MAX-PLANCK-GESELLSCHAFT

Max Planck Institute
for Astrophysics

Reconstructing signals from noisy data with unknown signal and noise covariance

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with
Georg Robbers and Torsten A. Enßlin

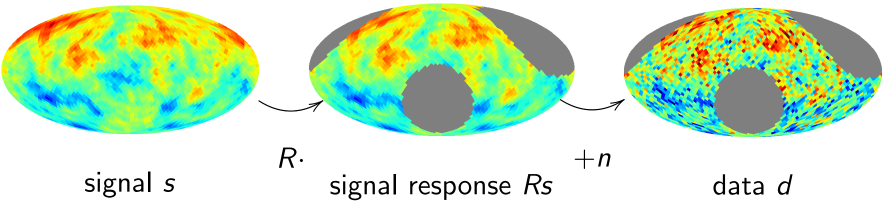
32nd International Workshop on Bayesian Inference and Maximum Entropy Methods in Science and Engineering
IPP Garching, 2012-07-19

An application of *Information Field Theory*
(Torsten Enßlin's talk at 11:50)

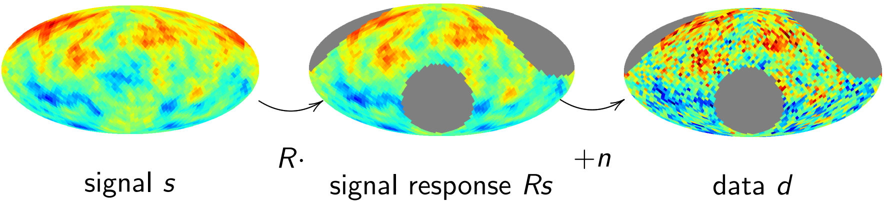
Outline

1. **The problem**
   a general setup
2. **The method**
   deriving the *extended critical filter*
3. **The application**
   making a map of the Galactic Faraday depth

The problem



$$\boxed{d = Rs + n}$$

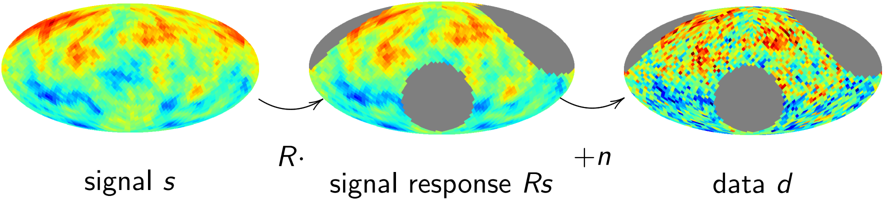


$$\boxed{d = Rs + n}$$

$$\mathcal{P}(s) = \mathcal{G}(s, S)$$

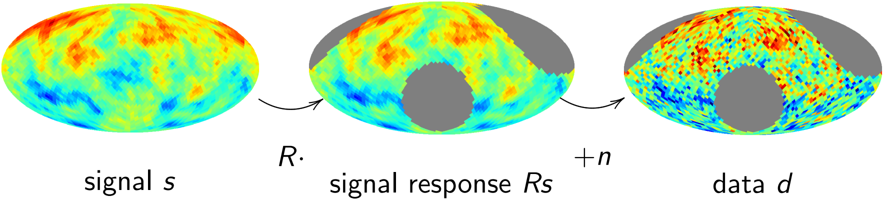
$$\mathcal{P}(n) = \mathcal{G}(n, N)$$

$$\mathcal{G}(s, S) = \frac{1}{|2\pi S|^{1/2}} \exp\left[\frac{1}{2} s^\dagger S^{-1} s\right]$$



$$d = Rs + n$$

$$m = \int \mathcal{D}s \; s \; \mathcal{P}(s|d)$$

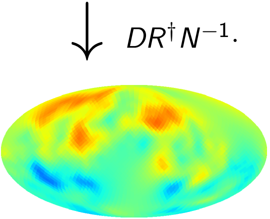


Wiener Filter

$$d = Rs + n$$

$$m = \int \mathcal{D}s \; s \; \mathcal{P}(s|d)$$

$$m = Dj, \;\text{where}\quad \begin{array}{l} j = R^\dagger N^{-1} d \\ D = \left(S^{-1} + R^\dagger N^{-1} R\right)^{-1} \end{array}$$



$$S(\hat{n}, \hat{n}') = \int \mathcal{D}s \; s(\hat{n}) s(\hat{n}') \mathcal{P}(s)$$

$$\Rightarrow S_{(\ell m),(\ell' m')} = \int \mathcal{D}s \; s_{\ell m} s^*_{\ell' m'} \mathcal{P}(s)$$

$$
\begin{aligned}
S(\hat{n}, \hat{n}') &= \int \mathcal{D}s \; s(\hat{n}) s(\hat{n}') \mathcal{P}(s) \\
&= S(\hat{n} \cdot \hat{n}') \\
\Rightarrow S_{(\ell m),(\ell' m')} &= \int \mathcal{D}s \; s_{\ell m} s^*_{\ell' m'} \mathcal{P}(s) \\
&= \delta_{\ell\ell'} \delta_{mm'} C_\ell
\end{aligned}
$$

$\hookrightarrow$ power spectrum

$$
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\end{aligned}
$$

$\hookrightarrow$ power spectrum

$$
N_{ij} = \delta_{ij} \eta_i \sigma_i^2
$$

$\hookrightarrow$ error variance correction factor

The method

$$S = \sum_{k=0}^{k_{\max}} p_k S_k \qquad N = \sum_{i=0}^{i_{\max}} \eta_i N_i$$

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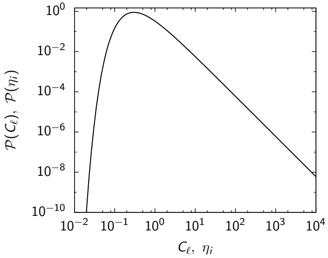
assume priors for parameters

$$\mathcal{P}\left((p_k)_k\right) = \prod_{k=0}^{k_{\max}} \frac{1}{q_k \Gamma(\alpha_k - 1)} \left(\frac{p_k}{q_k}\right)^{-\alpha_k} \exp\left(-\frac{q_k}{p_k}\right)$$

$$\mathcal{P}\left((\eta_i)_i\right) = \prod_{i=0}^{i_{\max}} \frac{1}{q_i \Gamma(\alpha_i - 1)} \left(\frac{\eta_i}{q_i}\right)^{-\alpha_i} \exp\left(-\frac{q_i}{\eta_i}\right)$$



$\Rightarrow$ marginalize over all possible parameters

Problem: $\mathcal{P}(s|d)$ is non-Gaussian.

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Solution: Find Gaussian $\mathcal{G}(s-m,D)$, that best approximates $\mathcal{P}(s|d)$.

- Minimize Kullback-Leibler divergence

$$d_{\mathrm{KL}} = \int \mathcal{D}s \; \mathcal{G}(s-m,D) \; \log\left(\frac{\mathcal{G}(s-m,D)}{\mathcal{P}(s|d)}\right)$$

or

- Minimize approximate Gibbs free energy

$$G = \left\langle H_{\mathcal{P}(s|d)} + \log\left(\mathcal{G}(s-m,D)\right)\right\rangle_{\mathcal{G}(s-m,D)}$$

Enßlin & Weig (2010)

Problem: $\mathcal{P}(s|d)$ is non-Gaussian.
Solution: Find Gaussian $\mathcal{G}(s-m, D)$, that best approximates $\mathcal{P}(s|d)$.

Extended Critical Filter

$$m = Dj, \quad D = \left[\sum_k p_k^{-1} S_k^{-1} + \sum_i \eta_i^{-1} R^\dagger N_i^{-1} R\right]^{-1},$$

$$j = \sum_i \eta_i^{-1} R^\dagger N_i^{-1} d$$

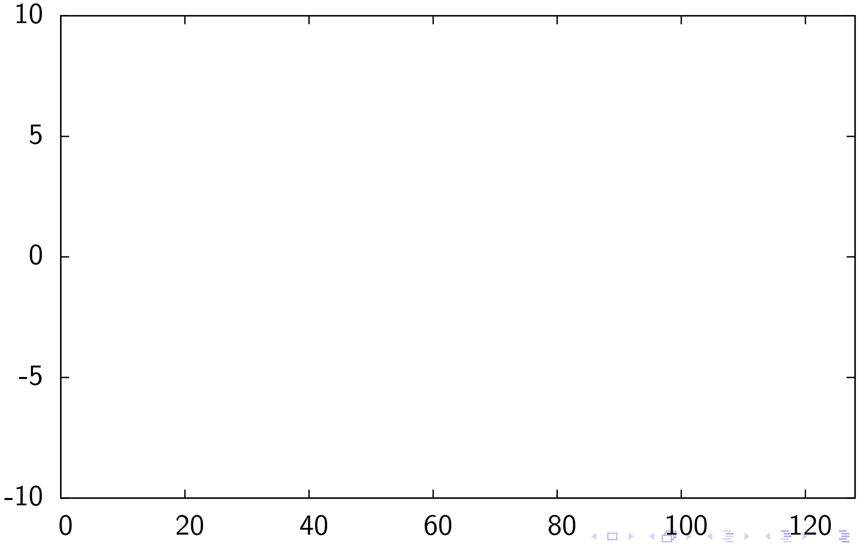
$$p_k = \frac{q_k + \frac{1}{2}\mathrm{tr}\left(\left(mm^\dagger + D\right) S_k^{-1}\right)}{\alpha_k - 1 + \mathrm{tr}\left(S_k S_k^{-1}\right)}$$

$$\eta_i = \frac{q_i + \frac{1}{2}\mathrm{tr}\left(\left(\left(d - Rm\right)\left(d - Rm\right)^\dagger + RDR^\dagger\right) N_i^{-1}\right)}{\alpha_i - 1 + \mathrm{tr}\left(N_i N_i^{-1}\right)}$$

1D test case

Assumptions:

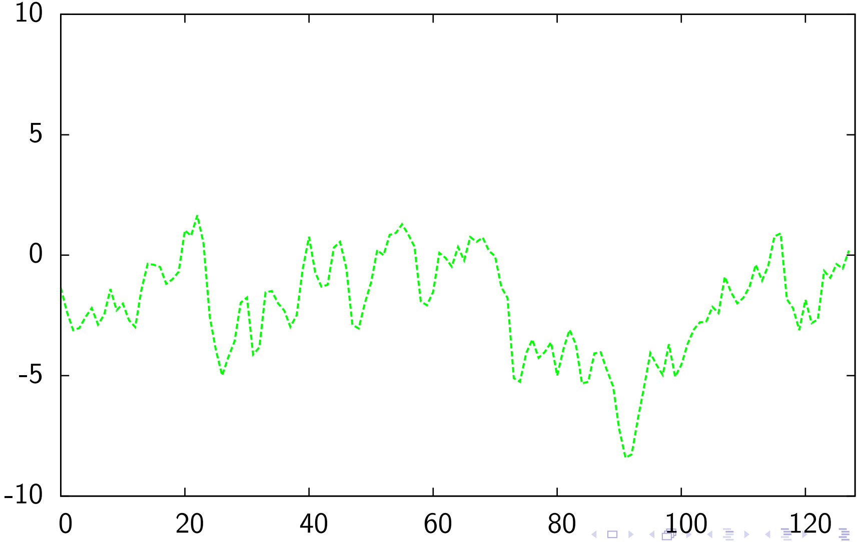
-
-



1D test case

Assumptions:

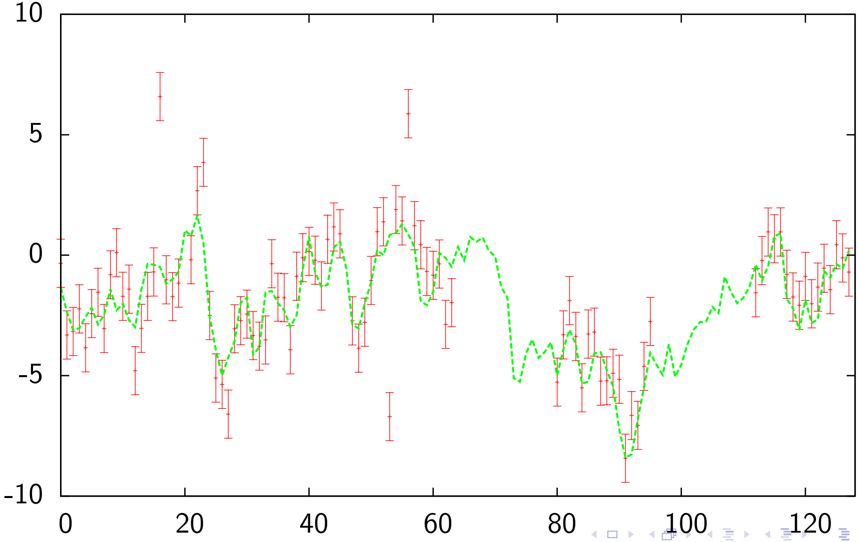
- signal field statistically homogeneous Gaussian random field
-



1D test case

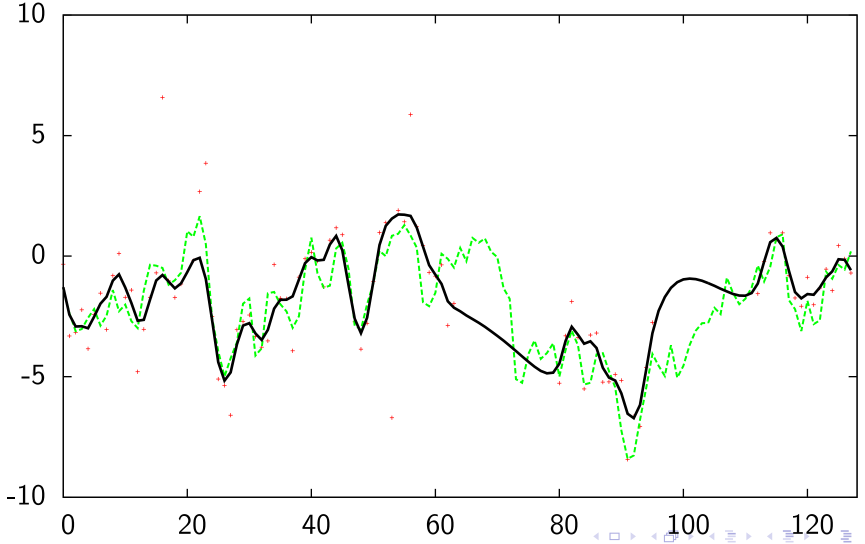
Assumptions:

- signal field statistically homogeneous Gaussian random field
- noise uncorrelated, Gaussian



1D test case

- Reconstruct (iteratively):
  signal, power spectrum, noise variance



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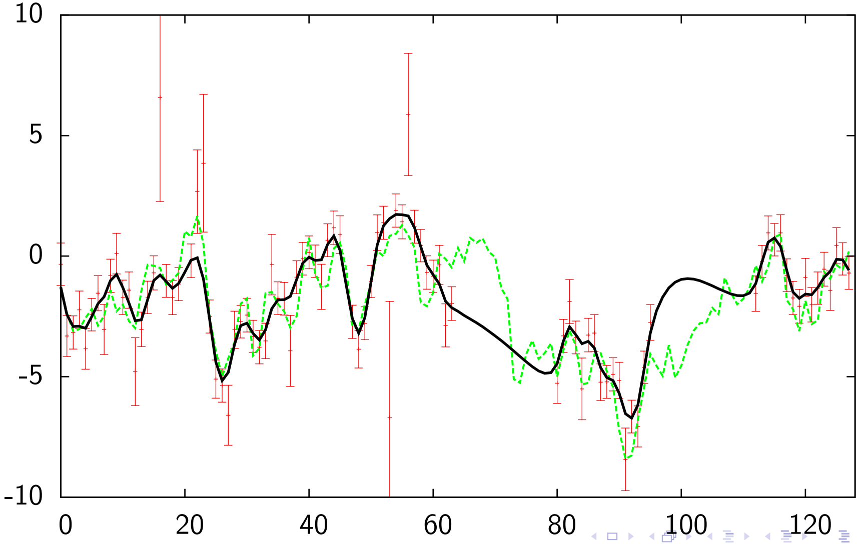
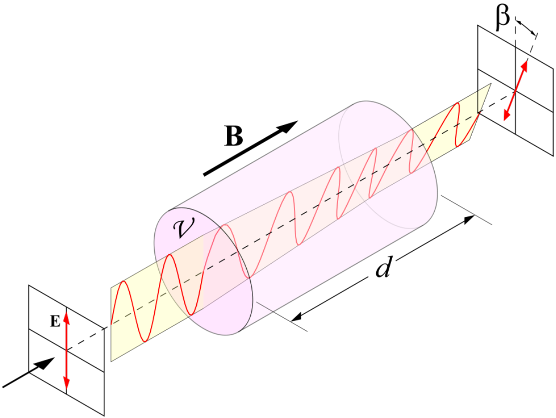
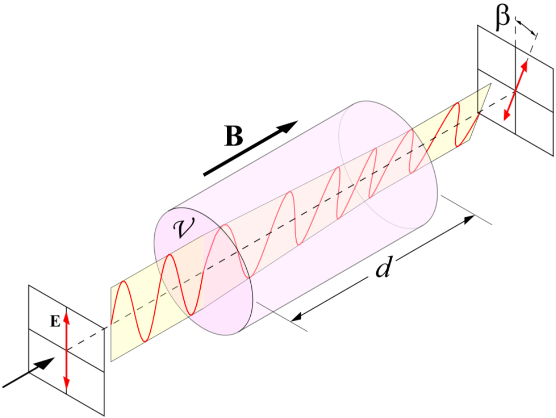


The application



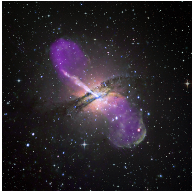
$$\mathrm{d}\beta \propto \lambda^2 n_\mathrm{e}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$$

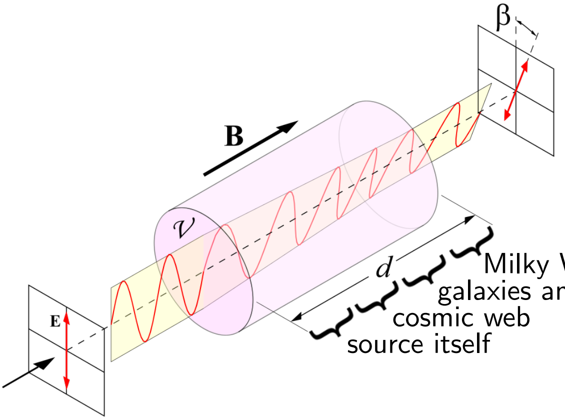
$$\Rightarrow \quad \beta \propto \lambda^2 \int_{r_\mathrm{source}}^{0} n_\mathrm{e}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$$



$$\text{Faraday depth: } \phi \propto \int_{r_{\rm source}}^{0} n_{\rm e}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$$

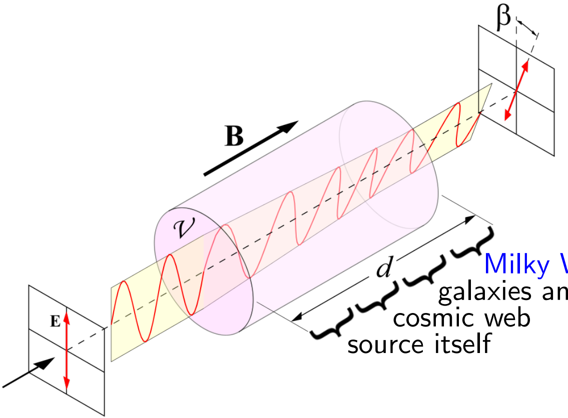
$$\beta = \phi \lambda^2$$

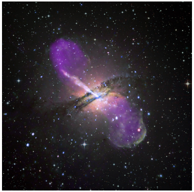




Faraday depth: $\phi \propto \int_{r_{\mathrm{source}}}^{0} n_{\mathrm{e}}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$

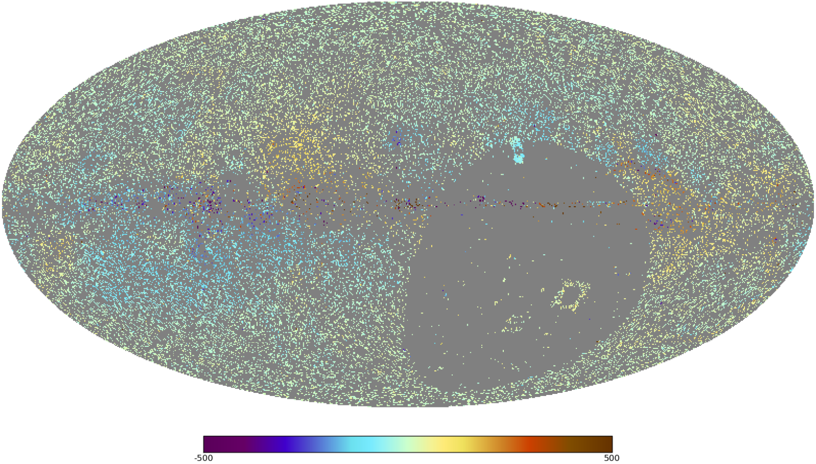
$$\beta = \phi \lambda^2$$



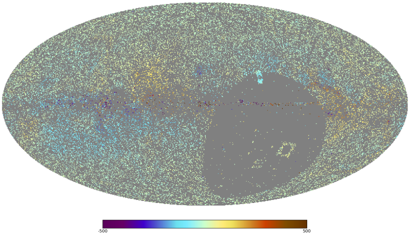


Galactic Faraday depth:

$$\phi \propto \int_{r_{\mathrm{MilkyWay}}}^{0} n_{\mathrm{e}}(\vec{x}) B_r(\vec{x}) \mathrm{d}r$$

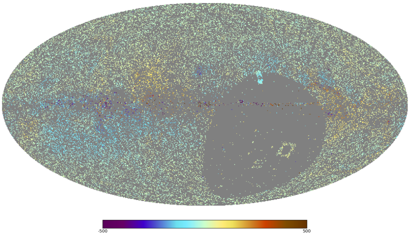


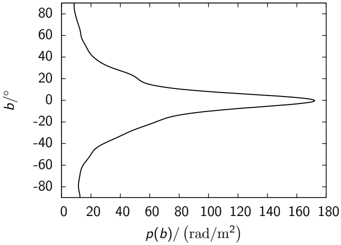
41 330 data points



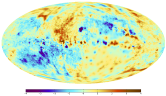
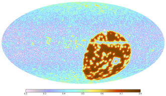
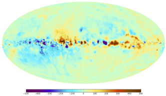
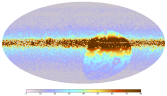
Challenges

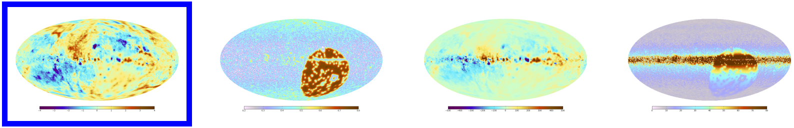
- Regions without data
- Uncertain error bars:
  - complicated observations
  - $n\pi$-ambiguity
  - extragalactic contributions unknown



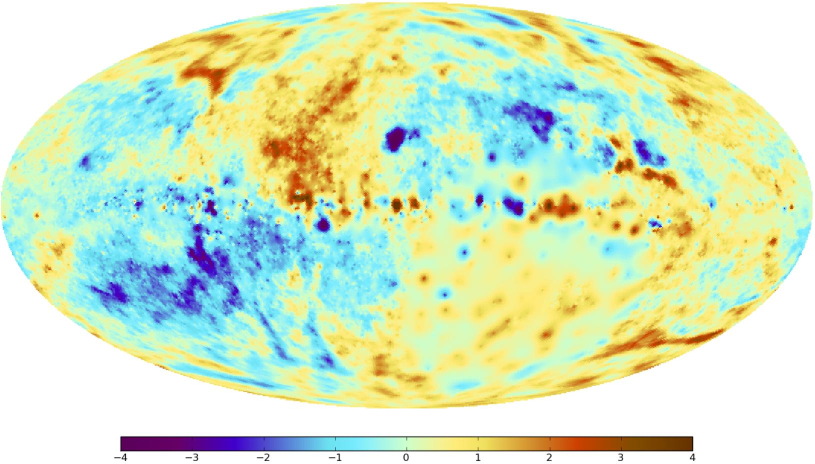


- Approximate $s(b, l) := \frac{\phi(b,l)}{p(b)}$ as a statistically isotropic Gaussian field
- R: multiplication with $p(b)$ and projection on directions of sources
- $N_{ij} = \delta_{ij}\eta_i\sigma_i^2$

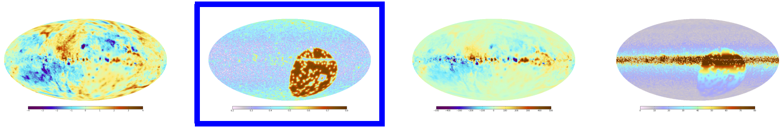







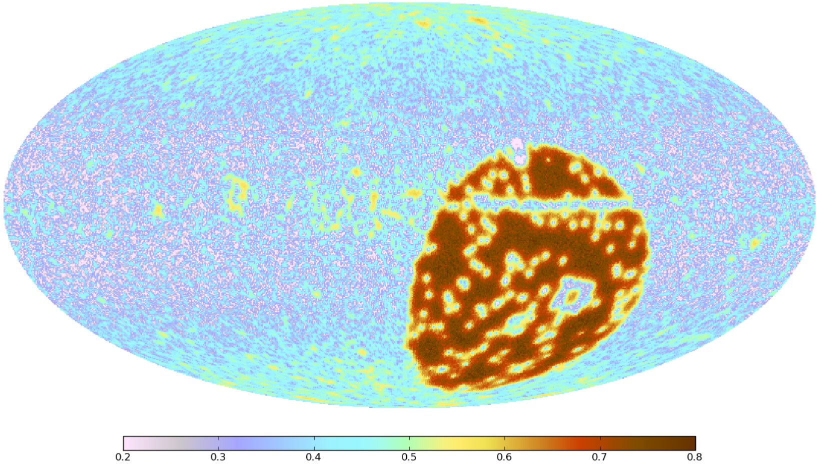
posterior mean of the signal



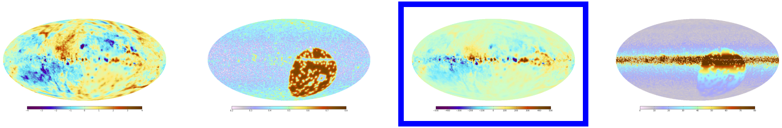
m



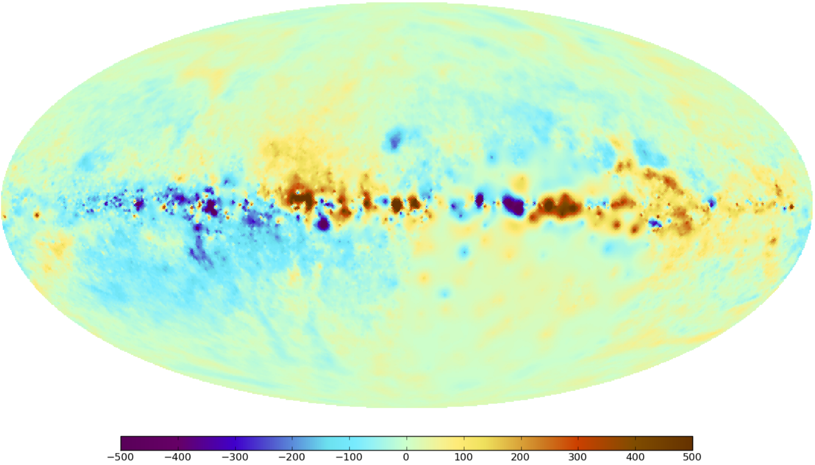
uncertainty of the signal map

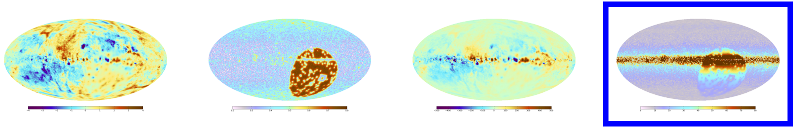


$$\sqrt{\mathrm{diag}\,(D)}$$

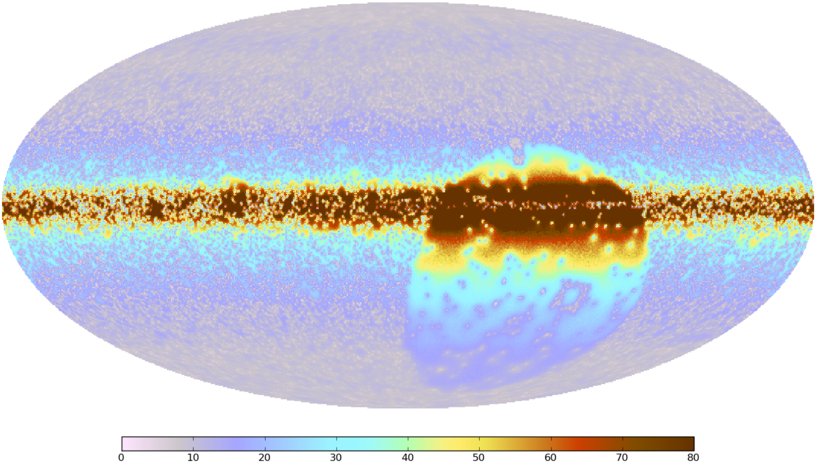


posterior mean of the Faraday depth





uncertainty of the Faraday depth



$$p\sqrt{\mathrm{diag}\,(D)}$$

Summary

1. The *extended critical filter* reconstructs
   - “smooth” signals
   - from data that are
     - noisy
     - and incomplete.
2. It makes use of the
   - signal covariance
   - and noise covariance

   even though they are unknown.

http://www.mpa-garching.mpg.de/ift/faraday/