}\right], \qquad (4)$$

and is a member of a one-parameter family of self-similar solutions

$$\Sigma = 0,$$

$$R = \sqrt{u^2 - uv - pv^2},$$

$$\Phi = \frac{1}{2}\ln\left[-\frac{2pv + u(1 - \sqrt{1+4p})}{2pv + u(1 + \sqrt{1+4p})}\right] + \varphi(p). \qquad (5)$$

Above $\varphi(p)$ is constant component of a scalar field, chosen so that $\Phi|_{v=0} = 0$, and the solution converges to the critical solution as p goes to zero:

$$\varphi(p) = -\frac{1}{2}\ln\left[\frac{\sqrt{1+4p}-1}{\sqrt{1+4p}+1}\right] = -\frac{1}{2}\ln p + p + O(p^2). \qquad (6)$$

For values $p > 0$ of control parameter, a black hole is formed; for $p < 0$ the field disperses, and the spacetime is flat again in region $u > 0$. (Our choice of sign of p is opposite to the one in [10]. Also note that the constant field term is omitted there.)

The existence and position of the apparent horizon is given by condition $A(u,v) = 0$, where

$$A(u,v) = u^{;\mu}_{;\mu} = R_{,v}. \qquad (7)$$

Due to the spherical symmetry of the problem it is possible to introduce a local mass $M(u,v)$ by

$$1 - \frac{2M}{R} = 2g^{uv}R_{,u}R_{,v}. \qquad (8)$$

The mass contained inside the apparent horizon is the mass of the black hole

$$M_{BH} = \frac{1}{2}R_{AH}, \qquad (9)$$

where R_{AH} is the radius of apparent horizon given by Eq. (7).

III. LINEAR PERTURBATIONS

We now apply the perturbation formalism [6–8]. It is convenient to analyze perturbations to the self-similar solution in new coordinates natural to the problem. For this purpose we introduce a ''spatial'' coordinate x and a ''scaling variable'' s,

$$x = \frac{1}{2}\ln\left[1 - \frac{v}{u}\right], \quad s = -\ln(-u), \qquad (10)$$

with the inverse transformation

$$u = -e^{-s}, \quad v = e^{-s}(e^{2x} - 1). \qquad (11)$$

The signs are chosen to make the arguments of the logarithm positive in the region of interest ($v > 0$, $u < 0$), where the field evolution occurs.

In these coordinates the metric (2) becomes

$$dS^2 = 2e^{2(\Sigma + x - s)}[(1 - e^{-2x})ds^2 - 2dsdx] + R^2d\Omega^2, \qquad (12)$$

and the critical solution (4) is simply

$$\Sigma_0 = 0, \quad R_0 = e^{x-s}, \quad \Phi_0 = x. \qquad (13)$$

We now consider small perturbations of critical solution (13). We fix the gauge by requiring that perturbations preserve the form of the metric (2). After imposing this gauge there still exists coordinate choice freedom, which we use to set $\Sigma|_{x=0} = 0$. We will deal with remaining coordinate freedom in v later.

Quite generally, the perturbation modes can be written as

$$\Sigma = \Sigma_0 + p\,\sigma(x)e^{ks},$$

$$R = R_0 + p\,r(x)e^{x-s}e^{ks},$$

$$\Phi = \Phi_0 + p\,\phi(x)e^{ks}, \qquad (14)$$

which amounts to doing Laplace transformation on general perturbations dependent on both x and s. It is understood that there can be many perturbation modes, with distinct σ, r, and ϕ for each (possibly complex) k in the perturbation spectrum. To recover a general perturbation we must take the sum of these modes. Modes with Re $k > 0$ (called relevant modes) grow and eventually lead to black hole formation, while modes with Re $k < 0$ decay and are irrelevant.

Let us discuss how the mass-scaling exponent β is related to the perturbation spectrum. The function A, defined by Eq. (7), takes the form

$$A(x,s,p)=A_0(x)-pa(x)e^{ks}, \qquad (15)$$

where A_0 is the self-similar background term. Recall that A vanishes at the apparent horizon. Even though there is no apparent horizon in the critical solution, i.e., $A_0(x)>0$, the exponentially growing perturbation will eventually make A zero, leading to black hole formation. The mode with the largest eigenvalue, $\kappa=\max\{\mathrm{Re}\,k\}$, will clearly dominate, giving the position of the apparent horizon $R_{\mathrm{AH}}(x_{\mathrm{AH}})$ by

$$A_0(x_{\mathrm{AH}})=pa(x_{\mathrm{AH}})\frac{e^{\kappa x_{\mathrm{AH}}}}{(R_{\mathrm{AH}})^\kappa}. \qquad (16)$$

Here we made use of the fact that $R=e^{x-s}$ to the zeroth order in p. Therefore the dependence of the black hole mass on p, taken to be the control parameter, is

$$M_{\mathrm{BH}}\propto R_{\mathrm{AH}}\propto p^{1/\kappa}. \qquad (17)$$

This is precisely the mass-scaling relation (1) with exponent $\beta=1/\kappa$. For a more detailed discussion of this approach and its validity see Ref. [7].

Perturbing the Einstein-scalar equations (3) by a mode (14), we obtain a system of linear partial differential equations. After a change of variables (u,v) to (x,s), the s dependence separates, and we are left with the following system of linear ordinary differential equations for σ, r, and ϕ in the independent variable x:

$$(1-e^{-2x})r''+2(k-e^{-2x})r'+4(k-1)r+4\sigma=0, \qquad (18a)$$

$$(1-e^{-2x})\sigma''+2(k+e^{-2x})\sigma'-4\sigma+2(1-e^{-2x})\phi'+2k\phi+2e^{-2x}r'+2(2-k)r=0, \qquad (18b)$$

$$r''-2\sigma'+2\phi'=0, \qquad (18c)$$

$$(e^{2x}-2+e^{-2x})r''-4k(1-e^{2x})r'-4k(1-ke^{2x})r+2(e^{2x}-e^{-2x})\sigma'$$
$$+4k(1+e^{2x})\sigma+2(e^{2x}-2+e^{-2x})\phi'-4k(1-e^{2x})\phi=0, \qquad (18d)$$

$$(1-e^{-2x})\phi''+2k\phi'+2k\phi+2(1-e^{-2x})r'+2kr=0. \qquad (18e)$$

Here a prime denotes the derivative with respect to x.

Boundary conditions for the system (18) are specified at $x=0$ and $x=\infty$. We require that perturbations grow slower than background at large x (i.e., that σ and r are bounded, and ϕ grows slower than x) for the perturbation expansion to be valid. Because the solution must be continuous at the hypersurface $v=0$, we have $r(0)=\phi(0)=0$. Since we used coordinate freedom in u to set Σ to zero on that surface, we also have $\sigma(0)=0$. Thus, the boundary conditions are

$$\sigma(0)=r(0)=\phi(0)=0,$$

$$\sigma(x),r(x),\phi(x) \text{ grow slowly as } x\rightarrow\infty. \qquad (19)$$

Equations (18) together with boundary conditions (19) constitute our eigenvalue problem.

The infinitesimal coordinate transformation $v\mapsto v-2\zeta(v)$ corresponding to the remaining coordinate freedom in v will give rise to unphysical gauge modes

$$\delta\Sigma=\zeta'(v)=\zeta'[e^{-s}(e^{2x}-1)],$$

$$\delta R=\frac{\zeta(v)}{(1-v/u)^{1/2}}=e^{-x}\zeta[e^{-s}(e^{2x}-1)],$$

$$\delta\Phi=\frac{\zeta(v)}{(u-v)}=-e^{s-2x}\zeta[e^{-s}(e^{2x}-1)]. \qquad (20)$$

Since on a $v=0$ hypersurface $\delta\Sigma$, δR, and $\delta\Phi$ must vanish identically, $\zeta(0)=\zeta'(0)=0$, so a Taylor expansion of $\zeta(v)$ around zero starts from the v^2 term. Therefore these gauge modes correspond to negative eigenvalues k, and are irrelevant.

IV. SOLUTION OF PERTURBATION EQUATIONS

Observe that there is an obvious solution of perturbation equations (18) which can be obtained from the self-similar solution (5) by expanding around $p=0$:

$$\Sigma=0,$$

$$R=\sqrt{u^2-uv}\left(1-\frac{p}{2}\frac{v^2}{u^2-uv}\right)=e^{x-s}(1-2p\,\sinh^2 x),$$

$$\Phi=\frac{1}{2}\ln\left[1-\frac{v}{u}\right]+\frac{p}{2}\frac{v^2-2uv}{u^2-uv}=x+p\,\sinh 2x. \qquad (21)$$

Clearly, the self-similar mode has eigenvalue $k=0$, hence any relevant mode will dominate the calculation of the mass-scaling exponent.

The perturbation equations (18) inherit their structure from general Einstein-scalar equations (3); Eqs. (18a), (18b), and (18e) are dynamical equations, while Eqs. (18c) and (18d) are constraints. Equation (18b) is redundant,

$$(18b) = [(18a)' - 2(18a) - (1 - e^{-2x})(18c)' - 2(e^{-2x} + k - 1)(18c) + 2(18e)]/2, \tag{22}$$

so it will be discarded. The simple integrable form of Eq. (18c) allows elimination of one of the unknowns, say σ, from the other equations by

$$\sigma = r'/2 + \phi + C. \tag{23}$$

Thus we are left with two second-order ordinary differential equations (ODE's) for r and ϕ:

$$(1 - e^{-2x})r'' + 2(k + 1 - e^{-2x})r' + 4(k-1)r + 4\phi + 4C = 0, \tag{24a}$$

$$(1 - e^{-2x})\phi'' + 2k\phi' + 2k\phi + 2(1 - e^{-2x})r' + 2kr = 0 \tag{24b}$$

with boundary conditions (19), and one constraint (18d), which we equivalently rewrite as first-order ODE with the use of Eq. (18a):

$$(k-2)(1 - e^{-2x})r' + 2(k^2 - ke^{-2x} - 2k + 2)r + 2(1 - e^{-2x})\phi' + 4(k-1)\phi + 2C(k + ke^{-2x} - 2) = 0. \tag{24c}$$

Equations (24a) and (24b) can be decoupled with introduction of the auxiliary function $z = r - \phi$, reducing to

$$(1 - e^{-2x})z'' + 2kz' + 2(k-2)z = -4C, \tag{25a}$$

$$(1 - e^{-2x})r'' + 2(k + 1 - e^{-2x})r' + 4kr = 4(z - C). \tag{25b}$$

We convert this into algebraic form using a change of variable

$$y = e^{2x}, \quad x = \frac{1}{2}\ln y. \tag{26}$$

Finally, Eqs. (25) and constraint (24c) become

$$y(1-y)\frac{d^2z}{dy^2} + [1 - (k+1)y]\frac{dz}{dy} - (k/2 - 1)z = C, \tag{27a}$$

$$y(1-y)\frac{d^2r}{dy^2} + [2 - (k+2)y]\frac{dr}{dy} - kr = C - z, \tag{27b}$$

$$2y(1-y)\frac{dz}{dy} - 2y(k-1)z - ky(1-y)\frac{dr}{dy} - k(1 - ky)r$$
$$+ C[k + (k-2)y] = 0, \tag{27c}$$

and boundary conditions (19) are specified at $y = 1$ and $y = \infty$ by

$$z(1) = r(1) = 0, \quad \frac{dr}{dy}(1) + C = 0,$$

$$z(y), r(y) \text{ grow slowly as } y \to \infty. \tag{28}$$

''Grow slowly'' means that $r(y)$ must be bounded, and $z(y)$ grows at most logarithmically, with $y\,dr/dy + r - z$ bounded, as $y \to \infty$. The last condition follows from σ being bounded at infinity.

Equation (27c) is indeed a constraint, since $d(27c)/dy = (27a) - k(27b)$, and is automatically satisfied for all y if

$$(k-1)[kr(1) - 2z(1) + 2C] = 0 \tag{29}$$

is satisfied initially at $y = 1$. Imposing initial conditions (28), the constraint (29) yields

$$C(k-1) = 0, \tag{30}$$

i.e., $C = 0$ unless $k = 1$.

The perturbation problem is now reduced to the study of decoupled ordinary differential equations (27a) and (27b) and constraint (30) with boundary conditions (28). We proceed solving it. Observe that Eqs. (27a) and (27b) are hypergeometric equations. Their properties are well known and described in a number of books on differential equations (see, for example, [13]), so the system (27) can be analytically solved and the perturbation spectrum exactly determined.

We start with Eq. (27a). It is a homogeneous (except when $k = 1$) hypergeometric equation

$$y(1-y)\frac{d^2z}{dy^2} + [c - (a + b + 1)y]\frac{dz}{dy} - abz = 0 \tag{31}$$

with coefficients

$$c = 1, \quad a + b = k, \quad ab = k/2 - 1,$$

$$a = 1/2(k - \sqrt{k^2 - 2k + 4}), \quad b = 1/2(k + \sqrt{k^2 - 2k + 4}). \tag{32}$$

It has singular points at $y = 0, 1, \infty$, and its general solution is a linear combination of any two different solutions from the set

$$z_1 = F(a, b; k; 1 - y),$$

$$z_2 = (1-y)^{1-k}F(1 - a, 1 - b; 2 - k; 1 - y),$$

$$z_3 = (-y)^{-a}F(a, a; a + 1 - b; y^{-1}),$$

$$z_4 = (-y)^{-b}F(b, b; b + 1 - a; y^{-1}), \tag{33}$$

where $F(a,b;c;y)$ is the hypergeometric function; $F(a,b;c;0)=1$. Any three of the functions (33) are linearly dependent with constant coefficients, for example,

$$z_2=\frac{\Gamma(2-k)\Gamma(b-a)}{\Gamma(1-a)^2}e^{-i\pi(1-b)}z_3$$

$$+\frac{\Gamma(2-k)\Gamma(a-b)}{\Gamma(1-b)^2}e^{-i\pi(1-a)}z_4. \qquad (34)$$

Functions z_1,z_2 and z_3,z_4 have the asymptotic behavior

$$z_1\simeq1,\quad z_2\simeq(1-y)^{1-k}\quad\text{near }y=1,$$

$$z_3\simeq(-y)^{-a},\quad z_4\simeq(-y)^{-b}\quad\text{near }y=\infty. \qquad (35)$$

Imposing boundary conditions (28) on solutions of Eqs. (27a) and (27b) leads to perturbation spectrum. It turns out that it is completely defined by the boundary value problem for z. As we will show, the boundary conditions for r are satisfied automatically if they are satisfied for z. When imposing boundary conditions, we use asymptotics (35) to analyze the perturbation spectrum.

The case when $k=1$ is special. In this case the solution (33) degenerates and z_2 defined by Eq. (33) coincides with z_1 identically. In order to deal with this situation we continue to use z_1 as earlier, but denote by z_2 a solution independent of z_1. It is easy to verify that it is logarithmically divergent at $y=1$. Note also that for $k=1$ we can have $C\neq0$, so that Eq. (27a) is inhomogeneous, and we must add the particular solution $z=C$ to the general solution above.

First, let us examine boundary conditions at $y=1$. Observe that at $y=1$, z_2 diverges as a power for Re $k>1$, logarithmically for $k=1$, does not have a limit for Re $k=1$, Im $k\neq0$, and converges (to zero) only for Re $k<1$. The solutions that satisfy the initial condition $z(1)=0$ are

$$z=cz_2,\quad\text{Re }k<1,$$

$$z=C(1-z_1),\quad k=1. \qquad (36)$$

From Eq. (34) we see that z_2 (and z_1) is connected to z_3,z_4 by a linear relation with nonzero coefficients, so the boundary conditions at infinity will only be satisfied if both z_3,z_4 do not blow up, i.e., if Re $a>0$, Re $b>0$. Re b is positive by construction, while the region Re $a>0$ of the complex k plane is divided from the region Re $a<0$ by the curve Re $a=0$. Written in terms the real and imaginary parts of k, it has the form

$$(\text{Im }k)^2=\frac{\text{Re }k(2-\text{Re }k)}{1-(2\text{ Re }k)^{-1}}, \qquad (37)$$

and is shown in Fig. 1: to the left of the solid curve Re $a<0$, and to the right Re $a>0$.

Combining restrictions on k placed by boundary conditions at $y=1$ and $y=\infty$ we see that to satisfy all boundary conditions (28), k must lie in the shaded region K of the complex plane, and for any $k\in K$,

$$z=(1-y)^{1-k}F(1-a,1-b;2-k;1-y) \qquad (38)$$

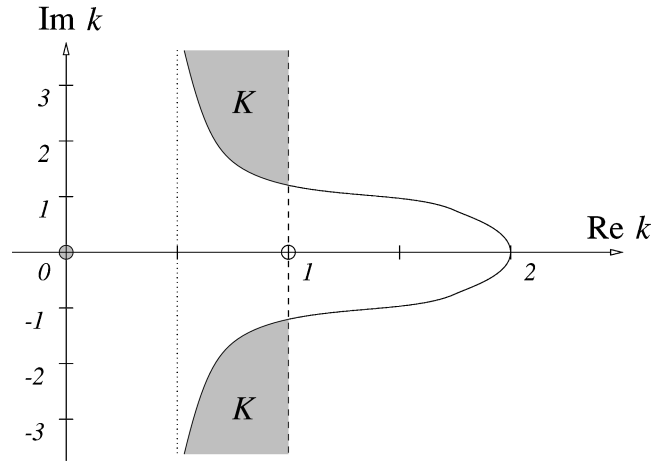is a solution of Eq. (27a) with boundary conditions (28).

FIG. 1. Complex perturbation spectrum. Values of k to the left of the solid line are prohibited by the boundary conditions at infinity, to the right of the broken line by the initial conditions at $y=1$. Values in the region of intersection (the shaded region K) are allowed, and constitute the perturbation spectrum.

Now we turn our attention to Eq. (27b) for r. It is also hypergeometric, with coefficients

$$c=2,\quad a=1,\quad b=k. \qquad (39)$$

The general solution of the homogeneous equation is

$$r_1=y^{-1},\quad r_2=y^{-1}(1-y)^{1-k}, \qquad (40)$$

and the solution of inhomogeneous equation is easily constructed from Eq. (40) by

$$r=-\frac{1}{y(1-k)}\left[\int_1^y z(\tilde{y})d\tilde{y}-(1-y)^{1-k}\int_1^y\frac{z(\tilde{y})d\tilde{y}}{(1-\tilde{y})^{1-k}}\right]. \qquad (41)$$

The limits of integration are chosen so that the boundary conditions at $y=1$ are satisfied. Let us verify that the boundary conditions for r do not place additional restrictions on perturbation spectrum. Asymptotic (35) and Eq. (41) give the behavior

$$r\simeq-\frac{y^{-1}(1-y)^{2-k}}{2-k}\quad\text{near }y=1. \qquad (42)$$

Clearly, $r(1)=dr/dy(1)=0$. Also, r is bounded at infinity since z, for the constraint (27c), requires $kdr/dy(\infty)-2dz/dy(\infty)=0$. Therefore, Eqs. (38) and (41) for $k\in K$ is a solution of perturbation equations (27) with boundary conditions (28), and the region K is the perturbation spectrum, which turns out to be continuous.

It should be emphasized that the calculations of perturbation spectrum were made using a special choice of gauge preserving the form of the metric (2). However, one can show that z coincides with gauge-independent quantity in our gauge, and hence the eigenvalue problem for the perturbation spectrum is gauge invariant.

V. CONCLUSION

In this paper we used linear perturbation analysis to investigate the critical behavior in the gravitational collapse of a massless scalar field. We have perturbed the continuously self-similar critical solution (Roberts solution), and solved the perturbation equations exactly. The perturbation spectrum was found to be not discrete, but continuous, and occupying the region of the complex plane

$$\frac{1}{2} < \text{Re } k < 1, \quad |\text{Im } k| > \sqrt{\frac{\text{Re } k(2 - \text{Re } k)}{1 - (2 \text{ Re } k)^{-1}}}, \quad (43)$$

which, according to [7], suggests nonuniversality of critical behavior for different ingoing wave packets. The complex oscillatory modes might lead to decay of a continuously self-similar solution (4) into discrete self-similar choptuon observed in [1].

The eigenvalues k approach sup Re $k = 1$, which corresponds to the mass-scaling exponent $\beta = 1$. This is different from the exponent $\beta = 1/2$ found in [10–12] using the self-similar solution.

The remarkable feature of the massless scalar field model is that it allows exact analytical treatment of perturbation modes as well as the critical solution. Because of its simplicity, the model considered in this paper can be used for further investigation of critical phenomena using perturbation analysis. In particular, one might hope that more general, for example, nonspherical, perturbations can be treated in the similar fashion.

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